Australian Mathematical Olympiad Committee

A department of the Australian Mathematics Trust

optiver
Official sponsor of the olympiad program.

2016 AUSTRALIAN MATHEMATICAL OLYMPIAD

DAY 1
Tuesday, 9 February 2016
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Find all positive integers n such that $2^n + 7^n$ is a perfect square.

2. Let ABC be a triangle. A circle intersects side BC at points U and V, side CA at points W and X, and side AB at points Y and Z. The points U, V, W, X, Y, Z lie on the circle in that order. Suppose that $AY = BZ$ and $BU = CV$.

Prove that $CW = AX$.

3. For a real number x, define $\lfloor x \rfloor$ to be the largest integer less than or equal to x, and define $\{x\} = x - \lfloor x \rfloor$.

(a) Prove that there are infinitely many positive real numbers x that satisfy the inequality

$$\{x^2\} - \{x\} > \frac{2015}{2016}.$$

(b) Prove that there is no positive real number x less than 1000 that satisfies this inequality.

4. A *binary sequence* is a sequence in which each term is equal to 0 or 1. We call a binary sequence *superb* if each term is adjacent to at least one term that is equal to 1. For example, the sequence $0, 1, 1, 0, 0, 1, 1, 1$ is a superb binary sequence with eight terms. Let B_n denote the number of superb binary sequences with n terms.

Determine the smallest integer $n \geq 2$ such that B_n is divisible by 20.

The Mathematics Olympiads are supported by the Australian Government Department of Education through the Mathematics and Science Participation Program.

2016 AUSTRALIAN MATHEMATICAL OLYMPIAD

DAY 2
Wednesday, 10 February 2016
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. Find all triples (x, y, z) of real numbers that simultaneously satisfy the equations

$$\begin{aligned} xy + 1 &= 2z \\ yz + 1 &= 2x \\ zx + 1 &= 2y. \end{aligned}$$

6. Let a, b, c be positive integers such that $a^3 + b^3 = 2^c$.

Prove that $a = b$.

7. Each point in the plane is assigned one of four colours.

Prove that there exist two points at distance 1 or $\sqrt{3}$ from each other that are assigned the same colour.

8. Three given lines in the plane pass through a point P.

(a) Prove that there exists a circle that contains P in its interior and intersects the three lines at six points A, B, C, D, E, F in that order around the circle such that $AB = CD = EF$.

(b) Suppose that a circle contains P in its interior and intersects the three lines at six points A, B, C, D, E, F in that order around the circle such that $AB = CD = EF$. Prove that

$$\frac{1}{2}\,\text{area}(\text{hexagon } ABCDEF) \geq \text{area}(\triangle APB) + \text{area}(\triangle CPD) + \text{area}(\triangle EPF).$$

The Mathematics Olympiads are supported by the Australian Government Department of Education through the Mathematics and Science Participation Program.