Tuesday 25 March 2014

Solutions

Junior Paper • Years 7 & 8

Intermediate Paper • Years 9 & 10

Senior Paper • Years 11 & 12

Australian Mathematics Trust
Tuesday 25 March 2014

Junior Paper • Years 7 & 8

Integrity of the Competition
To ensure the integrity of the competition and to identify outstanding students the competition organisers reserve the right to re-examine students before deciding whether to grant official status to their score.
Part A: Questions 1–6

Each question should be answered by a single choice from A to E. Questions are worth 3 points each.

1. Triangles

The Sierpinski triangle is a fractal that can be generated as shown.

The first four Sierpinski triangles are shown. How many downward-pointing (white) triangles are there in the 5th Sierpinski triangle?

(A) fewer than 30  (B) 30 – 34  (C) 35 – 39  
(D) 40 – 44  (E) more than 44

2. Lookouts

You are walking along a path near a clifftop and want to take photos from four of the seven lookouts on the way.

The numbers at the lookouts represent the time it will take in minutes for you to walk to the lookout and back on the tracks (dotted lines). The numbers on the path (dashed line) represent the time it takes to walk along the path without visiting the lookout.

It would take you 100 minutes to walk along the path without visiting any lookouts. What is the shortest time, in minutes, that you could take if you visit four lookouts?

(A) 125  (B) 126  (C) 127  (D) 128  (E) 129
3. Missing Planks

The footbridge above is missing two planks. To cross the river, you can step onto the next plank, or over one plank, or over a gap. For the footbridge above, the fewest number of planks you can step on is five. (For example, the 2nd, 3rd, 5th, 6th and 7th from the left.)

For the footbridge below, what is the fewest number of planks that you could step on?

(A) 15  (B) 16  (C) 17  (D) 18  (E) 19

4. Tardis

Your tardis (time machine) is playing up. It will only

\[ \text{A: take you forward one century,} \]
\[ \text{B: double the number of centuries you have already travelled from the present.} \]

For example, to get to the 6th century in the future you would need 4 jumps, \text{AAAB} or \text{ABAB}.

What is the smallest number of jumps needed to get to the 762nd century in the future?

(A) 12  (B) 16  (C) 20  (D) 21  (E) 22
5. Uranium Storage

For safety reasons, highly reactive uranium bars need to be stored in such a way that they are always as far away from others as possible, but once placed in storage they cannot be moved. An empty storage facility has 89 storage rooms all in a row. Unfortunately no-one has any idea how many bars will need to be stored. The first bar to arrive is put in room 1 and the second, in line with recommendations, is put in room 89. In which room could the 6th bar to arrive be placed?

(A) 11  (B) 21  (C) 33  (D) 56  (E) 77

6. Migrating Platypus

The endangered migrating platypus of SW Tasmania nest in burrows in the upper reaches of the Ornithorhynchus River. After the platypups hatch, they swim downstream to their feeding grounds. This requires them to swim over the Anatinus falls.

The Anatinus Falls consists of a series of drops of 3 or 6 metres onto horizontal ledges. The platypups can survive a drop of 3 metres, but not 6 metres. And they do not survive dropping onto rocks. As a conservation measure, barriers are to be constructed at one or both ends of some of the ledges.

What is the fewest number of barriers that would ensure the survival of all of the platypups?

(A) 2  (B) 3  (C) 4  (D) 5  (E) 6
7–9. Treasure Hunt

There are treasure chests at various points on a grid. Each treasure chest contains coins of a fixed value in roubles, denoted by the number on the chest in the diagrams below. Contestants start at $S$ and can only move right or down, finishing at $F$. They may only take one coin from a chest.

What is the greatest value in roubles of coins they can collect?

7.

8.
9.

10–12. Game Show

In a game show, contestants are given a number. They can increase or decrease a digit in the number by the click of a button. They are required to make all of the digits equal in as few clicks as possible. For instance, if they were given the number 114, the best they could do is three clicks (by decreasing the ones digit three times, giving 111).

For each of the following numbers, what is the fewest number of clicks required to make all digits equal?

10. 2393
11. 99478
12. 5559993321
13–15. Choreography

The choreographer of a new musical has a line of dancers dressed in red (R), green (G), or (sometimes) blue (B). She wants to put them in order R...G...B. For example, a line of dancers RGBGBGR should be arranged into the order RRGGGRRB. Her choreography requires that a pair of dancers dance out, round each other and then back to the other’s spot.

For each of the lines below, what is the smallest number of such swaps needed to get the dancers in order?

13. R G R R G G R G R G R


15. B R G B R R B B R G G B G B G
1. Triangles

The 1st Sierpinski triangle has 0 white triangles.
The 2nd Sierpinski triangle has 1 white triangle.
The 3rd Sierpinski triangle has 4 white triangles.
The 4th Sierpinski triangle has 13 white triangles.
The pattern is \( w_{n+1} = 3 \times w_n + 1 \), so there are \( 3 \times 13 + 1 = 40 \) white triangles in the 5th Sierpinski triangle. Hence (D).

2. Lookouts

The extra times to travel to the seven lookouts are \( 9 - 3 = 6 \), \( 11 - 4 = 7 \), \( 14 - 6 = 8 \),
\( 12 - 5 = 7 \), \( 17 - 9 = 8 \), \( 11 - 4 = 7 \) and \( 10 - 2 = 8 \) minutes. We choose the four smallest of these, 6, 7, 7 and 7 minutes, for a total of 27 minutes, visiting the 1st, 2nd, 4th and 6th lookouts. Hence (C).

3. Missing Planks

We can step on the fewest number of planks by stepping over the next plank unless it would land us on a gap. The planks we step on are shown in black.

![Diagram of planks]

We step on 16 planks. Hence (B).

4. Tardis

It is easier to work backwards from the 762nd century, subtracting 1 if the number is odd, and dividing by 2 if it is even. This gives
\[
762 \rightarrow 381 \rightarrow 380 \rightarrow 190 \rightarrow 95 \rightarrow 94 \rightarrow 47 \rightarrow 46 \rightarrow 23 \rightarrow 22 \rightarrow 11 \rightarrow 10 \rightarrow 5 \rightarrow 4 \rightarrow 2 \rightarrow 1 \rightarrow 0
\]
There are 16 jumps (→s). Hence (B).
5. Uranium Storage

The 3rd bar is stored in room 45, and the 4th and 5th bars are stored into rooms 23 and 67. The 6th bar could then be stored in room 12, 34, 56 or 78. Hence (D).

6. Migrating Platypus

Five barriers are required to ensure the survival of the platypups. There are several possible places where they could be built. One is indicated by the ×s on the diagram below. Hence (D).
7–9. Treasure Hunt

This is a breadth-first search problem. We determine the greatest value of coins that could have been collected on the way to each intersection. This will be the greater of the total collected by coming from the left and that by coming from above. In a computer program, these amounts would have to be calculated for each intersection. However the grids are sparse, and the amounts for some intersections only need be calculated. These are shown in circles below. The route that gives the greatest total value of coins is shown by the wavy lines.

7.

The best path allows coins totalling 24 roubles to be collected.

8.

The best path allows coins totalling 23 roubles to be collected. (There is an alternate route.)
9.

The best path allows coins totalling 20 roubles to be collected.

10–12. Game Show

The median of the digits in a number is the middle-sized digit in the number. For example, the median of the digits in 283 is 3. The fewest clicks is achieved by making all digits in the number equal to its median.

10. The median of the digits in 2393 is 3.
One click is required to increase the 2 to 3 and six to reduce the 9 to 3, a total of 7 clicks.

11. The median of the digits in 99478 is 8.
We require $1 + 1 + 4 + 1 + 0$ clicks to make all digits equal to 8, a total of 7 clicks.

12. The median of the digits in 555993321 is 5.
We require $0 + 0 + 0 + 4 + 4 + 4 + 2 + 2 + 3 + 4$ clicks to make all digits equal to 5, a total of 23 clicks.
13–15. Choreography

13. In $\text{RGRGGRGG}$, three of the $\text{Rs}$ and three of the $\text{Gs}$ are already in place, so the problem reduces to $\text{GGG}$ | $\text{RRR}$ requiring 3 swaps.

14. $\text{BRGBRG}$ | $\text{RBBRGR}$ | $\text{GGGRR}$ reduces to $\text{BGBG}$ | $\text{RBBR}$ | $\text{GGRR}$, requiring 6 (direct) swaps.

15. $\text{BRGB}$ | $\text{RRBBG}$ | $\text{GBGBGB}$ reduces to $\text{BGB}$ | $\text{RRBB}$ | $\text{GGGG}$.

After 3 direct swaps we have $\text{BB}$ | $\text{RR}$ | $\text{GG}$, requiring another $2 \times 2 = 4$ swaps, for a total of 7 swaps.
Tuesday 25 March 2014

Intermediate Paper • Years 9 & 10

Integrity of the Competition

To ensure the integrity of the competition and to identify outstanding students the competition organisers reserve the right to re-examine students before deciding whether to grant official status to their score.
Part A: Questions 1–6

Each question should be answered by a single choice from A to E. Questions are worth 3 points each.

1. Triangles

The Sierpinski triangle is a fractal that can be generated as shown.

![Sierpinski triangle sequence]

The first five Sierpinski triangles are shown. How many downward-pointing (white) triangles are there in the 6th Sierpinski triangle?

(A) fewer than 100  (B) 100 – 149  (C) 150 – 199
(D) 200 – 249  (E) more than 249

2. Tardis

Your tardis (time machine) is playing up. It will only

A: take you forward one century,

B: double the number of centuries you have already travelled from the present.

For example, to get to the 6th century in the future you would need 4 jumps, AAAB or ABAB.

What is the smallest number of jumps needed to get to the 762nd century in the future?

(A) 12  (B) 16  (C) 20  (D) 21  (E) 22
3. Gems

The number of gems to be found in a particular area is documented on the map below.

You are at the camp at C, and can only move horizontally or vertically away from the camp. What is the maximum number of gems that you could find in three moves?

(A) 11    (B) 12    (C) 13    (D) 14    (E) 15

4. Swap Three

You have a list of cards, each showing a single letter. You wish to sort them into alphabetical order (A B C...). A move consists of reversing the order of three consecutive cards. For example, B D C A → C D B A or B A C D.

Which of the following lists can be sorted into alphabetical order by a series of such moves?

(A) F B C D E A    (B) C F A B E G D    (C) C F A B G D E
    (D) C F G A E D H B    (E) H B C F B D E G A
5. Rectangles

A grid has been subdivided into non-overlapping rectangles. One cell in each rectangle has been labelled with the area of the rectangle. Your task is to determine the rectangles from the labels.

For instance,

if you were given

<table>
<thead>
<tr>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
</tr>
</tbody>
</table>

you would deduce

<table>
<thead>
<tr>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
</tr>
</tbody>
</table>

Here is the grid.

<p>| | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<td></td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>4</td>
</tr>
</tbody>
</table>

What is the area of the rectangle containing the \(×\)?

(A) 2 (B) 3 (C) 4 (D) 6 (E) 8
6. Aesthetic Skyline

The council’s planning committee has decided that the buildings in a new development should be arranged to provide an aesthetic skyline. This means that adjacent buildings should differ in height as much as possible. For example, consider the two arrangements of five buildings with heights of 8, 4, 3, 2 and 1 floors below:

The arrangement on the left has a total height difference of $4 + 3 + 1 + 1 = 9$ floors, whilst that on the right has a total height difference of $4 + 7 + 2 + 1 = 14$ floors. (But better arrangements can be found.)

A new development consisting of eight buildings with heights of 2, 3, 5, 2, 9, 6, 5 and 1 floors is planned.

What is the maximum total height difference for these eight buildings?

(A) 28  (B) 29  (C) 30  (D) 31  (E) 32
7–9. Rapids

You are at A and want to kayak down parts of one or more rivers. Each river runs left to right and has six rapids, denoted by thick black irregular segments of the line. Each rapid has a thrill factor, denoted by the number above the rapid. From each river to the next are seven paths, denoted by dotted lines.

You can only use a path to carry your kayak from one river to another, and once you have done so you cannot go back. This means that you will kayak down exactly six rapids.

You want to maximise the sum of your thrill factors. For each of the endpoints below, what is the maximum sum of your thrill factors if you start at A?

7. B
8. C
9. D
10–12. Robot Librarian

The school has acquired a robot to help the librarian. It can sort books on shelves, but only by taking a book out and placing it at either end of the shelf.

For instance, if the books ABC were on a shelf in the order BAC they could be sorted by moving the A to the front. If they were in the order CBA they could be sorted by moving the C to the end and then the A to the front, (or the A to the front and then the C to the end).

Each of the following lists represents a shelf of books. For each shelf what is the smallest number of books the robot must move to sort the books into alphabetical order?

10. FCAEBDE
11. DECAFGBH
12. DFAECIGBJH

13–15. Domino Loops

Dominoes are rectangular tiles with a digit written on each end. You are playing a game using domino tiles. In it you try to make a loop of three or more dominoes connected end-to-end, with the digits matching wherever the dominoes touch.

The diagrams below show domino loops of 3 and 4 dominoes. Note that dominoes can be turned upside down, as in making the second loop below.

At each turn you draw a new domino from the pile. The game ends when you are able to make a single domino loop out of all the dominoes drawn so far. For instance, if the domino pile was [1:4] [2:6] [2:4] [1:6] [3:0] and you drew dominoes from left to right, then you could finish the game after 4 turns, as is illustrated in the diagram above.

For each of the following domino piles, how many turns does it take you to finish the game, drawing dominoes from the left (top row first)?

13. 2 5 5 0 5 1 0 3 2 2 3 6 5 6 1 2 2 0 0 0 6 1
14. 2 4 6 6 3 2 3 1 0 4 1 4 5 6 2 0 6 4 1 2 1 1
   5 1 3 0 3 6 5 0 2 2 0 6 5 4 6 2
15. 2 4 4 4 2 6 3 1 3 5 1 5 4 6 1 6 6 6 1 0 3 4
   6 3 3 3 4 0 5 0 2 2 0 6 0 0 5 5
1. Triangles

The 1st Sierpinski triangle has 0 white triangles.
The 2nd Sierpinski triangle has 1 white triangle.
The 3rd Sierpinski triangle has 4 white triangles.
The 4th Sierpinski triangle has 13 white triangles.
The pattern is \( w_{n+1} = 3 \times w_n + 1 \),
so there are \( 3 \times 13 + 1 = 40 \) white triangles in the 5th Sierpinski triangle,
and \( 3 \times 40 + 1 = 121 \) white triangles in the 6th Sierpinski triangle.
Hence (B).

2. Tardis

It is easier to work backwards from the 762nd century, subtracting 1 if the number is odd, and dividing by 2 if it is even. This gives
762 \( \rightarrow \) 381 \( \rightarrow \) 380 \( \rightarrow \) 190 \( \rightarrow \) 95 \( \rightarrow \) 94 \( \rightarrow \) 47 \( \rightarrow \) 46 \( \rightarrow \) 23 \( \rightarrow \) 22 \( \rightarrow \) 11 \( \rightarrow \) 10 \( \rightarrow \) 5 \( \rightarrow \) 4 \( \rightarrow \) 2 \( \rightarrow \) 1 \( \rightarrow \) 0
There are 16 jumps (\( \rightarrow \)s). Hence (B).

3. Gems

The greatest number of gems that can be found in each cell is the second number.

The greatest number of gems that can be found in three moves is 13. One of the ways in which this can be done is shown by the arrows. Hence (C).
4. Swap Three

Consider the first list, F B C D E A. The A is in the 6th position. Reversing the last three letters would put it in the 4th position, and then reversing the 2nd, 3rd and 4th letters would put it in the 2nd position. But it would always end up in an even numbered position, and never an odd numbered position, including the first. So it is not possible to sort this list into alphabetical order.

We can extend this idea by putting each letter’s position in the alphabet under it. A list can only be sorted into alphabetical order if these numbers are odd – even – odd ...

F B C D E A | C F A B E G D | C F A B G D E | C F G A E D H B | H B C F B D E G A
6 2 3 4 5 1 | 3 6 1 2 5 7 4 | 3 6 1 2 7 4 5 | 3 6 7 1 5 4 8 2 | 8 2 3 6 2 4 5 7 1

Only the third list, C F A B G D E, has the odd – even pattern. Hence (C).

5. Rectangles

The rightmost 8 can only be part of the 4 × 2 rectangle with the 8 in the top-right corner. Then the lower 4 must be part of the 4 × 1 rectangle immediately below it. Continuing on, we get the rectangles below. The shading indicates one order in which the rectangles can be deduced (there are some ties).

The × is in the 3 × 1 rectangle of area 3. Hence (B).
6. Aesthetic Skyline

There are several ways of determining the best order. One algorithm is

Place the largest number in the centre.
Repeat
    place the smallest remaining number on the immediate right
    place the largest remaining number next on the right
    place the smallest remaining number on the immediate left
    place the largest remaining number on the next left
Until there are no numbers left.

This gives 3 5 2 9 1 6 2 5, giving a total height difference of 32. Hence (E).

An alternative algorithm is

Place the largest number in the centre.
Repeat
    place the remaining number that gives most difference on the end that gives
    the greater difference
Until there are no numbers left.

This gives 5 2 6 1 9 2 5 3, again giving a total height difference of 32.
7–9. Rapids

This is a breadth-first search problem. We find the greatest total thrill factor at the point at the beginning and end of each path. For the end of a path, this will be the greater of the thrill factors at the beginning of the path, and the thrill factors after coming down an earlier path and shooting the rapid immediately upstream.

We can take advantage of the thrill factors calculated for the second river in calculating the thrill factors in the third river, and those in the third river for the fourth.

The highest possible thrill factors are shown in circles on the diagram below.

7. B: The highest total thrill factor is 22.

8. C: The highest total thrill factor is 23.

9. D: The highest total thrill factor is 23.
10–12. Robot Librarian

The algorithm is to find the longest subsequence of contiguous ascending letters. These can be left where they are and the other books moved around them. The number of moves is then (the number of books on the shelf) − (the length of the longest subsequence).

10. In F C A B D E, the longest subsequence of contiguous increasing letters is C D E. So the robot can sort by F C A B D E → C A B D E F → B C A D E F → A B C D E F requiring 6 − 3 = 3 books to be moved.

11. In D E C A F B G H the longest subsequence is D E F G H so 8 − 5 = 3 books must be moved.

12. In D F A E C I G B J H the longest subsequence is F G H so 10 − 3 = 7 books must be moved.
13–15. Domino Loops

One or more domino loops can be constructed whenever there is an even number of each digit. This (first) happens after 8 dominoes in the first data set, 12 dominoes in the second data set and 7 dominoes in the third data set. In the first two data sets, a single loop can be constructed. However in the third data set two domino loops are required, and a further 7 dominoes must be drawn to construct a single domino loop.

13. A single domino loop can be made with the first 8 dominoes.

14. A single domino loop can be made with the first 12 dominoes.

15. The first 7 dominoes can be used to form two domino loops, but not one.

A further 7 dominoes are required to form one loop.

A single domino loop can be made with the first 14 dominoes.
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Senior Paper • Years 11 & 12

Integrity of the Competition
To ensure the integrity of the competition and to identify outstanding students the competition organisers reserve the right to re-examine students before deciding whether to grant official status to their score.
Part A: Questions 1–6

Each question should be answered by a single choice from A to E. Questions are worth 3 points each.

1. Trapdoors

In the board game ‘Trapdoors’, the players start at the top left and finish at the bottom left. Players move from 1 to 6 squares by rolling a dice. If they land on a trapdoor, they drop one, two or three rows. For example, rolling a 2 from $S$ means the player would drop to the second row, and rolling a 2 from the square marked $\times$ means she would drop to the third row.

Players move from left to right on the odd (shaded) rows and from right to left on the even (white) rows. At the end of a row they move down to the next row, so that rolling a 5 from the square marked $\times$ would also mean the player would drop to the third row.

On the board above, what is the fewest rolls of the dice required to get from $S$ to $F$?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10
2. Sapphires

You have inherited several emeralds, and your cousin has inherited several sapphires. You much prefer sapphires. Your cousin will swap sapphires for emeralds provided:

- Each swap is one sapphire for one emerald.
- The weight of the emerald (in carats) is at least the weight of the sapphire.

You agree to these conditions. Your first aim in swapping is to maximise the total weight of the sapphires you collect. Your second aim is to maximise the total weight of the emeralds that you retain.

For example, if you had an 8-carat emerald, you would swap it for a 6-carat sapphire rather than a 5-carat sapphire. And you would swap a 6-carat emerald rather than a 7-carat emerald for a 4-carat sapphire.

You have inherited 8 emeralds weighing 16, 12, 11, 10, 9, 5, 4 and 2 carats, and your cousin has inherited 7 sapphires weighing 15, 14, 13, 10, 10, 4 and 1 carats. How many carats will you lose in your swaps?

(A) 3  (B) 4  (C) 5  (D) 6  (E) 7

3. Maximum Flow

This diagram shows a network of pipes, with the numbers representing the maximum flow through each pipe.

What is the maximum flow from left to right through the network?

(A) 25  (B) 27  (C) 29  (D) 32  (E) 25
4. Aesthetic Skyline

The council’s planning committee has decided that the buildings in a new development should be arranged to provide an aesthetic skyline. This means that adjacent buildings should differ in height as much as possible. For example, consider the two arrangements of five buildings with heights of 8, 4, 3, 2 and 1 floors below:

The arrangement on the left has a total height difference of $4 + 3 + 1 + 1 = 9$ floors, whilst that on the right has a total height difference of $4 + 7 + 2 + 1 = 14$ floors. (But better arrangements can be found.)

A new development consisting of eight buildings with heights of 2, 3, 5, 2, 9, 6, 5 and 1 floors is planned.

What is the maximum total height difference for these eight buildings?

(A) 28  (B) 29  (C) 30  (D) 31  (E) 32
5. Largest Increasing Sequence Sum

A strictly increasing sequence of numbers is one where each number is greater than the previous number. For example, 2 5 10 11 is a strictly increasing sequence.

Find the largest total that can be formed by taking a strictly increasing sequence from left to right from

7 11 3 6 8 10 2 14 12 5 9 13

(A) 32   (B) 39   (C) 41   (D) 52   (E) 56

6. Shoelaces

The Shoelace Makers’ Guild has determined the rules for tying shoelaces.

- Start at the bottom left.
- Thread holes criss-cross fashion, left, right, left, right, . . .
- Progress upwards towards the top row, threading a hole next to or somewhere above the current one, but never below.
- On the return journey, only travel across or downwards.
- The lace must go through each hole exactly once.
- The lace must come out of the bottom-right hole at the finish.

There are exactly three ways of tying the laces in a shoe with three rows of holes.

How many ways are there of tying laces in a shoe with four rows of holes?

(A) 4   (B) 6   (C) 7   (D) 8   (E) 9
Part B: Questions 7–15

Each question should be answered by a number in the range 0–999. Questions are worth 2 points each.

7–9. Robot Librarian

The school has acquired a robot to help the librarian. It can sort books on shelves, but only by taking a book out and placing it at either end of the shelf.

For instance, if the books ABC were on a shelf in the order BAC they could be sorted by moving the A to the front. If they were in the order CBA they could be sorted by moving the C to the end and then the A to the front, (or the A to the front and then the C to the end).

Each of the following lists represents a shelf of books. For each shelf what is the smallest number of books the robot must move to sort the books into alphabetical order?

7. FCABDE
8. DECAF BGH
9. DFAEC IGBJH

10–12. Board Game

Anna and Brett are playing a game with a single counter that is moved along a board with numbers in a row. Anna starts by placing the counter on one of the two leftmost numbers. Then the players take turns, moving the counter one or two places to the right. Placing the counter on a number earns the player that number of points. The winning margin is (the winner’s score) – (the loser’s score). The aim of the game is to get as large a winning margin as possible.

If both players play as well as possible, what is Anna’s winning margin?

10. 

11. 

12. 

32
13–15. Domino Loops

Dominoes are rectangular tiles with a digit written on each end. You are playing a game using domino tiles. In it you try to make a loop of three or more dominoes connected end-to-end, with the digits matching wherever the dominoes touch.

The diagrams below show domino loops of 3 and 4 dominoes. Note that dominoes can be turned upside down, as in making the second loop below.

At each turn you draw a new domino from the pile. The game ends when you are able to make a single domino loop out of all the dominoes drawn so far. For instance, if the domino pile was [1:4] [2:6] [2:4] [1:6] [3:0] and you drew dominoes from left to right, then you could finish the game after 4 turns, as is illustrated in the diagram above.

For each of the following domino piles, how many turns does it take you to finish the game, drawing dominoes from the left (top row first)?

13. 2 5 5 0 5 1 0 3 2 2 3 6 5 6 1 2 2 0 0 6 1

14. 2 4 6 6 3 2 3 1 0 4 1 4 5 6 2 0 6 4 1 2 1 1
   5 1 3 0 3 6 5 0 2 2 0 6 5 4 6 2

15. 2 4 4 4 2 6 3 1 3 5 1 5 4 6 1 6 6 1 0 3 4
   6 3 3 3 4 0 5 0 2 2 0 6 0 0 5 5
1. Trapdoors

We label the squares that can be reached in one roll with 1, then the squares that can be reached with a further roll with 2, and so on. This gives

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</table>

Moving from $S$ to $F$ can be achieved in 7 rolls. This can be done by rolling 2, 6, 6, 4, 6, 6, and 2, so landing in the squares with the circled numbers. (There are other possibilities for the last three rolls.) The fewest number of rolls is 7. Hence (B).
2. Sapphires

The algorithm here is

\[
\text{For each sapphire, from heaviest to lightest} \\
\quad \text{if you have one or more emeralds at least as heavy} \\
\quad \quad \text{swap with the lightest of these emeralds.} \\
\text{endif} \\
\text{end for}
\]

This results in the following [emerald:sapphire] swaps:
[16:15], [10:10], [11:10], [4:4], [2:1].
You will then lose \(1 + 0 + 1 + 0 + 1 = 3\) carats. Hence (A).

3. Maximum Flow

The maximum flow through each pipe on the left is the smaller of the capacity of the pipe and the sum of the maximum flows coming into it. These are shown in circles above the pipe. The maximum flow through the pipes on the right is the smaller of the capacity of the pipe and the sum of the maximum flows exiting it. These are shown by circles below the pipe. Finally the maximum flow through the network is the smaller of the two flows at the central pipe.

The maximum flow through the network is 27. Hence (B).
4. Aesthetic Skyline
There are several ways of determining the best order. One algorithm is

Place the largest number in the centre.
Repeat
  place the smallest remaining number on the immediate right
  place the largest remaining number next on the right
  place the smallest remaining number on the immediate left
  place the largest remaining number on the next left
Until there are no numbers left.

This gives 3 5 2 9 1 6 2 5, giving a total height difference of 32. Hence (E).

An alternative algorithm is

Place the largest number in the centre.
Repeat
  place the remaining number that gives most difference on the end that gives
  the greater difference
Until there are no numbers left.

This gives 5 2 6 1 9 2 5 3, again giving a total height difference of 32.

5. Largest Increasing Sequence Sum
Let \( t[i] \) be the total of the LISS ending at index \( i \). Then \( t[0] = n[0], t[i] = n[i] + \max_{k<i} t[k] \).
Compute \( t[i] \) for \( i = 0 \) to \( n \). Take the maximum of those:

\[
\begin{array}{cccccccccccc}
 n[i] & 7 & 11 & 3 & 6 & 8 & 10 & 2 & 14 & 12 & 5 & 9 & 13 \\
 t[i] & 7 & 18 & 3 & 9 & 17 & 27 & 2 & 41 & 39 & 8 & 26 & 52 \\
\end{array}
\]

The largest longest increasing sequence sum is 52 \((3 + 6 + 8 + 10 + 12 + 13)\). Hence (D).

6. Shoelaces
Threading the lace upwards, there is no choice about the bottom row (must be threaded first on the left only) or the top row (must be threaded in both holes), but the ones in between each offer three choices:

- Skip the row.
- Thread one hole in the row.
- Thread two holes in the row.

The ‘return journey’ of the lace offers no choices either; there will be only one way of lacing all the remaining holes.
Therefore for \( n \) rows of holes there are \( 3^{n-2} \) ways of lacing the shoe. For four holes, that means 9 ways. Hence (E).
7–9. Robot Librarian

The algorithm is to find the longest subsequence of contiguous ascending letters. These can be left where they are and the other books moved around them. The number of moves is then

\[( \text{the number of books on the shelf}) - (\text{the length of the longest subsequence}).\]

7. In $\text{FCABDE}$, the longest subsequence of contiguous increasing letters is $\text{CDE}$. So the robot can sort by

$\text{FCABDE} \rightarrow \text{CABDEF} \rightarrow \text{BCADEF} \rightarrow \text{ABCDEF}$

requiring $6 - 3 = 3$ books to be moved.

8. In $\text{DECAFGBH}$ the longest subsequence is $\text{DEFGH}$ so $8 - 5 = 3$ books must be moved.

9. In $\text{DFAECIGBJH}$ the longest subsequence is $\text{FGH}$ so $10 - 3 = 7$ books must be moved.
10–12. Board Game

Consider the game \(4, 3\). If Anna moves onto the 4, Brett will move onto the 3 and the value of the 4 is only \(4 - 3 = 1\). So Anna would move onto the 3, and win by 3.

Now consider the game \(5, 4, 3\). If Anna moves onto the 5, Brett will move onto the 3 and the value of the 5 is only \(5 - 3 = 2\). Anna would still move onto the 5, and win the game by 2.

In general we work from right to left. The value of the rightmost number is the number itself. Then, working backwards, the value of each number is (the number – the larger of the two values to its right). Each player will choose the next number with the higher value. The outcome is determined by the value of the leftmost two numbers. The values of the numbers are shown below.

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<tr>
<th></th>
<th>2</th>
<th>0</th>
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<tbody>
<tr>
<td>10. Anna wins by 2 by placing the token on the first number. The token is placed on the shaded cells on the board below.</td>
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<td></td>
<td>Anna wins by 2.</td>
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<tr>
<td>11. Anna wins by 3 by placing the token on the second number. The token is placed on the shaded cells on the board below.</td>
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<td>Anna wins by 3.</td>
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<tr>
<td>12. Anna wins by 2 by placing the token on the second number. The token is placed on the shaded cells on the board below.</td>
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<td></td>
<td>Anna wins by 2.</td>
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13–15. Domino Loops

One or more domino loops can be constructed whenever there is an even number of each digit. This (first) happens after 8 dominoes in the first data set, 12 dominoes in the second data set and 7 dominoes in the third data set. In the first two data sets, a single loop can be constructed. However in the third data set two domino loops are required, and a further 7 dominoes must be drawn to construct a single domino loop.

13.  

```
1 2 2 2 5

5 0 0 3 3 6
```

A single domino loop can be made with the first 8 dominoes.

14.  

```
2 4 1 1 1 1 2 2 0

3 1 1 5 5 6 6 6 4
```

A single domino loop can be made with the first 12 dominoes.

15.  

The first 7 dominoes can be used to form two domino loops, but not one.

```
3 5

4 4

3 2 6
```

A further 7 dominoes are required to form one loop.

```
1 3 3 5 6 1 0 0 4 4

6 6 6 3 3 3 4 2 6
```

A single domino loop can be made with the first 14 dominoes.