PROBLEM 1

A road from Alphaville to Echoton passes through three other towns: Bravoton, Charlyville and Deltatown in that order. The distances along the road between pairs of these towns are given in kilometres. From smallest to largest, these ten distances are 2, 5, 6, *, *, *, *, 15, 18, 20. Find all possible missing distances and explain how you got them.

Solution

The distance between Alphaville \((A)\) and Echoton \((E)\) is 20 km. The following diagram indicates the five towns on the road.

\[
\begin{array}{cccccc}
A & B & C & D & E \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

Since 2 km is not the sum of any smaller distances, it must be the distance between consecutive towns. Either the distance from \(A\) to \(B\) or the distance from \(D\) to \(E\) must be 2 km otherwise all other distances between pairs of towns would be less than 18 km. Suppose the distance from \(A\) to \(B\) is 2 km.

\[
\begin{array}{cccccc}
A & B & C & D & E \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

Since 5 km is not the sum of any smaller distances, it too must be the distance between consecutive towns. The distance from \(D\) to \(E\) must be 5 km otherwise all remaining distances between pairs of towns would be less than 15 km.

\[
\begin{array}{cccccc}
A & B & C & D & E \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]
Now 6 km is not the sum of any smaller distances, so it must be the distance between B and C or between C and D. Thus we get either of the following diagrams.

The first diagram accounts for the following distances between pairs of towns: 2, 5, 6, 7, 8, 12, 13, 15, 18, 20. The required missing distances are: 7, 8, 12, 13.

The second diagram accounts for the following distances between pairs of towns: 2, 5, 6, 7, 9, 11, 13, 15, 18, 20. The required missing distances are: 7, 9, 11, 13.

From symmetry, the same answers are obtained if the distance from D to E is 2 km.
PROBLEM 2

a Some hexominoes may be made by joining a domino to a straight tetromino. Draw all hexominoes that can be made in this way, showing clearly how the two pieces are joined.

b Some hexominoes may be made by joining a domino to a ‘T’ tetromino. Draw all hexominoes that can be made in this way, showing clearly how the two pieces are joined.

Solution

a Seven hexominoes can be made from a domino and a straight tetromino:

[Diagrams of hexominoes are shown here.]

2
b Eleven hexominoes can be made from a domino and a ‘T’ tetromino:
PROBLEM 3

a Write down a rule for multiplying a number by 25 that is based on the fact that $25 \times 4 = 100$.

b Show how to use your rule to calculate $73459 \times 25$.

c Use digit sums to check your answer to Part b.

Solution

a Since $25 = 100 \div 4 = (100 \div 2) \div 2$, we may multiply a number by 25 by first multiplying it by 100, then halving the answer twice.

b $73459 \times 25 = (73459 \times 100) \div 2 = (7345900 \div 2) \div 2 = 3672950 \div 2 = 1836475$.

c The digit sum of 1836475 = the digit sum of 34 = 7.

   The digit sum of $(73459 \times 25)$
   $= (\text{the digit sum of 28}) \times (\text{the digit sum of 7})$
   $= (\text{the digit sum of 10}) \times 7 = 1 \times 7 = 7$.

   So the answer may be correct.
PROBLEM 4

The six numbers 2, 3, 4, 5, 6, 7 are arranged in a triangle as shown in the following example.

\[
\begin{array}{ccc}
6 & \ & \\
3 & 4 & \\
2 & 5 & 7
\end{array}
\]

a  Rearrange the numbers so that the sum of the numbers along each side of the triangle is 13.

b  Rearrange the numbers so that the sum of the numbers along each side of the triangle is 14.

c  Find the triangles with the largest and smallest common side sum and explain why the sums are the largest and smallest.

Solution

a

\[
\begin{array}{ccc}
2 &  & \\
5 & 7 & \\
6 & 3 & 4
\end{array}
\]

b

\[
\begin{array}{ccc}
3 &  & \\
4 & 6 & \\
7 & 2 & 5
\end{array}
\]

c Alternative i

Because the side sums are the same, the larger the sum of two corner numbers is, the smaller the number between them must be. So for each selection of numbers for the corners, the three middle numbers on the sides are decided and then we can check if the side sums are the same. We list all possibilities systematically in the following table. Reflections and rotations of a triangle are not regarded as different.
Since there are no other ways of arranging the numbers in a triangle, the largest common side sum is 15 and the smallest is 12. The triangles with these side sums are shown below.

<table>
<thead>
<tr>
<th>Corner numbers</th>
<th>Middle numbers</th>
<th>Constant side sum?</th>
<th>Corner numbers</th>
<th>Middle numbers</th>
<th>Constant side sum?</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 4 3</td>
<td>6 7 5</td>
<td>yes, 12</td>
<td>3 5 4</td>
<td>6 7 2</td>
<td>no</td>
</tr>
<tr>
<td>2 5 3</td>
<td>6 7 4</td>
<td>no</td>
<td>3 6 4</td>
<td>5 7 2</td>
<td>no</td>
</tr>
<tr>
<td>2 6 3</td>
<td>5 7 4</td>
<td>no</td>
<td>3 7 4</td>
<td>5 6 2</td>
<td>no</td>
</tr>
<tr>
<td>2 7 3</td>
<td>5 6 4</td>
<td>no</td>
<td>3 6 5</td>
<td>4 7 2</td>
<td>no</td>
</tr>
<tr>
<td>2 5 4</td>
<td>6 7 3</td>
<td>no</td>
<td>3 7 5</td>
<td>4 6 2</td>
<td>yes, 14</td>
</tr>
<tr>
<td>2 6 4</td>
<td>5 7 3</td>
<td>yes, 13</td>
<td>3 7 6</td>
<td>4 5 2</td>
<td>no</td>
</tr>
<tr>
<td>2 7 4</td>
<td>5 6 3</td>
<td>no</td>
<td>4 6 5</td>
<td>3 7 2</td>
<td>no</td>
</tr>
<tr>
<td>2 6 5</td>
<td>4 7 3</td>
<td>no</td>
<td>4 7 5</td>
<td>3 6 2</td>
<td>no</td>
</tr>
<tr>
<td>2 7 5</td>
<td>4 6 3</td>
<td>no</td>
<td>4 7 6</td>
<td>3 5 2</td>
<td>no</td>
</tr>
<tr>
<td>2 7 6</td>
<td>4 5 3</td>
<td>no</td>
<td>5 7 6</td>
<td>3 4 2</td>
<td>yes, 15</td>
</tr>
</tbody>
</table>

Since there are no other ways of arranging the numbers in a triangle, the largest common side sum is 15 and the smallest is 12. The triangles with these side sums are shown below.

![Triangles](attachment:image.png)
Alternative ii

The sum of all six numbers is $2 + 3 + 4 + 5 + 6 + 7 = 27$.

If we add the three side sums then we include each corner number twice. So the sum of the three side sums equals the sum of all numbers plus the three corner numbers.

The largest sum of the three corner numbers is $7 + 6 + 5 = 18$ and the smallest sum is $2 + 3 + 4 = 9$. Hence the sum of the three side sums is at most $27 + 18 = 45$ and at least $27 + 9 = 36$. Therefore the constant side sum is at most $45 \div 3 = 15$ and at least $36 \div 3 = 12$.

A triangle with constant side sum 15 must have the numbers 5, 6, 7 at the corners. A triangle with constant side sum 12 must have the numbers 2, 3, 4 at the corners. In each case there is only one way to place the other three numbers.

The resulting triangles are shown below.

![Triangles with side sums 15 and 12]
PROBLEM 5

The following diagram is a net for a certain polyhedron.

![Net Diagram](image)

a How many edges does the polyhedron have? Explain your answer.

b Use Euler’s formula to determine how many vertices the polyhedron has.

c How many of each kind of face meet at each vertex of the polyhedron? Explain your answer.

d Draw a net of the polyhedron with four faces attached to the small square.

Solution

a Alternative i

To determine the number of edges on the polyhedron we could cut out an enlarged copy of the net and fold it to make a model of the polyhedron. It should look like the polyhedron in the following diagram.
This shows that the polyhedron has 20 edges: 4 on the bottom square, 4 vertical edges, and 12 on the triangles.

**Alternative ii**

By considering one edge at a time on the net we can work out what other edge it will coincide with when the net is folded. We give two coinciding edges the same number. No number will appear more than twice because each edge of the polyhedron is shared by exactly two faces.
All edges on the net are numbered and 20 numbers are used so the polyhedron has 20 edges.

b The faces of the polyhedron are 1 large square, 4 pentagons, 4 triangles, and 1 small square: a total of 10 faces. Euler’s formula, $F + V = E + 2$, with $F = 10$ and $E = 20$ gives $10 + V = 20 + 2$. Hence the number of vertices, $V$, is 12.

c Alternative i

Again, we could cut out an enlarged copy of the net and fold it to make a model of the polyhedron. This would tell us that two pentagons and the large square meet at each of four vertices; two pentagons and one triangle meet at each of four other vertices; and two triangles, a pentagon and the small square meet at each of the remaining four vertices.

Alternative ii

We can use the numbered net in Part a to determine which vertices on the net coincide as vertices on the polyhedron. If two vertices will coincide we give them the same letter.

Thus two pentagons and the large square meet at each of the vertices $a$, $b$, $c$, $d$; two pentagons and one triangle meet at each of the vertices $e$, $f$, $g$, $h$; and two triangles, a pentagon and the small square meet at each of the vertices $i$, $j$, $k$, $m$. 

There are several possible nets with four faces attached to the small square. Here is one example.
PROBLEM 6

Consider the following square arrays of numbers.

\[
\begin{array}{cccc}
7 & 8 & 9 & 10 \\
7 & 8 & 9 & 6 & 1 & 2 & 11 \\
1 & 2 & 6 & 1 & 2 & 5 & 4 & 3 & 12 \\
1 & 4 & 3 & 5 & 4 & 3 & 16 & 15 & 14 & 13 \\
\end{array}
\]

\(S_1 \quad S_2 \quad S_3 \quad S_4\)

\(a\) Write down the next two squares, \(S_5\) and \(S_6\).

\(b\) What is the first square that contains the number 2010? Explain.

\(c\) For the first square that contains 2010, find the row and column in which 2010 appears.

**Solution**

\(a\)

\[
\begin{array}{cccc}
21 & 22 & 23 & 24 & 25 & 26 \\
21 & 22 & 23 & 24 & 25 & 20 & 7 & 8 & 9 & 10 & 27 \\
20 & 7 & 8 & 9 & 10 & 19 & 6 & 1 & 2 & 11 & 28 \\
19 & 6 & 1 & 2 & 11 & 18 & 5 & 4 & 3 & 12 & 29 \\
18 & 5 & 4 & 3 & 12 & 17 & 16 & 15 & 14 & 13 & 30 \\
17 & 16 & 15 & 14 & 13 & 36 & 35 & 34 & 33 & 32 & 31 \\
\end{array}
\]

\(S_5 \quad S_6\)

\(b\) The largest number in \(S_1\) is 1, in \(S_2\) is 4, in \(S_3\) is 9, in \(S_4\) is 16, and so on. In general, since there are \(n\) numbers along the side of the \(n\)th square, the largest number in the \(n\)th square is \(n \times n\).

Now \(44 \times 44 = 1936\), which is less than 2010, and \(45 \times 45 = 2025\), which is greater than 2010. So the first square that contains 2010 is the 45th square.
c The odd square numbers occur in the top right corner of the first square in which they appear. Thus 2025 occurs in the top right corner of the 45th square.

The 45th square has 45 numbers in the top row and $2025 - 2010 = 15$, so 2010 appears in the top row and in column 30 (from the left).
PROBLEM 7

Jack is exactly two years older than his sister Jill. Their mother opened an EastBank Bonus Saver account with $100 for each of them on their tenth birthdays. Jack deposited $38 in his account on the first business day of each month after his tenth birthday. Jill deposited $50 in her account on the first business day of each month after her tenth birthday. Both immediately spent any interest the bank paid into their accounts but made no other withdrawals.

a  Who had more money in the bank on Jill’s 13th birthday? Explain.

b  Who had more money in the bank on Jack’s 20th birthday? Explain.

c  How many months did it take for Jack and Jill to have the same amount in their accounts and what amount was that?

Solution

a  On Jill’s 13th birthday her account contained the 36 monthly deposits of $50 that she had made plus the original $100 her mother deposited, a total of $100 + 36 \times 50 = $1900. At the same time, Jack had been depositing $38 a month for five years so his total was $100 + 60 \times 38 = $2380. Thus Jack had more than Jill.

b  On Jack’s 20th birthday his account contained 10 \times 12 = 120 monthly deposits of $38 that he had made plus the original $100 his mother deposited, a total of $100 + 120 \times 38 = $4660. When Jack turned 20, Jill turned 18 so her account had $100 + 8 \times 12 \times 50 = $4900. Thus Jill had more than Jack.

c  Alternative i

At the time of Jill’s tenth birthday, Jack already had $100 + 24 \times 38 = $1012 in his account. We calculate both bank balances in several months, \( M \), after Jill’s tenth birthday and check to see if they are the same. If Jack’s is more than Jill’s we increase the number of months. If Jill’s is more than Jack’s we decrease the number of months. The results are arranged in the following table.
<table>
<thead>
<tr>
<th>$M$</th>
<th>Jack’s amount</th>
<th>Jill’s amount</th>
<th>Who has more?</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>$1012 + 38 \times 50 = $2912$</td>
<td>$100 + 50 \times 50 = $2600$</td>
<td>Jack</td>
</tr>
<tr>
<td>60</td>
<td>$1012 + 38 \times 60 = $3292$</td>
<td>$100 + 50 \times 60 = $3100$</td>
<td>Jack</td>
</tr>
<tr>
<td>70</td>
<td>$1012 + 38 \times 70 = $3672$</td>
<td>$100 + 50 \times 70 = $3600$</td>
<td>Jack</td>
</tr>
<tr>
<td>80</td>
<td>$1012 + 38 \times 80 = $4052$</td>
<td>$100 + 50 \times 80 = $4100$</td>
<td>Jill</td>
</tr>
<tr>
<td>75</td>
<td>$1012 + 38 \times 75 = $3862$</td>
<td>$100 + 50 \times 75 = $3850$</td>
<td>Jack</td>
</tr>
<tr>
<td>76</td>
<td>$1012 + 38 \times 76 = $3900$</td>
<td>$100 + 50 \times 76 = $3900$</td>
<td>same</td>
</tr>
</tbody>
</table>

Thus it took 76 months, or 6 years 4 months after her tenth birthday, for Jill to save the same amount as Jack. At that time they both had $3900.

**Alternative ii**

At the time of Jill’s tenth birthday, Jack already had $100 + 24 \times 38 = $1012 in his account and Jill had $100 in her account. After that Jill deposited $50 - 38 = $12 more than Jack each month. So she took $(1012 - 100) \div 12 = 912 \div 12 = 76$ months to catch up with Jack. At that time they both had $100 + 76 \times 50 = $3900.
PROBLEM 8

Insert two digits, one in the centre and one on the right of the number 451587, to make a number that is divisible by 132. Explain why there is only one such number.

Solution

Since \(132 = 3 \times 4 \times 11\), the new number must be divisible by 3, 4, and 11.

In order for the number to be divisible by 4, the last two digits must form a number that is divisible by 4. Hence the digit inserted on the right of 451587 must be 2 or 6. Thus the number we are looking for is 451x5872 or 451x5876 where \(x\) is some digit.

Alternative i

Since 451x5872 is divisible by 3, the sum of its digits is divisible by 3. Since \(4 + 5 + 1 + 5 + 8 + 7 + 2 = 32\), \(x\) must be 1, 4, or 7. In each case we test the number to see if it is divisible by 11.

<table>
<thead>
<tr>
<th>451x5872</th>
<th>Test</th>
<th>Multiple of 11?</th>
</tr>
</thead>
<tbody>
<tr>
<td>45115872</td>
<td>((4 + 1 + 5 + 7) - (5 + 1 + 8 + 2) = 1)</td>
<td>no</td>
</tr>
<tr>
<td>45145872</td>
<td>((4 + 1 + 5 + 7) - (5 + 4 + 8 + 2) = -2)</td>
<td>no</td>
</tr>
<tr>
<td>45175872</td>
<td>((4 + 1 + 5 + 7) - (5 + 7 + 8 + 2) = -5)</td>
<td>no</td>
</tr>
</tbody>
</table>

Since 451x5876 is divisible by 3, the sum of its digits is divisible by 3. Since \(4 + 5 + 1 + 5 + 8 + 7 + 6 = 36\), \(x\) must be 0, 3, 6, or 9. In each case we test the number to see if it is divisible by 11.

<table>
<thead>
<tr>
<th>451x5876</th>
<th>Test</th>
<th>Multiple of 11?</th>
</tr>
</thead>
<tbody>
<tr>
<td>45105876</td>
<td>((4 + 1 + 5 + 7) - (5 + 0 + 8 + 6) = -2)</td>
<td>no</td>
</tr>
<tr>
<td>45135876</td>
<td>((4 + 1 + 5 + 7) - (5 + 3 + 8 + 6) = -5)</td>
<td>no</td>
</tr>
<tr>
<td>45165876</td>
<td>((4 + 1 + 5 + 7) - (5 + 6 + 8 + 6) = -8)</td>
<td>no</td>
</tr>
<tr>
<td>45195876</td>
<td>((4 + 1 + 5 + 7) - (5 + 9 + 8 + 6) = -11)</td>
<td>yes</td>
</tr>
</tbody>
</table>

Thus the only new number is 45195876.
Alternative ii

Since $451x5872$ is divisible by 11, the number

$$(4 + 1 + 5 + 7) - (5 + x + 8 + 2) = 2 - x$$

must also be a multiple of 11. So $x = 2$ and the number is $45125872$. The sum of its digits is 34, which is not divisible by 3, so the number is not divisible by 3. Thus the new number does not end in 2.

Since $451x5876$ is divisible by 11, the number

$$(4 + 1 + 5 + 7) - (5 + x + 8 + 6) = -2 - x$$

must also be a multiple of 11. So $x = 9$ and the number is $45195876$. The sum of its digits is 45, which is divisible by 3, so the number is divisible by 3.

Thus the only new number is 45195876.