2015 Australian Intermediate Mathematics Olympiad - Questions

Time allowed: 4 hours. NO calculators are to be used.

Questions 1 to 8 only require their numerical answers all of which are non-negative integers less than 1000.
Questions 9 and 10 require written solutions which may include proofs.
The bonus marks for the Investigation in Question 10 may be used to determine prize winners.

1. A number written in base a is 123_a. The same number written in base b is 146_b. What is the minimum value of $a+b$? [**2 marks**]

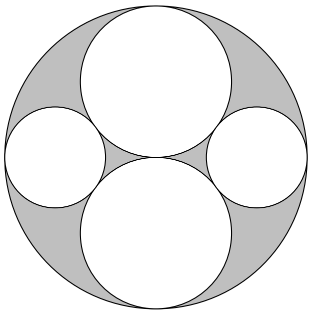
2. A circle is inscribed in a hexagon $ABCDEF$ so that each side of the hexagon is tangent to the circle. Find the perimeter of the hexagon if $AB = 6$, $CD = 7$, and $EF = 8$. [**2 marks**]

3. A selection of 3 whatsits, 7 doovers and 1 thingy cost a total of \$329. A selection of 4 whatsits, 10 doovers and 1 thingy cost a total of \$441. What is the total cost, in dollars, of 1 whatsit, 1 doover and 1 thingy? [**3 marks**]

4. A fraction, expressed in its lowest terms $\frac{a}{b}$, can also be written in the form $\frac{2}{n} + \frac{1}{n^2}$, where n is a positive integer. If $a + b = 1024$, what is the value of a? [**3 marks**]

5. Determine the smallest positive integer y for which there is a positive integer x satisfying the equation $2^{13} + 2^{10} + 2^x = y^2$. [**3 marks**]

6. The large circle has radius $30/\sqrt{\pi}$. Two circles with diameter $30/\sqrt{\pi}$ lie inside the large circle. Two more circles lie inside the large circle so that the five circles touch each other as shown. Find the shaded area.



[**4 marks**]

7. Consider a shortest path along the edges of a 7×7 square grid from its bottom-left vertex to its top-right vertex. How many such paths have no edge above the grid diagonal that joins these vertices? [**4 marks**]

8. Determine the number of non-negative integers x that satisfy the equation

$$\left\lfloor \frac{x}{44} \right\rfloor = \left\lfloor \frac{x}{45} \right\rfloor.$$

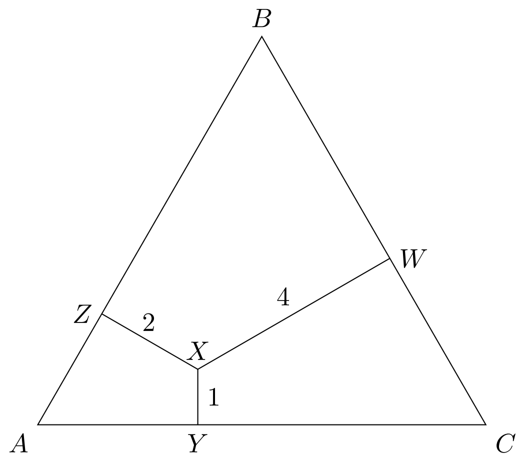
(Note: if r is any real number, then $\lfloor r \rfloor$ denotes the largest integer less than or equal to r.) [**4 marks**]

9. A sequence is formed by the following rules: $s_1 = a, s_2 = b$ and $s_{n+2} = s_{n+1} + (-1)^n s_n$ for all $n \geq 1$.

If $a = 3$ and b is an integer less than 1000, what is the largest value of b for which 2015 is a member of the sequence? Justify your answer. [**5 marks**]

10. X is a point inside an equilateral triangle ABC. Y is the foot of the perpendicular from X to AC, Z is the foot of the perpendicular from X to AB, and W is the foot of the perpendicular from X to BC.

The ratio of the distances of X from the three sides of the triangle is $1 : 2 : 4$ as shown in the diagram.



If the area of $AZXY$ is $13\,\text{cm}^2$, find the area of ABC. Justify your answer. [**5 marks**]

Investigation

If $XY : XZ : XW = a : b : c$, find the ratio of the areas of $AZXY$ and ABC. [**2 bonus marks**]

2015 Australian Intermediate Mathematics Olympiad - Solutions

1. *Method 1*

$$\begin{aligned} 123_a = 146_b &\iff a^2 + 2a + 3 = b^2 + 4b + 6 \\ &\iff (a+1)^2 + 2 = (b+2)^2 + 2 \\ &\iff (a+1)^2 = (b+2)^2 \\ &\iff a + 1 = b + 2 \ (a \text{ and } b \text{ are positive}) \\ &\iff a = b + 1 \end{aligned}$$

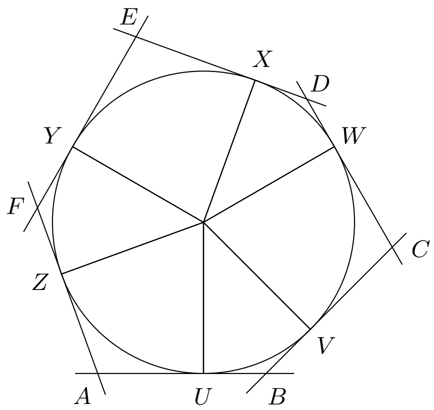
Since the minimum value for b is 7, the minimum value for $a + b$ is $8 + 7 = \mathbf{15}$.

Method 2

Since the digits in any number are less than the base, $b \geq 7$.
We also have $a > b$, otherwise $a^2 + 2a + 3 < b^2 + 4b + 6$.

If $b = 7$ and $a = 8$, then $a^2 + 2a + 3 = 83 = b^2 + 4b + 6$.
So the minimum value for $a + b$ is $8 + 7 = \mathbf{15}$.

2. Let AB, BC, CD, DE, EF, FA touch the circle at U, V, W, X, Y, Z respectively.



Since the two tangents from a point to a circle have equal length,
$UB = BV$, $VC = CW$, $WD = DX$, $XE = EY$, $YF = FZ$, $ZA = AU$.

The perimeter of hexagon $ABCDEF$ is
$AU + UB + BV + VC + CW + WD + DX + XE + EY + YF + FZ + ZA$
$= AU + UB + UB + CW + CW + WD + WD + EY + EY + YF + YF + AU$
$= 2(AU + UB + CW + WD + EY + YF)$
$= 2(AB + CD + EF) = 2(6 + 7 + 8) = 2(21) = \mathbf{42}$.

3. *Preamble*

Let the required cost be x. Then, with obvious notation, we have:

$$3w + 7d + t = 329 \qquad (1)$$
$$4w + 10d + t = 441 \qquad (2)$$
$$w + d + t = x \qquad (3)$$

Method 1

$3 \times (1) - 2 \times (2)$: $w + d + t = 3 \times 329 - 2 \times 441 = 987 - 882 = \mathbf{105}$.

Method 2

(2) − (1): $w + 3d = 112$.
(1) − (3): $2w + 6d = 329 - x = 2 \times 112 = 224$.

Then $x = 329 - 224 = \mathbf{105}$.

Method 3

$10 \times (1) - 7 \times (2)$: $w = (203 - 3t)/2$
$3 \times (2) - 4 \times (1)$: $d = (7 + t)/2$

Then $w + d + t = 210/2 - 2t/2 + t = \mathbf{105}$.

4. We have $\dfrac{2}{n} + \dfrac{1}{n^2} = \dfrac{2n+1}{n^2}$.

Since $2n + 1$ and n^2 are coprime, $a = 2n + 1$ and $b = n^2$.
So $1024 = a + b = n^2 + 2n + 1 = (n + 1)^2$, hence $n + 1 = 32$.

This gives $a = 2n + 1 = 2 \times 31 + 1 = \mathbf{63}$.

5. *Method 1*

$$\begin{aligned} 2^{13} + 2^{10} + 2^x = y^2 \quad &\iff \quad 2^{10}(2^3 + 1) + 2^x = y^2 \\ &\iff \quad (2^5 \times 3)^2 + 2^x = y^2 \\ &\iff \quad 2^x = y^2 - 96^2 \\ &\iff \quad 2^x = (y + 96)(y - 96). \end{aligned}$$

Since y is an integer, both $y + 96$ and $y - 96$ must be powers of 2.
Let $y + 96 = 2^m$ and $y - 96 = 2^n$. Then $2^m - 2^n = 192 = 2^6 \times 3$.

Hence $2^{m-6} - 2^{n-6} = 3$. So $2^{m-6} = 4$ and $2^{n-6} = 1$.
In particular, $m = 8$. Hence $y = 2^8 - 96 = 256 - 96 = \mathbf{160}$.

Method 2

We have $y^2 = 2^{13} + 2^{10} + 2^x = 2^{10}(2^3 + 1 + 2^{x-10}) = 2^{10}(9 + 2^{x-10})$.

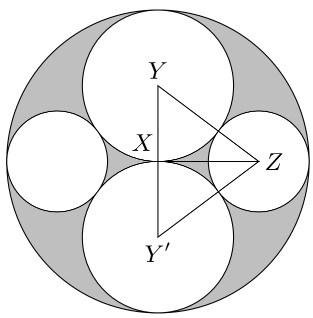
So we want the smallest value of $9 + 2^{x-10}$ that is a perfect square.
Since $9 + 2^{x-10}$ is odd and greater than 9, $9 + 2^{x-10} \geq 25$.

Since $9 + 2^{14-10} = 25$, $y = 2^5 \times 5 = 32 \times 5 = \mathbf{160}$.

Comment

Method 1 shows that $2^{13} + 2^{10} + 2^x = y^2$ has only one solution.

6. The centres Y and Y' of the two medium circles lie on a diameter of the large circle. By symmetry about this diameter, the two smaller circles are congruent. Let X be the centre of the large circle and Z the centre of a small circle.



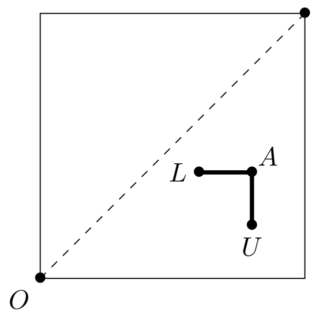
Let R and r be the radii of a medium and small circle respectively. Then $ZY = R + r = ZY'$. Since $XY = XY'$, triangles XYZ and $XY'Z$ are congruent. Hence $XZ \perp XY$.

By Pythagoras, $YZ^2 = YX^2 + XZ^2$. So $(R+r)^2 = R^2 + (2R-r)^2$.
Then $R^2 + 2Rr + r^2 = 5R^2 - 4Rr + r^2$, which simplifies to $3r = 2R$.

So the large circle has area $\pi(30/\sqrt{\pi}\,)^2 = 900$,
each medium circle has area $\pi(15/\sqrt{\pi}\,)^2 = 225$,
and each small circle has area $\pi(10/\sqrt{\pi}\,)^2 = 100$.
Thus the shaded area is $900 - 2 \times 225 - 2 \times 100 =$ **250**.

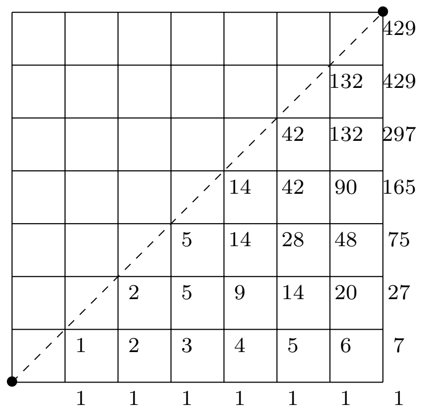
7. *Method 1*

Any path from the start vertex O to a vertex A must pass through either the vertex L left of A or the vertex U underneath A. So the number of paths from O to A is the sum of the number of paths from O to L and the paths from O to U.



There is only one path from O to any vertex on the bottom line of the grid.

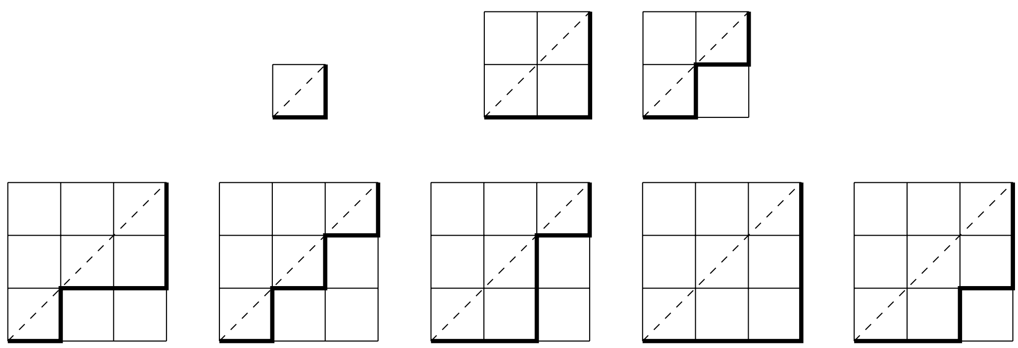
So the number of paths from O to all other vertices can be progressively calculated from the second bottom row upwards as indicated.



Thus the number of required paths is **429**.

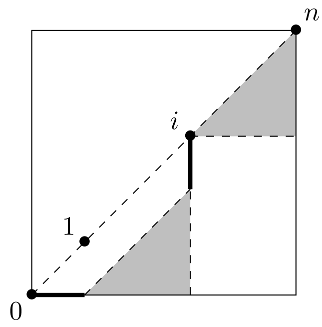
Method 2

To help understand the problem, consider some smaller grids.



Let $p(n)$ equal the number of required paths on an $n \times n$ grid and let $p(0) = 1$.

Starting with the bottom-left vertex, label the vertices of the diagonal 0, 1, 2, . . . , n.



Consider all the paths that touch the diagonal at vertex i but not at any of the vertices between vertex 0 and vertex i. Each such path divides into two subpaths.

One subpath is from vertex 0 to vertex i and, except for the first and last edge, lies in the lower triangle of the diagram above. Thus there are $p(i-1)$ of these subpaths.

The other subpath is from vertex i to vertex n and lies in the upper triangle in the diagram above. Thus there are $p(n-i)$ of these subpaths.

So the number of such paths is $p(i-1)\times p(n-i)$.

Summing these products from $i=1$ to $i=n$ gives all required paths. Thus

$$p(n)=p(n-1)+p(1)p(n-2)+p(2)p(n-3)+\cdots+p(n-2)p(1)+p(n-1)$$

We have $p(1)=1$, $p(2)=2$, $p(3)=5$. So

$p(4)=p(3)+p(1)p(2)+p(3)p(1)+p(3)=14$,
$p(5)=p(4)+p(1)p(3)+p(2)p(2)+p(3)p(1)+p(4)=42$,
$p(6)=p(5)+p(1)p(4)+p(2)p(3)+p(3)p(2)+p(1)p(4)+p(5)=132$, and
$p(7)=p(6)+p(1)p(5)+p(2)p(4)+p(3)p(3)+p(4)p(2)+p(5)p(1)+p(6)=\mathbf{429}$.

8. *Method 1*

Let $\left\lfloor \frac{x}{44} \right\rfloor = \left\lfloor \frac{x}{45} \right\rfloor = n$.

Since x is non-negative, n is also non-negative.

If $n=0$, then x is any integer from 0 to $44-1=43$: a total of 44 values.

If $n=1$, then x is any integer from 45 to $2\times 44-1=87$: a total of 43 values.

If $n=2$, then x is any integer from $2\times 45=90$ to $3\times 44-1=131$: a total of 42 values.

If $n=k$, then x is any integer from $45k$ to $44(k+1)-1=44k+43$: a total of $(44k+43)-(45k-1)=44-k$ values.

Thus, increasing n by 1 decreases the number of values of x by 1. Also the largest value of n is 43, in which case x has only 1 value.

Therefore the number of non-negative integer values of x is $44+43+\cdots+1=\frac{1}{2}(44\times 45)=\mathbf{990}$.

Method 2

Let n be a non-negative integer such that $\left\lfloor \frac{x}{44} \right\rfloor = \left\lfloor \frac{x}{45} \right\rfloor = n$.

Then $\left\lfloor \frac{x}{44} \right\rfloor = n \iff 44n \le x < 44(n+1)$ and $\left\lfloor \frac{x}{45} \right\rfloor = n \iff 45n \le x < 45(n+1)$.

So $\left\lfloor \frac{x}{44} \right\rfloor = \left\lfloor \frac{x}{45} \right\rfloor = n \iff 45n \le x < 44(n+1) \iff 44n+n \le x < 44n+44$.

This is the case if and only if $n<44$, and then x can assume exactly $44-n$ different values.

Therefore the number of non-negative integer values of x is

$(44-0)+(44-1)+\cdots+(44-43)=44+43+\cdots+1=\frac{1}{2}(44\times 45)=\mathbf{990}$.

Method 3

Let n be a non-negative integer such that $\left\lfloor \frac{x}{44} \right\rfloor = \left\lfloor \frac{x}{45} \right\rfloor = n$.

Then $x = 44n + r$ where $0 \leq r \leq 43$ and $x = 45n + s$ where $0 \leq s \leq 44$.

So $n = r - s$. Therefore $0 \leq n \leq 43$. Also $r = n + s$. Therefore $n \leq r \leq 43$.

Therefore the number of non-negative integer values of x is $44 + 43 + \cdots + 1 = \frac{1}{2}(44 \times 45) = \mathbf{990}$.

9. Working out the first few terms gives us an idea of how the given sequence develops:

n	s_{2n-1}	s_{2n}
1	a	b
2	$b-a$	$2b-a$
3	b	$3b-a$
4	$2b-a$	$5b-2a$
5	$3b-a$	$8b-3a$
6	$5b-2a$	$13b-5a$
7	$8b-3a$	$21b-8a$

It appears that the coefficients in the even terms form a Fibonacci sequence and, from the 5th term, every odd term is a repeat of the third term before it.

These observations are true for the entire sequence since, for $m \geq 1$, we have:

$$\begin{aligned} s_{2m+2} &= s_{2m+1} + s_{2m} \\ s_{2m+3} &= s_{2m+2} - s_{2m+1} &&= s_{2m} \\ s_{2m+4} &= s_{2m+3} + s_{2m+2} &&= s_{2m+2} + s_{2m} \end{aligned}$$

So, defining $F_1 = 1$, $F_2 = 2$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, we have $s_{2n} = bF_n - aF_{n-2}$ for $n \geq 3$. Since $a = 3$ and $b < 1000$, none of the first five terms of the given sequence equal 2015. So we are looking for integer solutions of $bF_n - 3F_{n-2} = 2015$ for $n \geq 3$.

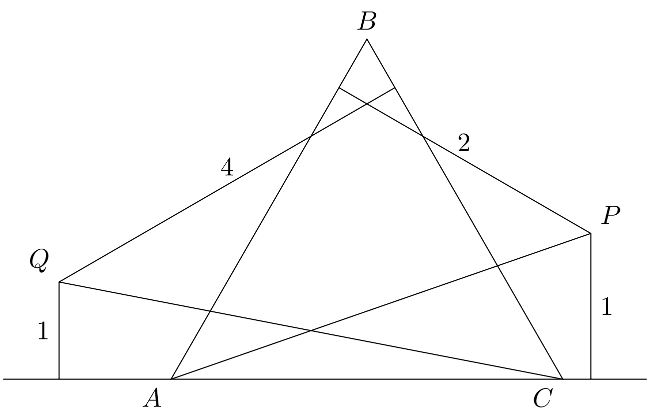
$s_6 = 3b - 3 = 2015$, has no solution.
$s_8 = 5b - 6 = 2015$, has no solution.
$s_{10} = 8b - 9 = 2015$ implies $b = 253$.

For $n \geq 6$ we have $b = 2015/F_n + 3F_{n-2}/F_n$. Since F_n increases, we have $F_n \geq 13$ and $F_{n-2}/F_n < 1$ for $n \geq 6$. Hence $b < 2015/13 + 3 = 158$. So the largest value of b is $\mathbf{253}$.

10. *Method 1*

We first show that X is uniquely defined for any given equilateral triangle ABC.

Let P be a point outside $\triangle ABC$ such that its distances from AC and AB are in the ratio 1:2. By similar triangles, any point on the line AP has the same property. Also any point between AP and AC has the distance ratio less than 1:2 and any point between AP and AB has the distance ratio greater than 1:2.

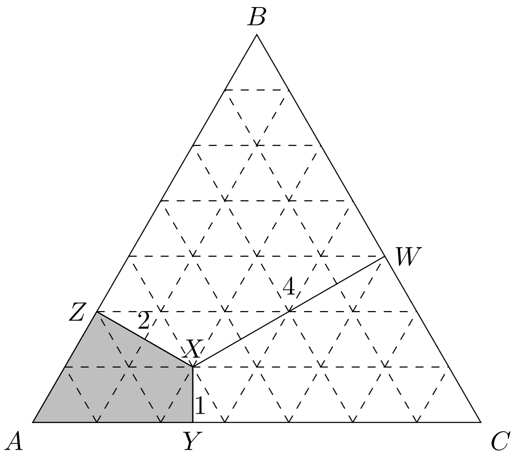


Let Q be a point outside $\triangle ABC$ such that its distances from AC and BC are in the ratio 1:4. By an argument similar to that in the previous paragraph, only the points on CQ have the distance ratio equal to 1:4.

Thus the only point whose distances to AC, AB, and BC are in the ratio 1:2:4 is the point X at which AP and CQ intersect.

Scaling if necessary, we may assume that the actual distances of X to the sides of $\triangle ABC$ are 1, 2, 4. Let h be the height of $\triangle ABC$. Letting $|\ |$ denote area, we have
$|ABC| = \frac{1}{2}h \times AB$ and
$|ABC| = |AXB| + |BXC| + |CXA| = \frac{1}{2}(2AB + 4BC + AC) = \frac{1}{2}AB \times 7$.
So $h = 7$.

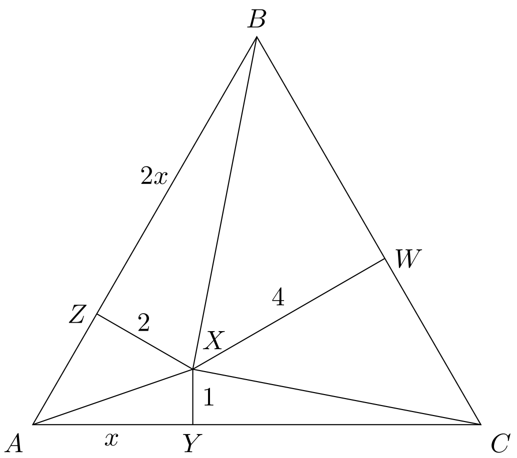
Draw a 7-layer grid of equilateral triangles each of height 1, starting with a single triangle in the top layer, then a trapezium of 3 triangles in the next layer, a trapezium of 5 triangles in the next layer, and so on. The boundary of the combined figure is $\triangle ABC$ and X is one of the grid vertices as shown.



There are 49 small triangles in ABC and 6.5 small triangles in $AZXY$. Hence, after rescaling so that the area of $AZXY$ is $13\,\mathrm{cm}^2$, the area of ABC is $13 \times 49/6.5 = \mathbf{98}\,\mathrm{cm}^2$.

Method 2

Join AX, BX, CX. Since $\angle YAZ = \angle ZBW = 60°$, the quadrilaterals $AZXY$ and $BWXZ$ are similar. Let XY be 1 unit and AY be x. Then $BZ = 2x$.



By Pythagoras: in $\triangle AXY$, $AX = \sqrt{1+x^2}$ and in $\triangle AXZ$, $AZ = \sqrt{x^2-3}$. Hence $BW = 2\sqrt{x^2-3}$.

Since $AB = AC$, $YC = x + \sqrt{x^2-3}$.

By Pythagoras: in $\triangle XYC$, $XC^2 = 1 + (x + \sqrt{x^2-3})^2 = 2x^2 - 2 + 2x\sqrt{x^2-3}$
and in $\triangle XWC$, $WC^2 = 2x^2 - 18 + 2x\sqrt{x^2-3}$.

Since $BA = BC$, $2x + \sqrt{x^2-3} = 2\sqrt{x^2-3} + \sqrt{2x^2 - 18 + 2x\sqrt{x^2-3}}$.

So $2x - \sqrt{x^2-3} = \sqrt{2x^2 - 18 + 2x\sqrt{x^2-3}}$.

Squaring gives $4x^2 + x^2 - 3 - 4x\sqrt{x^2-3} = 2x^2 - 18 + 2x\sqrt{x^2-3}$, which simplifies to $3x^2 + 15 = 6x\sqrt{x^2-3}$.

Squaring again gives $9x^4 + 90x^2 + 225 = 36x^4 - 108x^2$. So $0 = 3x^4 - 22x^2 - 25 = (3x^2 - 25)(x^2 + 1)$, giving $x = \frac{5}{\sqrt{3}}$.

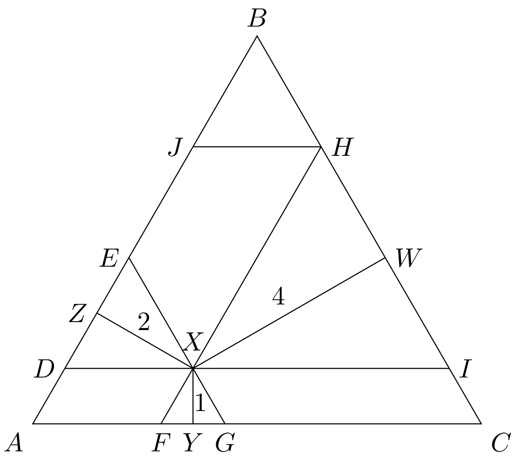
Hence, area $AZXY = \frac{x}{2} + \sqrt{x^2-3} = \frac{5}{2\sqrt{3}} + \frac{4}{\sqrt{3}} = \frac{13}{2\sqrt{3}}$ and

area $ABC = \frac{\sqrt{3}}{4}(2x + \sqrt{x^2-3})^2 = \frac{\sqrt{3}}{4}\Big(\frac{10}{\sqrt{3}} + \frac{4}{\sqrt{3}}\Big)^2 = \frac{49}{\sqrt{3}}$.

Since the area of $AZXY$ is $13\,\mathrm{cm}^2$, the area of ABC is $(\frac{49}{\sqrt{3}}/\frac{13}{2\sqrt{3}}) \times 13 = \mathbf{98}\,\mathrm{cm}^2$.

Method 3

Let DI be the line through X parallel to AC with D on AB and I on BC.
Let EG be the line through X parallel to BC with E on AB and G on AC.
Let FH be the line through X parallel to AB with F on AC and H on BC.
Let J be a point on AB so that HJ is parallel to AC.
Triangles XDE, XFG, XHI, BHJ are equilateral, and triangles XDE and BHJ are congruent.



The areas of the various equilateral triangles are proportional to the square of their heights. Let the area of $\triangle FXG = 1$. Then, denoting area by | |, we have:

$|DEX| = 4$, $|XHI| = 16$, $|AEG| = 9$, $|DBI| = 36$, $|FHC| = 25$.

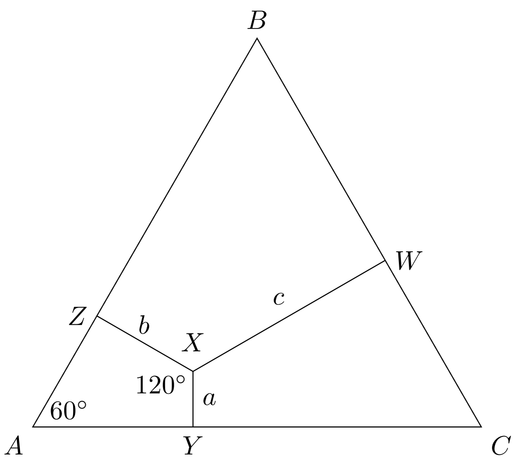
$|ABC| = |AEG| + |FHC| + |DBI| - |FXG| - |DEX| - |XHI| = 9 + 25 + 36 - 1 - 4 - 16 = 49$.

$|AZXY| = |AEG| - \frac{1}{2}(|FXG| + |DEX|) = 9 - \frac{1}{2}(1 + 4) = 6.5$.

Since the area of $AZXY$ is $13\,\mathrm{cm}^2$, the area of ABC is $2 \times 49 = \mathbf{98}\,\mathrm{cm}^2$.

Method 4

Consider the general case where $XY = a$, $XZ = b$, and $XW = c$.



Projecting AY onto the line through ZX gives $AY \sin 60° - a \cos 60° = b$.
Hence $AY = (a + 2b)/\sqrt{3}$. Similarly, $AZ = (b + 2a)/\sqrt{3}$.

Letting | | denote area, we have

$$\begin{aligned} |AZXY| &= |YAZ| + |YXZ| \\ &= \tfrac{1}{2}(AY)(AZ)\sin 60° + \tfrac{1}{2}ab\sin 120° \\ &= \tfrac{\sqrt{3}}{4}((AY)(AZ) + ab) \\ &= \tfrac{\sqrt{3}}{12}((a + 2b)(b + 2a) + 3ab) \\ &= \tfrac{\sqrt{3}}{12}(2a^2 + 2b^2 + 8ab) \\ &= \tfrac{\sqrt{3}}{6}(a^2 + b^2 + 4ab) \end{aligned}$$

Similarly, $|CYXW| = \frac{\sqrt{3}}{6}(a^2 + c^2 + 4ac)$ and $|BWXZ| = \frac{\sqrt{3}}{6}(b^2 + c^2 + 4bc)$.

Hence $|ABC| = \frac{\sqrt{3}}{6}(2a^2 + 2b^2 + 2c^2 + 4ab + 4ac + 4bc) = \frac{\sqrt{3}}{3}(a + b + c)^2$.

So $|ABC|/|AZXY| = 2(a + b + c)^2/(a^2 + b^2 + 4ab)$.

Letting $a = k$, $b = 2k$, $c = 4k$, and $|AZXY| = 13\,\mathrm{cm}^2$, we have $|ABC| = 26(49k^2)/(k^2 + 4k^2 + 8k^2) = \mathbf{98}\,\mathrm{cm}^2$.

Investigation

Method 4 gives $|ABC|/|AZXY| = 2(a + b + c)^2/(a^2 + b^2 + 4ab)$.

Alternatively, as in Method 3,

$$\begin{aligned} |ABC| &= |AEG| + |FHC| + |DBI| - |FXG| - |DEX| - |XHI| \\ &= (a + b)^2 + (a + c)^2 + (b + c)^2 - a^2 - b^2 - c^2 = (a + b + c)^2. \end{aligned}$$

Also
$$\begin{aligned} |AZXY| &= |AEG| - \tfrac{1}{2}(|FXG| + |DEX|) \\ &= (a + b)^2 - \tfrac{1}{2}(a^2 + b^2) \\ &= 2ab + \tfrac{1}{2}(a^2 + b^2). \end{aligned}$$

So $|ABC|/|AZXY| = 2(a + b + c)^2/(a^2 + b^2 + 4ab)$.