Australian Mathematical Olympiad Committee

A DEPARTMENT OF THE AUSTRALIAN MATHEMATICS TRUST

2017 AUSTRALIAN MATHEMATICAL OLYMPIAD

DAY 1
Tuesday, 14 February 2017
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. For which integers $n \geq 2$ is it possible to write the numbers $1, 2, 3, \ldots, n$ in a row in some order so that any two numbers written next to each other in the row differ by 2 or 3?

2. Given five distinct integers, consider the ten differences formed by pairs of these numbers. (Note that some of these differences may be equal.)

Determine the largest integer that is certain to divide the product of these ten differences, regardless of which five integers were originally given.

3. Determine all functions f defined for real numbers and taking real numbers as values such that

$$f(x^2 + f(y)) = f(xy)$$

for all real numbers x and y.

4. Suppose that S is a set of 2017 points in the plane that are not all collinear.

Prove that S contains three points that form a triangle whose circumcentre is not a point in S.

The Mathematics/Informatics Olympiads are supported by the Australian Government through the National Innovation and Science Agenda.

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A department of the Australian Mathematics Trust

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DAY 2
Wednesday, 15 February 2017
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. Determine the number of positive integers n less than $1\,000\,000$ for which the sum

$$\frac{1}{2\times\lfloor\sqrt{1}\rfloor+1}+\frac{1}{2\times\lfloor\sqrt{2}\rfloor+1}+\frac{1}{2\times\lfloor\sqrt{3}\rfloor+1}+\cdots+\frac{1}{2\times\lfloor\sqrt{n}\rfloor+1}$$

is an integer.

(Note that $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x.)

6. The circles K_1 and K_2 intersect at two distinct points A and M. Let the tangent to K_1 at A meet K_2 again at B, and let the tangent to K_2 at A meet K_1 again at D. Let C be the point such that M is the midpoint of AC.

Prove that the quadrilateral $ABCD$ is cyclic.

7. There are 1000 athletes standing equally spaced around a circular track of length 1 kilometre.

(a) How many ways are there to divide the athletes into 500 pairs such that the two members of each pair are 335 metres apart around the track?

(b) How many ways are there to divide the athletes into 500 pairs such that the two members of each pair are 336 metres apart around the track?

8. Let $f(x) = x^2 - 45x + 2$.

Find all integers $n \geq 2$ such that exactly one of the numbers

$$f(1), f(2), \ldots, f(n)$$

is divisible by n.

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