PROBLEM 1

Four friends, Ali, Joy, Kay, and Lyn, decided to celebrate Kay’s birthday with dinner at the Clever Cafe. Each chose one main course (duck, beef, pork, or fish) and one dessert (mousse, torte, fruit, or crepe) but no two chose the same dish. Their surnames, in no particular order, are Gorman, Foster, Fields, and Devlin. Use the following clues to determine their full names and what they had to eat.

1. Joy never eats pork but Ali had crepe.
2. The beef-eater has the letter ‘a’ in her first name.
3. Ms Gorman had fruit dessert but is not the birthday girl.
4. The fish-eater had torte but she wasn’t Ms Devlin.
5. Ms Foster had duck but not mousse.

Solution

We keep track of the information by using a table, putting T for true and F for false in appropriate squares. The five clues give:

<table>
<thead>
<tr>
<th></th>
<th>Gorman</th>
<th>Foster</th>
<th>Fields</th>
<th>Devlin</th>
<th>duck</th>
<th>beef</th>
<th>pork</th>
<th>fish</th>
<th>mousse</th>
<th>torte</th>
<th>fruit</th>
<th>crepe</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ali</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>T</td>
</tr>
<tr>
<td>Joy</td>
<td></td>
<td></td>
<td>F</td>
<td></td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kay</td>
<td></td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lyn</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>mousse</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>torte</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>T</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>fruit</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>crepe</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>duck</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>T</td>
</tr>
<tr>
<td>beef</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>pork</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>fish</td>
<td></td>
<td></td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
If T appears in a $4 \times 4$ box, then F must appear in the remaining three squares of the box’s row and column that contain the T. This gives:

<table>
<thead>
<tr>
<th></th>
<th>Gorman</th>
<th>Foster</th>
<th>Fields</th>
<th>Devlin</th>
<th>duck</th>
<th>beef</th>
<th>pork</th>
<th>fish</th>
<th>mousse</th>
<th>torte</th>
<th>fruit</th>
<th>crepe</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ali</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>Joy</td>
<td></td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>Kay</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>Lyn</td>
<td></td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>mousse</td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>torte</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>fruit</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>crepe</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>duck</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>beef</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>pork</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>fish</td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
</tbody>
</table>

From the bottom row, the fish-eater was Ms Gorman or Ms Fields. From Clue 4, the fish-eater had torte. From row 6, the torte-eater was not Ms Gorman. So Ms Fields had fish and torte. This gives:

<table>
<thead>
<tr>
<th></th>
<th>Gorman</th>
<th>Foster</th>
<th>Fields</th>
<th>Devlin</th>
<th>duck</th>
<th>beef</th>
<th>pork</th>
<th>fish</th>
<th>mousse</th>
<th>torte</th>
<th>fruit</th>
<th>crepe</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ali</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td></td>
</tr>
<tr>
<td>Joy</td>
<td></td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>Kay</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>Lyn</td>
<td></td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>mousse</td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>torte</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>fruit</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>crepe</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>duck</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>beef</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>pork</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>fish</td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>F</td>
</tr>
</tbody>
</table>
From row 5, Ms Devlin had mousse. So Ms Foster had crepe and is therefore Ali from Clue 1. From Clue 5, she also had duck. This gives:

<table>
<thead>
<tr>
<th></th>
<th>Gorman</th>
<th>Foster</th>
<th>Fields</th>
<th>Devlin</th>
<th>duck</th>
<th>beef</th>
<th>pork</th>
<th>fish</th>
<th>mousse</th>
<th>torte</th>
<th>fruit</th>
<th>crepe</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ali</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>Joy</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>Kay</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>Lyn</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>mousse</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>torte</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>fruit</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>crepe</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>duck</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>beef</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>pork</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>fish</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
</tr>
</tbody>
</table>

From Row 2, Joy had fish. Hence, from the bottom row, she is Ms Fields. This gives:
From Column 1, Lyn’s surname is Gorman and form Row 4 she had pork. This gives:

<table>
<thead>
<tr>
<th></th>
<th>Gorman</th>
<th>Foster</th>
<th>Fields</th>
<th>Devlin</th>
<th>duck</th>
<th>beef</th>
<th>pork</th>
<th>fish</th>
<th>mousse</th>
<th>torte</th>
<th>fruit</th>
<th>crepe</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ali</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td></td>
</tr>
<tr>
<td>Joy</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>Kay</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>Lyn</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
<td>F</td>
</tr>
<tr>
<td>mousse</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td></td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>torte</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>fruit</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td></td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>crepe</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td></td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>duck</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>beef</td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>pork</td>
<td>F</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>fish</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Thus Ali Foster had duck and crepe, Joy Fields had fish and torte, Kay Devlin had beef and mousse, and Lyn Gorman had pork and fruit.

**Comment**

It would be interesting to drop one of the five given conditions in the problem to see

1. if the dropped condition is necessary, that is, if any other solutions would be possible if it was dropped, and

2. if the dropped condition could be replaced with another condition to give a unique solution that is the same or different to the original solution.
PROBLEM 2

a  Use dot paper to show how each of these polygons can tessellate by itself. Draw at least 12 copies of each polygon to show how it tessellates.

\begin{center}
\begin{tikzpicture}
\fill[lightgray] (0,0) rectangle (3,3);
\draw (0,1) -- (1,2) -- (2,1) -- (1,0) -- cycle;
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
\fill[lightgray] (0,0) rectangle (3,3);
\draw (0,1) -- (1,2) -- (2,1) -- (1,0) -- cycle;
\end{tikzpicture}
\end{center}

b  For each of the following polygons, change the shape of side $AB$ to make a new polygon that tessellates by itself. Draw at least 12 copies of the new polygon to show how it tessellates.
In Part i display two different tessellations.
In Part ii change $AB$ so that the new polygon has exactly nine sides.

\begin{center}
\begin{tikzpicture}
\fill[lightgray] (0,0) rectangle (3,3);
\draw (0,1) -- (1,2) -- (2,1) -- (1,0) -- cycle;
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
\fill[lightgray] (0,0) rectangle (3,3);
\draw (0,1) -- (1,2) -- (2,1) -- (1,0) -- cycle;
\end{tikzpicture}
\end{center}
Solution

a i

ii
Comment

The other way to change side $AB$ in Part b ii is as follows. The resulting polygon has 10 sides and it also tessellates by itself.
PROBLEM 3

a Construct the addition table in base 8 and use it to construct the multiplication table in base 8.

b Use the tables from Part a to calculate

i $2736_8 \times 35_8$

ii $42752_8 \div 6_8$

Solution

a Each table is symmetric about its main diagonal so we only need to calculate approximately half the entries. For each row in the addition table we start with the row number and simply add 1 until we reach the diagonal.

$$
\begin{array}{cccccccc}
+ & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 10 \\
2 & 2 & 3 & 4 & 5 & 6 & 7 & 10 & 11 \\
3 & 3 & 4 & 5 & 6 & 7 & 10 & 11 & 12 \\
4 & 4 & 5 & 6 & 7 & 10 & 11 & 12 & 13 \\
5 & 5 & 6 & 7 & 10 & 11 & 12 & 13 & 14 \\
6 & 6 & 7 & 10 & 11 & 12 & 13 & 14 & 15 \\
7 & 7 & 10 & 11 & 12 & 13 & 14 & 15 & 16
\end{array}
$$

For each column in the multiplication table we start with 0 then add the number of the column until we reach the diagonal. Remember to add 1 to the 8’s digit each time the sum of the units digits exceeds 7.

$$
\begin{array}{cccccccc}
\times & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
2 & 0 & 2 & 4 & 6 & 10 & 12 & 14 & 16 \\
3 & 0 & 3 & 6 & 11 & 14 & 17 & 22 & 25 \\
4 & 0 & 4 & 10 & 14 & 20 & 24 & 30 & 34 \\
5 & 0 & 5 & 12 & 17 & 24 & 31 & 36 & 43 \\
6 & 0 & 6 & 14 & 22 & 30 & 36 & 44 & 52 \\
7 & 0 & 7 & 16 & 25 & 34 & 43 & 52 & 61
\end{array}
$$
Comment

It is interesting to use the base 8 multiplication table to compare some of the divisibility rules in base 10 with the corresponding rules in base 8.

For example, in base 10, a number is divisible by 2 if and only if its unit digit is divisible by 2. This is also true in base 8 (except the units digit can only be 0, 2, 4, or 6).

A number in base 10 is divisible by 3 if and only if the sum of its digits is divisible by 3. This is not true in base 8. For example, $3 \times_8 3 = 11$ but 3 does not divide $1 +_8 1 = 2$.

A number in base 10 is divisible by 5 if and only if its unit digit is 0 or 5. In base 8, this rule translates to ‘a number is divisible by 4 if and only if its unit digit is 0 or 4’. This is also true.

A number in base 10 is divisible by 9 if and only if the sum of its digits is divisible by 9. In base 8, this rule translates to ‘a number is divisible by 7 if and only if the sum of its digits is divisible by 7’. This is also true.
PROBLEM 4

If a number contains the uninterrupted sequence 123, then we call it a 123-number. For example, 351238 is a 123-number but 631245 and 719235 are not.

a Starting with the smallest, what are the first, second, third, and fourth 123-numbers?

b Find the 4000th 123-number.

Solution

a 123, 1123, 1230, 1231.

b We start to count the 123-numbers by first considering the 3-digit and 4-digit numbers.

The only 3-digit 123-number is 123.

The 4-digit 123-numbers have the form \(a123\) or \(123a\) where \(a\) is a single digit. If \(a\) is the first digit, then it is any digit except 0. If it is the last digit, then it can be any digit. Thus there are nine 123-numbers of the first kind and ten of the second kind.

So the number of 123-numbers so far is \(1 + 9 + 10 = 20\).

The 5-digit 123-numbers have the form \(ab123\) or \(a123b\) or \(123ab\) where \(a\) and \(b\) are single digits. There are \(9 \times 10 = 90\) numbers of the first kind, \(9 \times 10 = 90\) of the second kind, and \(10 \times 10 = 100\) of the third kind. Thus the number of 5-digit 123-numbers is \(90 + 90 + 100 = 280\).

So the number of 123-numbers so far is \(20 + 280 = 300\).

The 6-digit 123-numbers have the form \(abc123\) or \(ab123c\) or \(a123bc\) or \(123abc\) where \(a\), \(b\), \(c\) are single digits. There are \(9 \times 10 \times 10 = 900\) numbers of the first kind, \(9 \times 10 \times 10 = 900\) of the second kind, \(9 \times 10 \times 10 = 900\) of the third kind, and \(10 \times 10 \times 10 = 1000\) of the fourth kind. But the number 123123 is both of the first and fourth kind, so it is counted twice. Thus the number of 6-digit 123-numbers is \(900 + 900 + 900 + 1000 - 1 = 3700 - 1 = 3699\).

So the number of 123-numbers so far is \(300 + 3699 = 3999\).

The largest 6-digit 123-number is 999123. The next largest 123-number must have 7 digits and is therefore 1000123. So the 4000th 123-number is 1000123.
Comment

This problem illustrates the technique of simplifying the enumeration of objects by dividing them into collections that are easier to count. When we do this however, we must be careful that (a) we have not missed any objects and (b) no object is counted twice.

For example, when we counted the 6-digit 123-numbers in the solution above, the number 123123 was counted twice: once as a number of the form $abc123$ and once as a number of the form $123abc$.

If we wanted to count all of the 7-digit 123-numbers, then we would have to be even more careful of double counting. All ten of the 7-digit numbers $123123d$ would be counted twice: once as a number of the form $123abcd$ and once as a number of the form $abc123d$. Similarly, all ten of the 7-digit numbers $123a123$ would be counted twice and all nine of the 7-digit numbers $a123123$ would be counted twice.
PROBLEM 5

Jack and Jill live on a farm. They decided to go to the swimming hole on the river 4 km away. The only way of getting there was riding Jill’s bike or walking. They decided to do both.

Starting from the farmhouse at the same time, Jill rode her bike while Jack walked. After one kilometre Jill left her bike at the side of the track and walked for the next kilometre. When Jack reached Jill’s bike he rode it for a kilometre and left it for Jill. They continued in this way until they reached the river.

Jack walks at an average speed of 6 km/h and rides at an average speed of 24 km/h. Jill walks a little slower than Jack at 5 km/h but rides much faster at 30 km/h.

**a** At what time did Jill reach the river and what was her average speed?

**b** At what time did Jack reach the river and what was his average speed?

**c** How far from the farmhouse, to the nearest metre, did Jack and Jill first meet along the track?

**Solution**

**a** The time taken for Jill to ride her bike for 1 km at 30 km/h was $1 \div 30$ hours $= \left(\frac{1}{30}\right) \times 60$ minutes $= 2$ minutes. The time taken for Jill to walk for 1 km at 5 km/h was $1 \div 5$ hours $= \left(\frac{1}{5}\right) \times 60$ minutes $= 12$ minutes.

So Jill took $2 + 12 + 2 + 12 = 28$ minutes to reach the river and her average speed was $\frac{4}{28} \times 60$ km/h $= \frac{60}{7}$ km/h $\approx 8.6$ km/h.

**b** The time taken for Jack to ride the bike for 1 km at 24 km/h was $1 \div 24$ hours $= \left(\frac{1}{24}\right) \times 60$ minutes $= 2.5$ minutes. The time taken for Jack to walk for 1 km at 6 km/h was $1 \div 6$ hours $= \left(\frac{1}{6}\right) \times 60$ minutes $= 10$ minutes.

So Jack took $10 + 2.5 + 10 + 2.5 = 25$ minutes to reach the river and his average speed was $\frac{4}{25} \times 60$ km/h $= \frac{48}{5}$ km/h $= 9.6$ km/h.

**c** From Parts **a** and **b**, Jill reached the 1 km point in 2 minutes and Jack in 10 minutes. Jill reached the 2 km point in 14 minutes and Jack reached it in 12.5 minutes. So they first met somewhere between the 1 km and 2 km points.
Alternative i

We guess the distance from the 1 km point where they met and check the times they took to reach that distance.

<table>
<thead>
<tr>
<th>Distance (metres)</th>
<th>Jill’s time in minutes = 2 +</th>
<th>Jack’s time in minutes = 10 +</th>
<th>First there</th>
</tr>
</thead>
<tbody>
<tr>
<td>1500</td>
<td>$(0.5)\frac{60}{5} = 8$</td>
<td>$(0.5)\frac{60}{24} = 11.25$</td>
<td>Jill</td>
</tr>
<tr>
<td>1800</td>
<td>$(0.8)\frac{60}{5} = 11.6$</td>
<td>$(0.8)\frac{60}{24} = 12$</td>
<td>Jill</td>
</tr>
<tr>
<td>1850</td>
<td>$(0.85)\frac{60}{5} = 12.2$</td>
<td>$(0.85)\frac{60}{24} = 12.125$</td>
<td>Jack</td>
</tr>
<tr>
<td>1840</td>
<td>$(0.84)\frac{60}{5} = 12.08$</td>
<td>$(0.84)\frac{60}{24} = 12.1$</td>
<td>Jill</td>
</tr>
<tr>
<td>1845</td>
<td>$(0.845)\frac{60}{5} = 12.14$</td>
<td>$(0.845)\frac{60}{24} = 12.11$</td>
<td>Jack</td>
</tr>
<tr>
<td>1842</td>
<td>$(0.842)\frac{60}{5} = 12.104$</td>
<td>$(0.842)\frac{60}{24} = 12.105$</td>
<td>Jill</td>
</tr>
<tr>
<td>1842.5</td>
<td>$(0.8425)\frac{60}{5} = 12.11$</td>
<td>$(0.8425)\frac{60}{24} = 12.106$</td>
<td>Jack</td>
</tr>
</tbody>
</table>

Thus, when Jack and Jill first met, they were approximately 1842 metres from the farmhouse.

Alternative ii

Suppose the distance from the farmhouse to their first meeting point was $1 + d$ km. Both took the same time to get there. Jill’s time in minutes was $2 + (d \div 5) \times 60$ while Jack took $10 + (d \div 24) \times 60$ minutes. So we have

$$2 + \left(\frac{d}{5}\right)60 = 10 + \left(\frac{d}{24}\right)60$$

$$\left(\frac{d}{5}\right)60 - \left(\frac{d}{24}\right)60 = 8$$

$$12d - 2.5d = 8$$

$$9.5d = 8$$

Hence $d = \frac{8}{9.5} \approx 0.8421$ km. Thus, when Jack and Jill first met, they were approximately 1842 metres from the farmhouse.
Alternative iii

When Jack reached the 1 km point, Jill had been walking at 5 km/h for $10 - 2 = 8$ minutes. So her distance beyond the 1 km point at that time was $5 \times \frac{8}{60} = \frac{2}{3}$ km. Jack then closed this gap between himself and Jill at a speed of $24 - 5 = 19$ km/h. So the time taken to catch up was $\frac{2}{3} \div 19 = \frac{2}{57}$ hours. Hence the distance that Jack rode from the 1 km point to their first meeting point was $24 \times \frac{2}{57} = \frac{16}{19} \approx 0.8421$ km. Thus, when Jack and Jill first met, they were approximately 1842 metres from the farmhouse.

Comment

The answers to this problem seem to go against common sense. Although Jill walks a little slower than Jack, she cycles much faster. Since they walk and cycle the same distances, this suggests that she should reach the river before Jack. Why does Jack get there first?

The cycle and walk distances may be the same but the cycle and walk times are quite different. The ratio of Jack and Jill’s times over any kilometre leg is the inverse ratio of their speeds for that leg. So Jill’s walking time for 1 km is $\frac{6}{5}$ths of Jack’s time, that is, $\frac{6}{5} \times 10 = 12$ minutes. Jack’s cycling time for 1 km is $\frac{30}{24}$ths of Jill’s time, that is, $\frac{30}{24} \times 2 = 2.5$ minutes. Although Jill’s walking time is 20% more than Jack’s and Jack’s cycling time is 25% more than Jill’s, these percentages apply to significantly different amounts of time: 10 minutes and 2 minutes respectively. So the cycling legs contribute a smaller time difference than the walking legs. Hence Jack beats Jill to the river.

This is an illustration of the speeding paradox: time delays on route are practically impossible to make up by speeding.
PROBLEM 6

Congruent rhombuses have been used to make the following sequence of patterns. The length of each side of every rhombus is 1 cm.

We define the following terms for the $n$th pattern $R_n$:

- $d_n =$ the number of dots in $R_n$
- $b_n =$ the number of rhombuses in $R_n$
- $e_n =$ the number of rhombus sides in $R_n$
- $p_n =$ the perimeter of $R_n$ in centimetres
- $s_n =$ the number of internal (non-perimeter) rhombus sides in $R_n$
- $c_n =$ the number of internal (non-perimeter) dots in $R_n$.

a Find $d_4$, $b_4$, $e_4$, $p_4$, $s_4$, and $c_4$.

b Show how $b_n$ and $c_n$ are related and find a formula for each in terms of $n$.

c Show how $d_n$, $p_n$, $c_n$ are related and find a formula for each of $d_n$ and $p_n$ in terms of $n$.

d Show how $e_n$, $p_n$, $s_n$ are related and find a formula for each of $e_n$ and $s_n$ in terms of $n$. 

Solution

\[ a \] \( d_4 = 8 + d_3 = 10 + 27 = 37, \quad b_4 = 6 + b_3 = 6 + 15 = 21, \)
\[ e_4 = 16 + e_3 = 16 + 41 = 57, \quad p_4 = 8 + p_3 = 8 + 22 = 30, \)
\[ s_4 = 8 + s_3 = 8 + 19 = 27, \quad c_4 = 2 + c_3 = 2 + 5 = 7. \]

\[ b \] Alternative i

In any \( R_n \), there are three unique rhombuses around each internal dot and there are no other rhombuses. So, for all \( n \), \( b_n = 3 \times c_n \). The \( c_n \) sequence starts with 1 and increases in steps of 2. Hence \( c_n = 2n - 1 \). Therefore \( b_n = 3(2n - 1) = 6n - 3 \).

Alternative ii

The \( b_n \) sequence starts with 3 and increases in steps of 6. Hence \( b_n = 6n - 3 \). The \( c_n \) sequence starts with 1 and increases in steps of 2. Hence \( c_n = 2n - 1 \). Therefore \( b_n = 3c_n \).

\[ c \] Alternative i

The total number of dots in \( R_n \) is the number of dots on the perimeter plus the number of internal dots. Hence \( d_n = p_n + c_n \). From Part b, \( c_n = 2n - 1 \). The \( p_n \) sequence starts with 6 and increases in steps of 8. Hence \( p_n = 8n - 2 \). Therefore \( d_n = (8n - 2) + (2n - 1) = 10n - 3 \).

Alternative ii

The \( d_n \) sequence starts with 7 and increases in steps of 10. Hence \( d_n = 10n - 3 \). The \( p_n \) sequence starts with 6 and increases in steps of 8. Hence \( p_n = 8n - 2 \). From Part b, \( c_n = 2n - 1 \). Hence \( d_n = p_n + c_n \).

\[ d \] Alternative i

The total number of rhombus sides in \( R_n \) is the number of rhombus sides on the perimeter plus the number of internal rhombus sides. Hence \( e_n = p_n + s_n \). From Part c, \( p_n = 8n - 2 \). The \( s_n \) sequence starts with 3 and increases in steps of 8. Hence \( s_n = 8n - 5 \). Therefore \( e_n = (8n - 2) + (8n - 5) = 16n - 7 \).

Alternative ii

The \( e_n \) sequence starts with 9 and increases in steps of 16. Hence \( e_n = 16n - 7 \). From Part c, \( p_n = 8n - 2 \). The \( s_n \) sequence starts with 3 and increases in steps of 8. Hence \( s_n = 8n - 5 \). Hence \( e_n = p_n + s_n \).
Comment

It is sometimes tricky to find the correct formula for the $n$th term in a sequence. One method, which is useful in simple cases, is to guess the formula from the first few terms in the sequence and then check that it works. Two other methods are explained below. The method used in the problem above is basically a variation of the recursion method.

1. Translation method. If you notice that the terms increase by a constant amount $c$, then they will all have the same remainder $r$ when you divide by $c$. So subtracting $r$ from every term will convert the terms into consecutive multiples of $c$. This is called translating the sequence. Then adding or subtracting $c$ a fixed number of times from each term will convert the terms into consecutive multiples of $c$ with the first term equal to $c$. So the $n$th term will now be $n \times c$. Working backwards then gives the formula for the $n$th term in the original sequence.

For example, in the problem above, the $e_n$ sequence starts with 9, 25, 41, 57. These terms increase by 16. Their remainders are 9 when they are divided by 16. Subtracting 9 from each term gives 0, 16, 32, 48. Adding 16 to each of these terms gives 16, 32, 48, 64, ..., 16. So $e_n = 16n - 16 + 9 = 16n - 7$.

2. Recursion method. If you notice that the terms increase by a constant amount $c$, then the next term is always $c$ plus the one before it, hence $2c$ plus the second term before it, hence $3c$ plus the third term before it, and so on. Thus the $n$th term is $n \times c$ plus the $n$th term before it. But the $n$th term before the $n$th term is the term before the first term and therefore doesn’t exist! Well, we can make one by subtracting $c$ from the first term.

For example, in the problem above, the $e_n$ sequence starts with 9, 25, 41, 57. These terms increase by 16. So the term we need before the first term is $9 - 16$. Hence $e_n = 16n + (9 - 16) = 16n - 7$. 

19
PROBLEM 7

Rana loves hopscotch. When hopping along a row of squares, she can hop a distance of up to three squares at a time. She always hops from a smaller number to a larger number. For example, from square 5, she can hop to square 6, 7, or 8. There are 4 ways she can hop from ‘home’ to square 3: 1, 2, 3; 1, 3; 2, 3; or 3. Calculate how many ways she can hop to the tenth square from ‘home’.

Solution

It is difficult keeping track of all possible hopping sequences that start from home. It is easier to work backwards from square 10.

Rana can reach square 10 in one hop from squares 7, 8, or 9. Thus the number of ways of reaching square 10 equals the number of ways of reaching square 9 plus the number of ways of reaching square 8 plus the number of ways of reaching square 7.

Similarly, the number of ways of reaching any square after square 3 is the sum of the number of ways of reaching the previous three squares.

There is only one way Rana may hop from home to square 1, two ways from home to square 2, and four ways from home to square 3. Hence the number of ways she may hop from home to square 4 is $1 + 2 + 4 = 7$.

The number of ways Rana may hop from home to each square is summarised in the following table.

<table>
<thead>
<tr>
<th>Destination square</th>
<th>Number of ways to hop</th>
<th>Destination square</th>
<th>Number of ways to hop</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>6</td>
<td>$4 + 7 + 13 = 24$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>7</td>
<td>$7 + 13 + 24 = 44$</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>8</td>
<td>$13 + 24 + 44 = 81$</td>
</tr>
<tr>
<td>4</td>
<td>$1 + 2 + 4 = 7$</td>
<td>9</td>
<td>$24 + 44 + 81 = 149$</td>
</tr>
<tr>
<td>5</td>
<td>$2 + 4 + 7 = 13$</td>
<td>10</td>
<td>$44 + 81 + 149 = 274$</td>
</tr>
</tbody>
</table>

Thus there are 274 ways Rana may hop from home to square 10.
This problem is an example of enumeration by *recursion*. Suppose we let \( h_n \) denote the number of hopping patterns from home to destination square \( n \). Instead of calculating \( h_n \) directly in terms of \( n \), we looked ‘backwards’ and expressed \( h_n \) in terms of a few previous values of this function, such as \( h_{n-1}, h_{n-2}, \) and so on. The equation that relates these values is called a *recurrence relation*. The recurrence relation we found for \( h_n \) was

\[
h_n = h_{n-1} + h_{n-2} + h_{n-3}.
\]

Notice that the recurrence relation by itself is not enough to calculate specific values of \( h_n \). We also need some *initial values* in order to start the calculations. Initial values are usually quite easy to find. In this problem, for example, we needed to know \( h_3, h_2, h_1 \) in order to calculate \( h_4 \). Then we were able to calculate \( h_5 \), and so on.
PROBLEM 8

a  Find the repetend for the fraction $1/103$.

b  Find the five largest fractions amongst $n/103$ for $n = 1, 2, 3, ..., 102$ whose repetends have the same cyclic order as the repetend for $1/103$.

c  Find three fractions amongst $n/103$ for $n = 1, 2, 3, ..., 102$ whose repetends do not have the same cyclic order.

d  Represent the three different repetends in Part c as $A$, $B$, $C$. Determine which of these repetends each of the fractions $n/103$ has for $n = 1, 2, 3, ..., 10$ and $93, 94, 95, ..., 102$. What pattern do you notice?

Solution

a  From a calculator, 

$1/103 \approx 0.009708738,$

$0.738 \times 103 = 76.014,$

$0.078 \times 103 = 8.034,$

$0.903 \times 103 = 93.009,$

$0.621 \times 103 = 63.963,$

$0.223 \times 103 = 22.969,$

Hence $1/103 = 0.009708738640776699029126213592233$.

b  The five largest cyclic rotations of the repetend for $1/103$ in decreasing order of size are:

$99029126213592233009708737864077669902912621359223300,$

$9708737864077669902912621359223300,$

$9223300970873786407766990291262135,$

$9126213592233009708737864077669902,$

$9029126213592233009708737864077669.$

Since $0.990 \times 103 = 101.97$, the largest fraction is $102/103$.

Since $0.970 \times 103 = 99.91$, the next fraction is $100/103$.

Since $0.922 \times 103 = 94.966$, the next fraction is $95/103$.

Since $0.912 \times 103 = 93.936$, the next fraction is $94/103$.

Since $0.903 \times 103 = 93.009$, the next fraction is $93/103$.

c  We already have the repetend for $1/103$. Its length is 34, which is even, so we may use the ‘rule of 9’ to find the other two repetends as they will have the same length.
From a calculator, \( \frac{2}{103} \approx 0.019417476 \),
\[ 0.476 \times 103 = 49.028, \quad 49/103 \approx 0.475728155, \]
\[ 0.155 \times 103 = 15.965, \quad 16/103 \approx 0.155339806. \]

Hence the first 17 digits in the repetend for \( \frac{2}{103} \) are
01941747572815533. By the ‘rule of 9’,
\[ 2/103 = 0.0194174757281553398058252427184466. \]

From a calculator, \( \frac{3}{103} \approx 0.029126214 \). Since the sequence
02912621 occurs in the repetend for \( \frac{1}{103} \), this suggests the repete-

dend for \( \frac{3}{103} \) has the same cyclic order as the repetend for \( \frac{1}{103} \). So try 4/103.

From a calculator, \( \frac{4}{103} \approx 0.038834951 \),
\[ 0.951 \times 103 = 97.953, \quad 98/103 \approx 0.951456311, \]
\[ 0.311 \times 103 = 32.033, \quad 32/103 \approx 0.310679612. \]

Hence the first 17 digits in the repetend for \( \frac{2}{103} \) are
03883495145631067. By the ‘rule of 9’,
\[ 4/103 = 0.0388349514563106796116504854368932. \]

Thus the fractions \( \frac{1}{103}, \frac{2}{103}, \) and \( \frac{4}{103} \) give three repete nds with
different cyclic orders.

\( \mathrm{d} \) Let \( A = 0097087378640776699029126213592233, \)
\[ B = 0194174757281553398058252427184466, \]
\[ C = 0388349514563106796116504854368932. \]

From Parts a, b, c, we already know the repetends for \( n = 1, 2, 3, 4, \)
93, 94, 95, 98, 100, 102.

\[
\begin{array}{cccccccccccc}
| n | 1 | 2 | 3 & 4 | 5 & 6 & 7 & 8 & 9 | 10 \\
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>93</td>
<td>94</td>
<td>95</td>
<td>96</td>
<td>97</td>
<td>98</td>
<td>99</td>
<td>100</td>
<td>101</td>
<td>102</td>
<td></td>
</tr>
</tbody>
</table>
\end{array}
\]

For the remaining values of \( n \), we calculate the first few digits in the
decimal expansion of \( n/103 \). This determines their repetends. For
example, \( 5/103 \approx 0.048543689 \) and this sequence occurs only in \( C \),
hence its repetend is \( C \). This gives

\[
\begin{array}{cccccccccccc}
| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 \\
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>A</td>
<td>C</td>
<td>C</td>
<td>B</td>
<td>C</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td></td>
</tr>
<tr>
<td>93</td>
<td>94</td>
<td>95</td>
<td>96</td>
<td>97</td>
<td>98</td>
<td>99</td>
<td>100</td>
<td>101</td>
<td>102</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>A</td>
<td>A</td>
<td>C</td>
<td>B</td>
<td>C</td>
<td>C</td>
<td>A</td>
<td>B</td>
<td>A</td>
<td></td>
</tr>
</tbody>
</table>
\end{array}
\]
The repetends are the same for each of the following pairs of values for \( n \): (1, 102), (2, 101), (3, 100), (4, 99), (5, 98), etc.

**Comment**

Some students may go a little further with Part d to explain the pattern that the repetends for the \( k \)th smallest fraction and the \( k \)th largest fraction have the same cyclic order for \( k = 1, 2, 3, \) etc.

Basically the reason for this pattern is the ‘rule of 9’. If the \( k \)th smallest fraction is \( \frac{n}{103} \), then the \( k \)th largest fraction is \( 1 - \frac{n}{103} = 0.\overline{9} - \frac{n}{103} \). So, to get the decimal expansion of the \( k \)th largest fraction from the \( k \)th smallest fraction we simply subtract each digit in the \( k \)th smallest fraction from 9. This is exactly what we do under the ‘rule of 9’ to get the last half of the repetend for the \( k \)th smallest fraction from its first half. Thus the repentend for the \( k \)th smallest fraction and the \( k \)th largest fraction have the same cyclic order.