2013 Australian Intermediate Mathematics Olympiad

Time allowed: 4 hours. NO calculators are to be used.

Questions 1 to 8 only require their numerical answers all of which are non-negative integers less than 1000.
Questions 9 and 10 require written solutions which may include proofs.
The bonus marks for the Investigation in Question 10 may be used to determine prize winners.

1. Find the area in cm^2 of a rhombus whose side length is 29 cm and whose diagonals differ in length by 2 cm. [**2 marks**]

2. How many 4-digit numbers are there whose digit product is 60? [**2 marks**]

3. A base 7 three-digit number has its digits reversed when written in base 9. Find the decimal representation of the number. [**3 marks**]

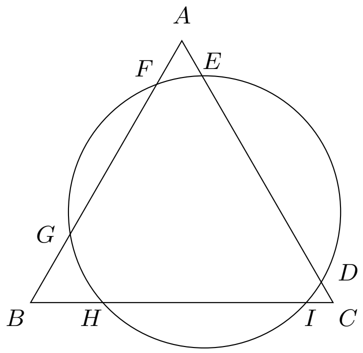
4. The prime numbers p, q, r satisfy the simultaneous equations $pq + pr = 80$ and $pq + qr = 425$. Find the value of $p + q + r$. [**3 marks**]

5. How many pairs of 3-digit palindromes are there such that when they are added together, the result is a 4-digit palindrome? For example, $232 + 989 = 1221$ gives one such pair. [**4 marks**]

6. ABC is an equilateral triangle with side length $2013\sqrt{3}$. Find the largest diameter for a circle in one of the regions between $\triangle ABC$ and its inscribed circle. [**4 marks**]

7. If a, b, c, d are positive integers with sum 63, what is the maximum value of $ab + bc + cd$? [**4 marks**]

8. A circle meets the sides of an equilateral triangle ABC at six points D, E, F, G, H, I as shown. If $AE = 4$, $ED = 26$, $DC = 2$, $FG = 14$, and the circle with diameter HI has area πb, find b.



[**4 marks**]

9. A box contains some identical tennis balls. The ratio of the total volume of the tennis balls to the volume of empty space surrounding them in the box is $1 : k$, where k is an integer greater than 1.

A prime number of tennis balls is removed from the box. The ratio of the total volume of the remaining tennis balls to the volume of empty space surrounding them in the box is $1 : k^2$. Find the number of tennis balls that were originally in the box. [**5 marks**]

10. I have a $1\,\mathrm{m} \times 1\,\mathrm{m}$ square, which I want to cover with three circular discs of equal size (which are allowed to overlap). Show that this is possible if the discs have diameter 1008 mm. [**4 marks**]

Investigation

Two discs of equal diameter cover a $1\,\mathrm{m} \times 1\,\mathrm{m}$ square. Find their minimum diameter. [**3 bonus marks**]

2013 Australian Intermediate Mathematics Olympiad - Solutions

1. *Method 1*

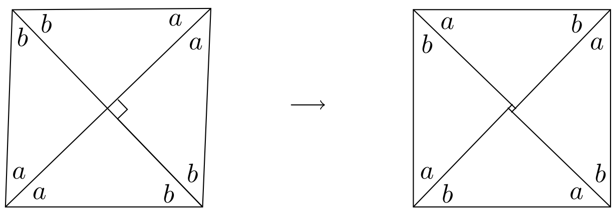
Let the diagonals be $2x$ and $2x + 2$. The diagonals of a rhombus bisect each other at right angles. Hence they partition the rhombus into four congruent right-angled triangles each with hypotenuse 29. Thus the area of the rhombus is $4 \times \frac{1}{2}x(x+1) = 2x^2 + 2x$.

From Pythagoras, $x^2 + (x+1)^2 = 29^2 = 841$. So $2x^2 + 2x = \mathbf{840}$.

Method 2

The diagonals of a rhombus bisect each other at right angles. Hence they partition the rhombus into four congruent right-angled triangles each with hypotenuse 29 and short sides differing by one.

Because the small angles of these triangles are complementary, the triangles can be rearranged to form a square of side 29 with a unit square hole.



Thus the area of the rhombus is $29^2 - 1 = \mathbf{840}$.

2. Since $60 = 2^2 \times 3 \times 5$, only the digits 1, 2, 3, 4, 5 and 6 can be used. The only combinations of four of these digits whose product is 60 are (1, 2, 5, 6), (1, 3, 4, 5), (2, 2, 3, 5).

There are 24 ways to arrange four different digits and 12 ways to arrange four digits of which two are the same. So the total number of required 4-digit numbers is $24 + 24 + 12 = \mathbf{60}$.

3. Call the number $(abc)_7$, where a, b, c are digits less than 7.
Then $(abc)_7 = (cba)_9$.

Hence $49a + 7b + c = 81c + 9b + a$.
So $48a = 80c + 2b$ or $24a = 40c + b$.
Then $b = 24a - 40c = 8(3a - 5c)$.
Since b is both a digit less than 7 and a multiple of 8, it must be 0.

Now we have $3a - 5c = 0$, or $3a = 5c$.
Since a and c are digits less than 7, $a = 5$ and $c = 3$.
So the number is $(503)_7 = 5 \times 49 + 3 \times 1 = \mathbf{248}$.

4. *Method 1*

We have $p(q + r) = 80 = 2^4 \times 5$, $q(p + r) = 425 = 5^2 \times 17$, $r(q - p) = 345 = 3 \times 5 \times 23$.

So $p = 2$ or 5, $q = 5$ or 17, $r = 3$, 5, or 23.
From $p(q + r) = 80$, if $p = 5$ then $q + r = 16$, which has no solution.

So $p = 2$, then $q + r = 40$, hence $q = 17$ and $r = 23$.
Therefore $p + q + r$ is $2 + 17 + 23 = \mathbf{42}$.

Method 2

We have $p(q + r) = 80 = 2^4 \times 5$ and $q(p + r) = 425 = 5^2 \times 17$.

So $p = 2$ or 5 and $q = 5$ or 17.
From $q(p + r) = 425$, if $q = 5$ then $p + r = 85$.
So $r = 83$ or 80, which both contradict $p(q + r) = 80$.

Hence $q = 17$ and $p + r = 25$. Since r is prime, $p = 2$ and $r = 23$.
Therefore $p + q + r$ is $2 + 17 + 23 = \mathbf{42}$.

5. Write the sum as follows:

$$\begin{array}{r} aba \\ \underline{cdc} \\ \underline{effe} \end{array}$$

It is clear from the 1000s column that $e = 1$.

So from the units column, $a + c = 1$ or $a + c = 11$.
But the carry from the 100s column means that $a + c = 11$.

Method 1

Without loss of generality, take $a < c$.
So the possible solution pairs for (a, c) are: (2, 9), (3, 8), (4, 7), (5, 6).
Now the 10s column gives $b + d + 1 = f$ or $b + d + 1 = f + 10$.

Case 1: $b + d + 1 = f$.

Since there is no carry to the 100s column and $a + c = 11$, we have $f = 1$.
Thus $b + d = 0$, hence $b = d = 0$.
So this gives four solutions: $202 + 909 = 1111$, $303 + 808 = 1111$, $404 + 707 = 1111$, $505 + 606 = 1111$.

Case 2: $b + d + 1 = f + 10$.

Since there is a carry to the 100s column, $a + c + 1 = 12$.
Then $f = 2$, hence $b + d = 11$.
For each pair of values of a and c, there are then eight solution pairs for (b, d): (2, 9), (3, 8), (4, 7), (5, 6), (6, 5), (7, 4), (8, 3), (9, 2).
So this gives $4 \times 8 = 32$ solutions:
($222 + 999 = 1221$, $232 + 989 = 1221$, etc).

Hence the number of solution pairs overall is $4 + 32 = \mathbf{36}$.

Method 2

We know that $e = 1$, $a + c = 11$, and the carry from the 10s column is at most 1.
Hence $f = 1$ or 2. Therefore $b + d = 0$ or 11 respectively.

For each pair of values of a and c, there are then nine solution pairs for (b, d): (0, 0), (2, 9), (3, 8), (4, 7), (5, 6), (6, 5), (7, 4), (8, 3), (9, 2).

Thus the number of required pairs of 3-digit palindromes is $4 \times 9 = \mathbf{36}$.

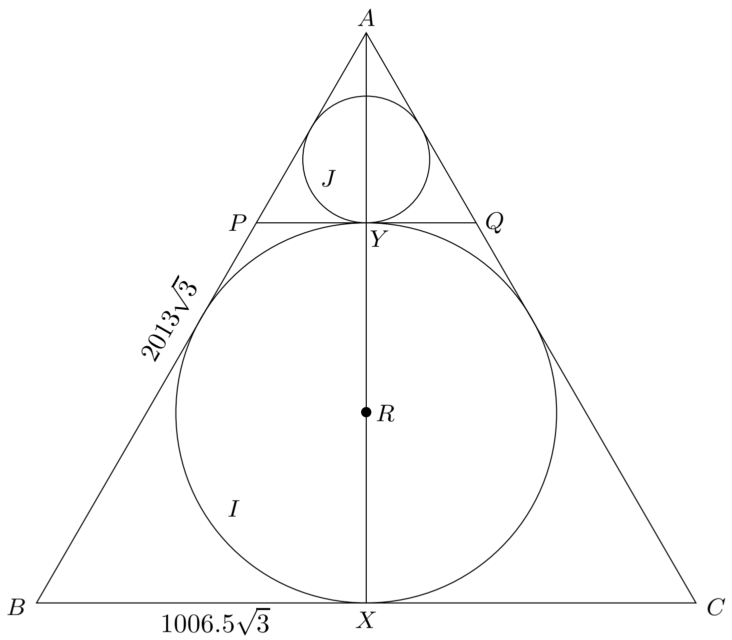
6. *Method 1*

Let I be the incircle of $\triangle ABC$ and let J be the largest circle in the top region between $\triangle ABC$ and I.

Let R be the incentre of $\triangle ABC$. Then AR bisects $\angle BAC$. Extend AR to meet BC at X. Since $\triangle ABC$ is equilateral, X is the midpoint of BC. By symmetry, R lies on all medians of $\triangle ABC$. Hence $RX = \frac{1}{3}AX$. AX is also perpendicular to BC.

Since J touches AB and AC, its centre is also on AX. Hence I and J touch at some point Y on AX. Let their common tangent meet AB at P and AC at Q. Then PQ and BC are parallel. Hence $\triangle APQ$ is similar to $\triangle ABC$.

So $\triangle APQ$ is equilateral and its altitude $AY = AX - YX = AX - \frac{2}{3}AX = \frac{1}{3}AX$. Since J is the incircle of $\triangle APQ$, its radius is $\frac{1}{3}AY = \frac{1}{9}AX$. Since $\triangle ABX$ is 30-60-90, $AX = \sqrt{3} \times 1006.5\sqrt{3} = 3 \times 1006.5$. So the diameter of $J = 2 \times \frac{1}{9}AX = 2 \times 1006.5/3 = \mathbf{671}$.



Method 2

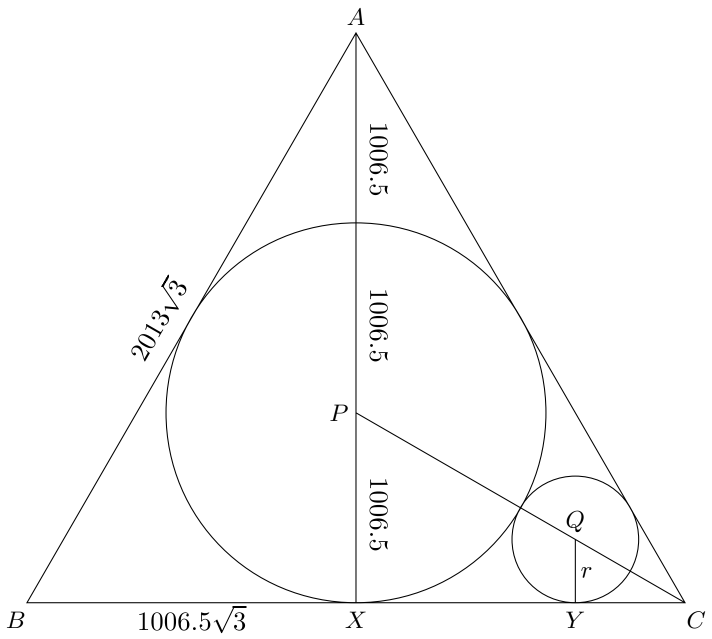
Let r be the radius of the smaller circle.

Let P be the incentre of $\triangle ABC$. Then AP bisects $\angle BAC$ and CP bisects $\angle ACB$. Extend AP to meet BC at X. Since $\triangle ABC$ is equilateral, X is the midpoint of BC, AX is perpendicular to BC, and $PA = PC$.

Thus $\triangle ABX$ and $\triangle CPX$ are 30-60-90. Hence $AX = \sqrt{3} \times 1006.5\sqrt{3} = 3 \times 1006.5$ and $PX = XC/\sqrt{3} = 1006.5$. So $AP = 2PX$.

Since the smaller circle touches AC and BC, its centre, Q, lies on CP. Let Y be the point where the smaller circle touches BC. Then QY is perpendicular to BC. Hence $\triangle CYQ$ is similar to $\triangle CXP$.

So $\dfrac{r}{XP} = \dfrac{CQ}{CP} = \dfrac{CQ}{AP} = \dfrac{CQ}{2PX}$.
Hence $2r = CQ = CP - QP = AP - QP = 2 \times 1006.5 - (1006.5 + r)$.
Therefore $3r = 1006.5$ and $2r = 2013/3 = \mathbf{671}$.



7. *Method 1*

For all real numbers x and y, we have $(x-y)^2 \geq 0$. So $x^2 + y^2 \geq 2xy$, hence $(x+y)^2 \geq 4xy$ and $xy \leq (x+y)^2/4$.

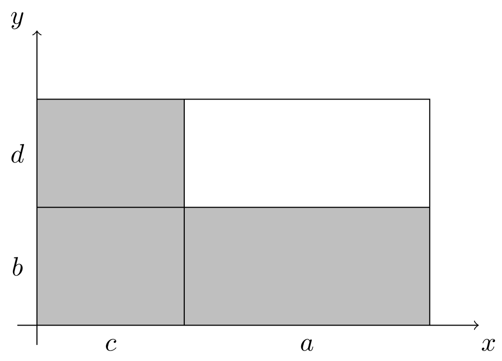
Letting $x = a+c$ and $y = b+d$ gives $(a+c)(b+d) \leq (a+b+c+d)^2/4$. So $ab + bc + cd + da \leq 63^2/4 = 3969/4 = 992.25$.

Since a, b, c, d are positive integers, the last inequality can be written as $ab + bc + cd + da \leq 992$. Hence $ab + bc + cd \leq 992 - da \leq 991$.

It remains to show that 991 is achievable. Suppose $ab+bc+cd = 991$ and $a = d = 1$. Then $(1 + b)(1 + c) = 992 = 2^5 \times 31$. So $b = 30$ and $c = 31$ is a solution. Thus the maximum value of $ab+bc+cd$ is **991**.

Method 2

Consider the rectangles $a \times b$, $b \times c$, $c \times d$, $a \times d$ arranged as follows. We wish to maximise the shaded area.



Let $u = a + c$ and $v = b + d$. For fixed u and v, the shaded area is maximum when $a = d = 1$. So, to maximise the shaded area, we need to maximise the area of the $u \times v$ rectangle with $u = c+1$, $v = b+1$, and $u + v = 63$.

The area of the $u \times v$ rectangle is $uv = u(63 - u)$. The graph of $y = x(63 - x)$ is a parabola with its maximum at $x = 63/2$. Hence the maximum value of uv is attained when u is as close as possible to $63/2$. Thus $u = 31$ and $v = 32$ or vice versa.

So the maximum shaded area is $31 \times 32 - 1 =$ **991**.

Method 3

From symmetry we may assume $b \leq c$.

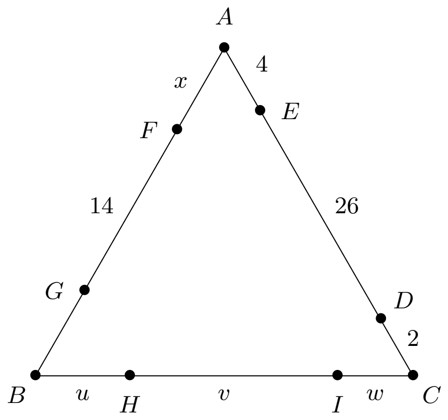
Then $ab + bc + cd \leq ac + bc + cd = c(a + b + d) = c(63 - c)$.

The graph of $y = x(63 - x)$ is a parabola with its maximum at $x = 63/2$. Hence the maximum value of $c(63 - c)$ is attained when c is as close as possible to $63/2$. Thus $c = 31$ or 32.

If $b = c$, then $a + b + c + d \geq 1 + 31 + 31 + 1 = 64$, a contradiction. So $b < c$ and we have $ab + bc + cd < (31)(32) = 992$.

It remains to show that 991 is achievable. If $c = 31$, $b = 30$, and $a = d = 1$, then $ab + bc + cd = 991$. Thus the maximum value of $ab + bc + cd$ is **991**.

8. Let $BH = u$, $HI = v$, $IC = w$, and $AF = x$.



The intersecting secant theorem at A gives $4 \times 30 = x(x + 14)$. Hence $x^2 + 14x - 120 = 0$. So $(x + 20)(x - 6) = 0$ and $x = 6$. Therefore $GB = 12$.

The intersecting secant theorem at B gives

$$12 \times 26 = u(u + v) = u(32 - w). \qquad (1)$$

The intersecting secant theorem at C gives

$$2 \times 28 = w(w + v) = w(32 - u). \qquad (2)$$

Subtracting (2) from (1) gives:

$$\begin{aligned} 256 &= 32(u - w) \\ u &= w + 256/32 = w + 8 \\ v &= 32 - u - w = 24 - 2w \end{aligned}$$

Substituting in (2) gives:

$$\begin{aligned}
56 &= w(24-w)\\
w^2-24w+56 &= 0\\
w &= \left(24 \pm \sqrt{24^2-224}\right)/2\\
v &= 4\sqrt{22}
\end{aligned}$$

Hence $\pi b = \pi(2\sqrt{22})^2 = \pi 88$ and $b = \mathbf{88}$.

9. Let the volume of the box be V_B and the volume of a single tennis ball be V_T. Suppose there are N tennis balls to begin.

The total volume of the tennis balls is NV_T and the volume of empty space surrounding them is $V_B - NV_T$. From the ratio given, $V_B - NV_T = kNV_T$, so $V_B = NV_T + kNV_T = (k+1)NV_T$.

Let P be the number of balls removed, where P is a prime. The total volume of the remaining tennis balls is $(N-P)V_T$ and the volume of empty space surrounding them is $V_B - (N-P)V_T = (k+1)NV_T - (N-P)V_T = kNV_T + PV_T = (kN+P)V_T$.

From the ratio given, $(kN+P)V_T = k^2(N-P)V_T$. So $kN + P = k^2N - k^2P$, hence $N = P(k^2+1)/(k^2-k)$.

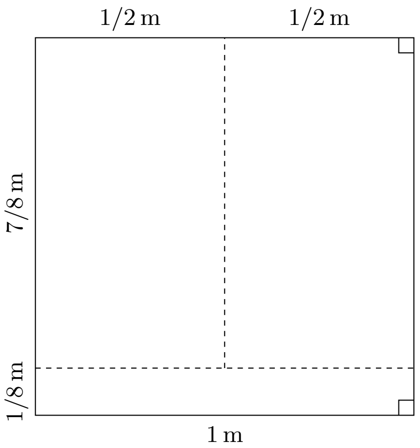
Now N is an integer and k and k^2+1 are relatively prime, so k divides P. But P is a prime and $k > 1$ so $k = P$. Thus $N = (P^2+1)/(P-1) = P + 1 + 2/(P-1)$.

Since N is an integer, $P-1$ divides 2. So $P = 2$ or $P = 3$. Either way $N = 5$. Thus the number of tennis balls originally in the box is **5**.

Comment. The algebra can be simplified by rescaling to let $V_T = 1$.

10. *Method 1*

Subdivide the square into three rectangles: one measuring $1\,\text{m} \times \frac{1}{8}\,\text{m}$ and each of the other two measuring $\frac{1}{2}\,\text{m} \times \frac{7}{8}\,\text{m}$.



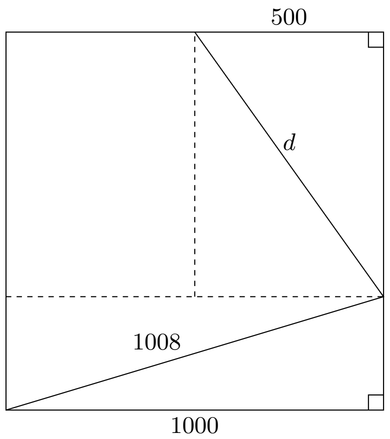
Note that $\sqrt{1 + (\frac{1}{8})^2} = \sqrt{\frac{65}{64}}$ and $\sqrt{(\frac{1}{2})^2 + (\frac{7}{8})^2} = \sqrt{\frac{65}{64}}$. So each of the three rectangles have diagonals of length $\sqrt{\frac{65}{64}}$ and can therefore be covered by a disc with this diameter.

Now $\sqrt{\frac{65}{64}} < 1.008 \Leftrightarrow 65 < 64(1.008)^2$ and $64(1 + 0.008)^2 > 64 \times 1.016 = 64(1 + 0.01 + 0.006) = 64 + 0.64 + 0.384 = 65.024 > 65$.

So $\sqrt{\frac{65}{64}} < 1.008$. Therefore it is possible for three discs each with diameter 1008 mm to cover the square.

Method 2

Construct two right-angled triangles in the 1 m × 1 m square as shown.



We need to show $d < 1008$.

Pythagoras gives:

$$\begin{aligned} d^2 &= 500^2 + \left(1000 - \sqrt{1008^2 - 1000^2}\right)^2 \\ &= 500^2 + \left(1000 - \sqrt{(1008-1000)(1008+1000)}\right)^2 \\ &= 500^2 + \left(1000 - \sqrt{8 \times 2008}\right)^2 \\ &= 250000 + 1000000 + 16064 - 2000\sqrt{16064} \end{aligned}$$

So $d^2 - 1008^2 = d^2 - 1016064 = 250000 - 2000\sqrt{16064}$.

Hence $d < 1008 \Leftrightarrow 125^2 < 16064$, which is true since $125^2 = 15625$.

Investigation

The minimum diameter is $500\sqrt{5} \approx 1118\,\text{mm}$.

Divide the square into two rectangles each $1000\,\text{mm} \times 500\,\text{mm}$. The length of the diagonal of each rectangle is $\sqrt{1000^2 + 500^2} = 500\sqrt{5}\,\text{mm}$. Hence the square can be covered by two discs of this diameter.

Suppose the square can be covered by two discs of shorter diameter. One of the discs must cover at least two of the vertices of the square. If two of these vertices were diagonally opposite on the square, then the diameter of the disc would be at least the length of the diagonal of the square, which is approximately $1400\,\text{mm}$. So the disc covers exactly two vertices of the square and they are on the same side of the square.

Denote the square by $ABCD$. We may assume the disc covers A and B, and intersects AD at a point X. The disc covers BX and its diameter is less than $500\sqrt{5}$. Hence BX is less than $500\sqrt{5}$, so $AX^2 < (500\sqrt{5})^2 - 1000^2 = (125 - 100)10000 = 250000$ and hence $AX < 500$. The second disc must cover points X, C, and D. Since $XD > 500$, $XC > 500\sqrt{5}$. Then the diameter of the second disc is greater than $500\sqrt{5}$, a contradiction.

So the diameters of the two covering discs cannot be less than $500\sqrt{5}$.