Australian Intermediate Mathematics Olympiad 2016

Questions

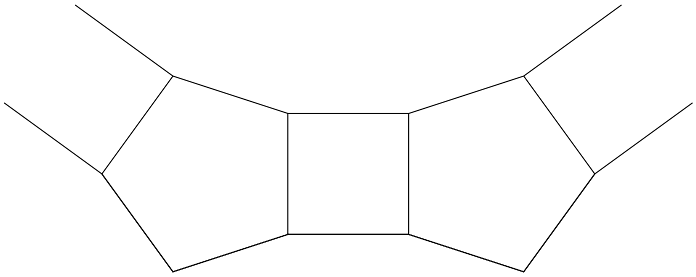
1. Find the smallest positive integer x such that $12x = 25y^2$, where y is a positive integer.

[**2 marks**]

2. A 3-digit number in base 7 is also a 3-digit number when written in base 6, but each digit has increased by 1. What is the largest value which this number can have when written in base 10?

[**2 marks**]

3. A ring of alternating regular pentagons and squares is constructed by continuing this pattern.



How many pentagons will there be in the completed ring?

[**3 marks**]

4. A sequence is formed by the following rules: $s_1 = 1, s_2 = 2$ and $s_{n+2} = s_n^2 + s_{n+1}^2$ for all $n \geq 1$. What is the last digit of the term s_{200}?

[**3 marks**]

5. Sebastien starts with an 11×38 grid of white squares and colours some of them black. In each white square, Sebastien writes down the number of black squares that share an edge with it.

Determine the maximum sum of the numbers that Sebastien could write down.

[**3 marks**]

6. A circle has centre O. A line PQ is tangent to the circle at A with A between P and Q. The line PO is extended to meet the circle at B so that O is between P and B. $\angle APB = x^\circ$ where x is a positive integer. $\angle BAQ = kx^\circ$ where k is a positive integer. What is the maximum value of k?

[**4 marks**]

PLEASE TURN OVER THE PAGE FOR QUESTIONS 7, 8, 9 AND 10

7. Let n be the largest positive integer such that $n^2 + 2016n$ is a perfect square. Determine the remainder when n is divided by 1000.

[**4 marks**]

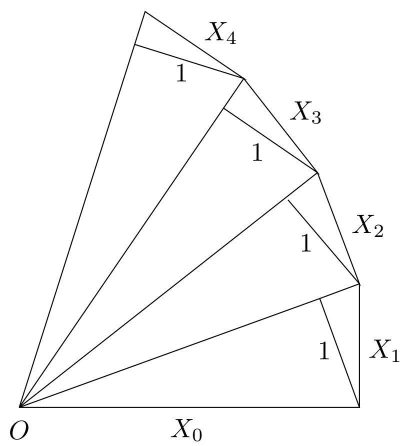
8. Ann and Bob have a large number of sweets which they agree to share according to the following rules. Ann will take one sweet, then Bob will take two sweets and then, taking turns, each person takes one more sweet than what the other person just took. When the number of sweets remaining is less than the number that would be taken on that turn, the last person takes all that are left. To their amazement, when they finish, they each have the same number of sweets.

They decide to do the sharing again, but this time, they first divide the sweets into two equal piles and then they repeat the process above with each pile, Ann going first both times. They still finish with the same number of sweets each.

What is the maximum number of sweets less than 1000 they could have started with?

[**4 marks**]

9. All triangles in the spiral below are right-angled. The spiral is continued anticlockwise.



Prove that $X_0^2 + X_1^2 + X_2^2 + \cdots + X_n^2 = X_0^2 \times X_1^2 \times X_2^2 \times \cdots \times X_n^2$.

[**5 marks**]

10. For $n \geq 3$, consider $2n$ points spaced regularly on a circle with alternate points black and white and a point placed at the centre of the circle.

The points are labelled $-n$, $-n+1$, $\ldots$, $n-1$, n so that:

(a) the sum of the labels on each diameter through three of the points is a constant s, and

(b) the sum of the labels on each black-white-black triple of consecutive points on the circle is also s.

Show that the label on the central point is 0 and $s = 0$.

[**5 marks**]

Investigation

Show that such a labelling exists if and only if n is even.

[**3 bonus marks**]

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Australian Intermediate Mathematics Olympiad 2016

Solutions

1. *Method 1*

We have $2^2 \times 3x = 5^2y^2$ where x and y are integers. So 3 divides y^2.
Since 3 is prime, 3 divides y. $\boxed{1}$

Hence 3 divides x. Also 25 divides x. So the smallest value of x is $3 \times 25 = \mathbf{75}$. $\boxed{1}$

Method 2

The smallest value of x will occur with the smallest value of y.
Since 12 and 25 are relatively prime, 12 divides y^2. $\boxed{1}$

The smallest value of y for which this is possible is $y = 6$.
So the smallest value of x is $(25 \times 36)/12 = \mathbf{75}$. $\boxed{1}$

2. $abc_7 = (a+1)(b+1)(c+1)_6$.

This gives $49a + 7b + c = 36(a+1) + 6(b+1) + c + 1$. Simplifying, we get $13a + b = 43$. Remembering that $a+1$ and $b+1$ are less than 6, and therefore a and b are less than 5, the only solution of this equation is $a = 3$, $b = 4$. $\boxed{1}$

Hence the number is $34c_7$ or $45(c+1)_6$. But $c+1 \leq 5$ so, for the largest such number, $c = 4$.

Hence the number is $344_7 = \mathbf{179}$. $\boxed{1}$

3. *Method 1*

The interior angle of a regular pentagon is $108°$. So the angle inside the ring between a square and a pentagon is $360° - 108° - 90° = 162°$. Thus on the inside of the completed ring we have a regular polygon with n sides whose interior angle is $162°$. $\boxed{1}$

The interior angle of a regular polygon with n sides is $180°(n-2)/n$.
So $162n = 180(n-2) = 180n - 360$. Then $18n = 360$ and $n = 20$. $\boxed{1}$

Since half of these sides are from pentagons, the number of pentagons in the completed ring is **10**. $\boxed{1}$

Method 2

The interior angle of a regular pentagon is $108°$. So the angle inside the ring between a square and a pentagon is $360° - 108° - 90° = 162°$. $\boxed{1}$

Thus on the inside of the completed ring we have a regular polygon with n sides whose exterior angle is $180° - 162° = 18°$. Hence $18n = 360$ and $n = 20$. $\boxed{1}$

Since half of these sides are from pentagons, the number of pentagons in the completed ring is **10**. $\boxed{1}$

Method 3

The interior angle of a regular pentagon is $108°$. So the angle inside the ring between a square and a pentagon is $360° - 108° - 90° = 162°$. Thus on the inside of the completed ring we have a regular polygon whose interior angle is $162°$. $\boxed{1}$

The bisectors of these interior angles form congruent isosceles triangles on the sides of this polygon. So all these bisectors meet at a point, O say. $\boxed{1}$

The angle at O in each of these triangles is $180° - 162° = 18°$. If n is the number of pentagons in the ring, then $18n = 360/2 = 180$. So $n = \mathbf{10}$. $\boxed{1}$

4. Working modulo 10, we can make a sequence of last digits as follows:

$$1,\ 2,\ 5,\ 9,\ 6,\ 7,\ 5,\ 4,\ 1,\ 7,\ 0,\ 9,\ 1,\ 2,\ \ldots$$ $\boxed{1}$

Thus the last digits repeat after every 12 terms. Now $200 = 16 \times 12 + 8$. Hence the 200th last digit will the same as the 8th last digit. $\boxed{1}$

So the last digit of s_{200} is **4**. $\boxed{1}$

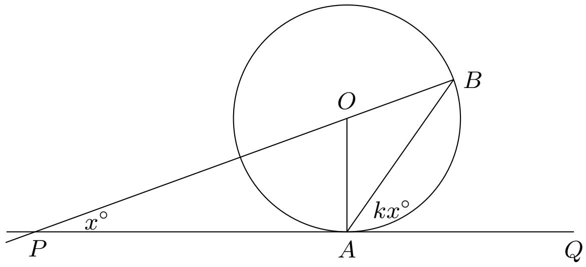
5. For each white square, colour in red the edges that are adjacent to black squares. Observe that the sum of the numbers that Sebastien writes down is the number of red edges. $\boxed{1}$

The number of red edges is bounded above by the number of edges in the 11×38 grid that do not lie on the boundary of the grid. The number of such horizontal edges is 11×37, while the number of such vertical edges is 10×38. Therefore, the sum of the numbers that Sebastien writes down is bounded above by $11 \times 37 + 10 \times 38 = 787$. $\boxed{1}$

Now note that this upper bound is obtained by the usual chessboard colouring of the grid. So the maximum sum of the numbers that Sebastien writes down is **787**. $\boxed{1}$

6. *Method 1*

Draw OA.



$\boxed{1}$

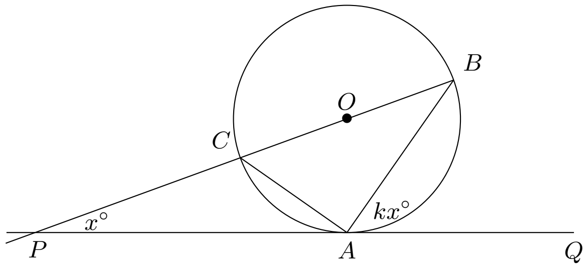
Since OA is perpendicular to PQ, $\angle OAB = 90^\circ - kx^\circ$.
Since $OA = OB$ (radii), $\angle OBA = 90^\circ - kx^\circ$. $\boxed{1}$

Since $\angle QAB$ is an exterior angle of $\triangle PAB$, $kx = x + (90 - kx)$.
Rearranging gives $(2k-1)x = 90$. $\boxed{1}$

For maximum k we want $2k - 1$ to be the largest odd factor of 90.
Then $2k - 1 = 45$ and $k = \mathbf{23}$. $\boxed{1}$

Method 2

Let C be the other point of intersection of the line PB with the circle.



$\boxed{1}$

By the Tangent-Chord theorem, $\angle ACB = \angle QAB = kx$. Since BC is a diameter, $\angle CAB = 90^\circ$.
By the Tangent-Chord theorem, $\angle PAC = \angle ABC = 180 - 90 - kx = 90 - kx$. $\boxed{1}$

Since $\angle ACB$ is an exterior angle of $\triangle PAC$, $kx = x + 90 - kx$.
Rearranging gives $(2k-1)x = 90$. $\boxed{1}$

For maximum k we want $2k - 1$ to be the largest odd factor of 90.
Then $2k - 1 = 45$ and $k = \mathbf{23}$. $\boxed{1}$

7. *Method 1*

If $n^2 + 2016n = m^2$, where n and m are positive integers, then $m = n + k$ for some positive integer k. Then $n^2 + 2016n = (n + k)^2$. So $2016n = 2nk + k^2$, or $n = k^2/(2016 - 2k)$. Since both n and k^2 are positive, we must have $2016 - 2k > 0$, or $2k < 2016$. Thus $1 \leq k \leq 1007$. $\boxed{1}$

As k increases from 1 to 1007, k^2 increases and $2016 - 2k$ decreases, so n increases. Conversely, as k decreases from 1007 to 1, k^2 decreases and $2016 - 2k$ increases, so n decreases. If we take $k = 1007$, then $n = 1007^2/2$, which is not an integer. If we take $k = 1006$, then $n = 1006^2/4 = 503^2$. So $n \leq 503^2$. $\boxed{1}$

If $k = 1006$ and $n = 503^2$, then $(n+k)^2 = (503^2+1006)^2 = (503^2+2\times503)^2 = 503^2(503+2)^2 = 503^2(503^2 + 4 \times 503 + 4) = 503^2(503^2 + 2016) = n^2 + 2016n$. So $n^2 + 2016n$ is indeed a perfect square. Thus 503^2 is the largest value of n such that $n^2 + 2016n$ is a perfect square. $\boxed{1}$

Since $503^2 = (500+3)^2 = 500^2 + 2 \times 500 \times 3 + 3^2 = 250000 + 3000 + 9 = 253009$, the remainder when n is divided by 1000 is **9**. $\boxed{1}$

Method 2

If $n^2 + 2016n = m^2$, where n and m are positive integers, then $m^2 = (n + 1008)^2 - 1008^2$. So $1008^2 = (n + 1008 + m)(n + 1008 - m)$ and both factors are even and positive. Hence $n + 1008 + m = 1008^2/(n + 1008 - m) \leq 1008^2/2$. $\boxed{1}$

Since m increases with n, maximum n occurs when $n + 1008 + m$ is maximum. If $n + 1008 + m = 1008^2/2$, then $n + 1008 - m = 2$. Adding these two equations and dividing by 2 gives $n + 1008 = 504^2 + 1$ and $n = 504^2 - 1008 + 1 = (504 - 1)^2 = 503^2$. $\boxed{1}$

If $n = 503^2$, then $n^2 + 2016n = 503^2(503^2 + 2016)$. Now $503^2 + 2016 = (504 - 1)^2 + 2016 = 504^2 + 1008 + 1 = (504 + 1)^2 = 505^2$. So $n^2 + 2016n$ is indeed a perfect square. Thus 503^2 is the largest value of n such that $n^2 + 2016n$ is a perfect square. $\boxed{1}$

Since $503^2 = (500+3)^2 = 500^2 + 2 \times 500 \times 3 + 3^2 = 250000 + 3000 + 9 = 253009$, the remainder when n is divided by 1000 is **9**. $\boxed{1}$

Method 3

If $n^2 + 2016n = m^2$, where n and m are positive integers, then solving the quadratic for n gives $n = (-2016 + \sqrt{2016^2 + 4m^2})/2 = \sqrt{1008^2 + m^2} - 1008$. So $1008^2 + m^2 = k^2$ for some positive integer k. Hence $(k-m)(k+m) = 1008^2$ and both factors are even and positive. Hence $k + m = 1008^2/(k - m) \leq 1008^2/2$. $\boxed{1}$

Since m, n, k increase together, maximum n occurs when $m + k$ is maximum. If $k + m = 1008^2/2$, then $k - m = 2$. Subtracting these two equations and dividing by 2 gives $m = 504^2 - 1$ and $1008^2 + m^2 = 1008^2 + (504^2 - 1)^2 = 4 \times 504^2 + 504^4 - 2 \times 504^2 + 1 = 504^4 + 2 \times 504^2 + 1 = (504^2 + 1)^2$. So $n = 504^2 + 1 - 2 \times 504 = (504 - 1)^2 = 503^2$. $\boxed{1}$

If $n = 503^2$, then $n^2 + 2016n = 503^2(503^2 + 2016)$. Now $503^2 + 2016 = (504 - 1)^2 + 2016 = 504^2 + 1008 + 1 = (504 + 1)^2 = 505^2$. So $n^2 + 2016n$ is indeed a perfect square. Thus 503^2 is the largest value of n such that $n^2 + 2016n$ is a perfect square. $\boxed{1}$

Since $503^2 = (500+3)^2 = 500^2 + 2 \times 500 \times 3 + 3^2 = 250000 + 3000 + 9 = 253009$, the remainder when n is divided by 1000 is **9**. $\boxed{1}$

8. Suppose Ann has the last turn. Let n be the number of turns that Bob has. Then the number of sweets that he takes is $2+4+6+\cdots+2n = 2(1+2+\cdots+n) = n(n+1)$. So the total number of sweets is $2n(n+1)$. $\boxed{1}$

Suppose Bob has the last turn. Let n be the number of turns that Ann has. Then the number of sweets that she takes is $1+3+5+\cdots+(2n-1) = n^2$. So the total number of sweets is $2n^2$. $\boxed{1}$

So half the number of sweets is $n(n+1)$ or n^2. Applying the same sharing procedure to half the sweets gives, for some integer m, one of the following four cases:

1. $n(n+1) = 2m(m+1)$
2. $n(n+1) = 2m^2$
3. $n^2 = 2m(m+1)$
4. $n^2 = 2m^2$.

In the first two cases we want n such that $n(n+1) < 500$. So $n \leq 21$.

In the first case, since 2 divides m or $m+1$, we also want 4 to divide $n(n+1)$. So $n \leq 20$. Since $20 \times 21 = 420 = 2 \times 14 \times 15$, the total number of sweets could be $2 \times 420 = 840$. $\boxed{1}$

In the second case $\frac{1}{2}n(n+1)$ is a perfect square. So $n < 20$.

In the last two cases we look for n so that $n^2 > 840/2 = 420$.
We also want n even and $n^2 < 500$. So $n = 22$.

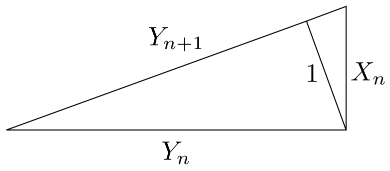
In the third case, $m(m+1) = \frac{1}{2} \times 22^2 = 242$ but $15 \times 16 = 240$ while $16 \times 17 = 272$.

In the fourth case, $m^2 = 242$ but 242 is not a perfect square.

So the maximum total number of sweets is **840**. $\boxed{1}$

9. *Method 1*

For each large triangle, one leg is X_n. Let Y_n be the other leg and let Y_{n+1} be the hypotenuse. Note that $Y_1 = X_0$.



$\boxed{1}$

By Pythagoras,

$$\begin{aligned} Y_{n+1}^2 &= X_n^2 + Y_n^2 \\ &= X_n^2 + X_{n-1}^2 + Y_{n-1}^2 \\ &= X_n^2 + X_{n-1}^2 + X_{n-2}^2 + Y_{n-2}^2 \\ &= X_n^2 + X_{n-1}^2 + X_{n-2}^2 + \cdots + X_1^2 + Y_1^2 \\ &= X_n^2 + X_{n-1}^2 + X_{n-2}^2 + \cdots + X_1^2 + X_0^2 \qquad \boxed{1} \end{aligned}$$

The area of the triangle shown is given by $\frac{1}{2}Y_{n+1}$ and by $\frac{1}{2}X_nY_n$. Using this or similar triangles we have $\boxed{1}$

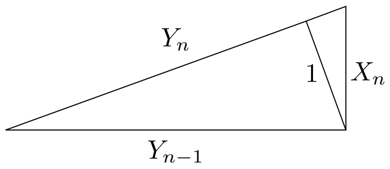
$$\begin{aligned} Y_{n+1} &= X_n \times Y_n \\ &= X_n \times X_{n-1} \times Y_{n-1} \\ &= X_n \times X_{n-1} \times X_{n-2} \times Y_{n-2} \\ &= X_n \times X_{n-1} \times X_{n-2} \times \cdots \times X_1 \times Y_1 \\ &= X_n \times X_{n-1} \times X_{n-2} \times \cdots \times X_1 \times X_0 \qquad \boxed{1} \end{aligned}$$

So

$$X_0^2 + X_1^2 + X_2^2 + \cdots + X_n^2 = X_0^2 \times X_1^2 \times X_2^2 \times \cdots \times X_n^2 \qquad \boxed{1}$$

Method 2

For each large triangle, one leg is X_n. Let Y_{n-1} be the other leg and let Y_n be the hypotenuse. Note that $Y_0 = X_0$.



$\boxed{1}$

From similar triangles we have $Y_1/X_1 = X_0/1$. So $Y_1 = X_0 \times X_1$.

By Pythagoras, $Y_1^2 = X_0^2 + X_1^2$. So $X_0^2 + X_1^2 = Y_1^2 = X_0^2 \times X_1^2$. $\boxed{1}$

Assume for some $k \geq 1$

$$Y_k^2 = X_0^2 + X_1^2 + X_2^2 + \cdots + X_k^2 = X_0^2 \times X_1^2 \times X_2^2 \times \cdots \times X_k^2 \qquad \boxed{1}$$

From similar triangles we have $Y_{k+1}/X_{k+1} = Y_k/1$. So $Y_{k+1} = Y_k \times X_{k+1}$.

By Pythagoras, $Y_{k+1}^2 = X_{k+1}^2 + Y_k^2$. So $X_{k+1}^2 + Y_k^2 = Y_{k+1}^2 = Y_k^2 \times X_{k+1}^2$. Hence

$$X_0^2 + X_1^2 + X_2^2 + \cdots + X_k^2 + X_{k+1}^2 = X_0^2 \times X_1^2 \times X_2^2 \times \cdots \times X_k^2 \times X_{k+1}^2 \qquad \boxed{1}$$

By induction,

$$X_0^2 + X_1^2 + X_2^2 + \cdots + X_n^2 = X_0^2 \times X_1^2 \times X_2^2 \times \cdots \times X_n^2$$

for all $n \geq 1$. $\boxed{1}$

10. *Method 1*

Let b and w denote the sum of the labels on all black and white vertices respectively. Let c be the label on the central vertex. Then

$$b + w + c = 0 \qquad (1)$$ $\boxed{1}$

Summing the labels over all diameters gives

$$b + w + nc = ns \qquad (2)$$ $\boxed{1}$

Summing the labels over all black-white-black arcs gives

$$2b + w = ns \qquad (3)$$ $\boxed{1}$

From (1) and (2),

$$(n-1)c = ns \qquad (4)$$

Hence n divides c. Since $-n \le c \le n$, $c = 0$, $-n$, or n. $\boxed{1}$

Suppose $c = \pm n$. From (2) and (3), $b = nc = \pm n^2$.
Since $|b| \le 1 + 2 + \cdots + n < n^2$, we have a contradiction.
So $c = 0$. From (4), $s = 0$. $\boxed{1}$

Method 2

For any label x not at the centre, let x' denote the label diametrically opposite x. Let the centre have label c. Then $\boxed{1}$

$$x + c + x' = s.$$

If x, y, z are any three consecutive labels where x and z are on black points, then we have

$$x + c + x' = y + c + y' = z + c + z' = s.$$ $\boxed{1}$

Adding these yields

$$x + y + z + 3c + x' + y' + z' = 3s.$$

Since there are an even number of points on the circle, diametrically opposite points have the same colour. So

$$x + y + z = s = x' + y' + z' \quad \text{and} \quad s = 3c.$$ $\boxed{1}$

Hence $p + p' = 2c$ for any label p on the circle. Since there are n such diametrically opposite pairs, the sum of all labels on the circle is $2nc$. $\boxed{1}$

Since the sum of all the labels is zero, we have $0 = 2nc + c = c(2n + 1)$. Thus $c = 0$, and $s = 3c = 0$. $\boxed{1}$

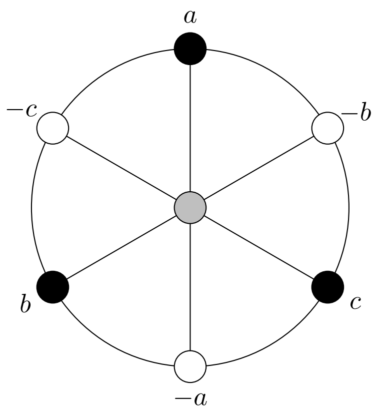
Investigation

Since $c = 0 = s$, for each diameter, the label at one end is the negative of the label at the other end.

Let n be an odd number.

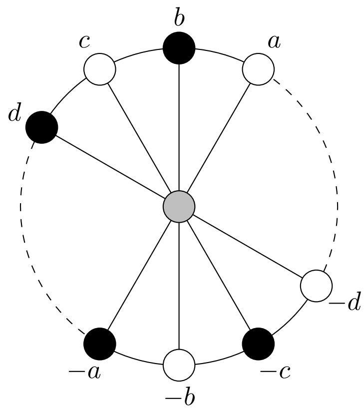
Each diameter is from a black point to a white point.

If $n = 3$, we have:



Hence $a + b - c = 0 = a - b + c$. So $b = c$, which is disallowed.

If $n > 3$, we have:



Hence $b + c + d = 0 = -a - b - c = a + b + c$. So $a = d$, which is disallowed.

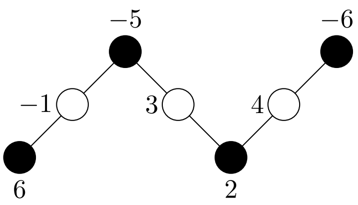
So the required labelling does not exist for odd n. $\boxed{\text{bonus 1}}$

Now let n be an even number.

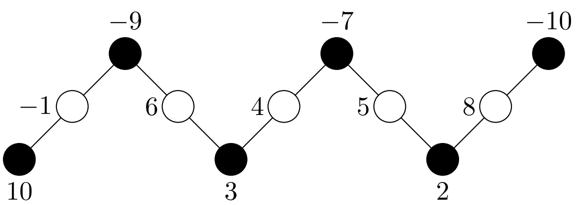
We show that a required labelling does exist for $n = 2m \geq 4$. It is sufficient to show that $n+1$ consecutive points on the circle from a black point to a black point can be assigned labels from ± 1, ± 2, ... , $\pm n$, so that the absolute values of the labels are distinct except for the two end labels, and the sum of the labels on each black-white-black arc is 0. We demonstrate such labellings with a zigzag pattern for clarity. Essentially, with some adjustments at the ends and in small cases, we try to place the odd labels on the black points, which are at the corners of the zigzag, and the even labels on the white points in between.

Case 1. m odd.

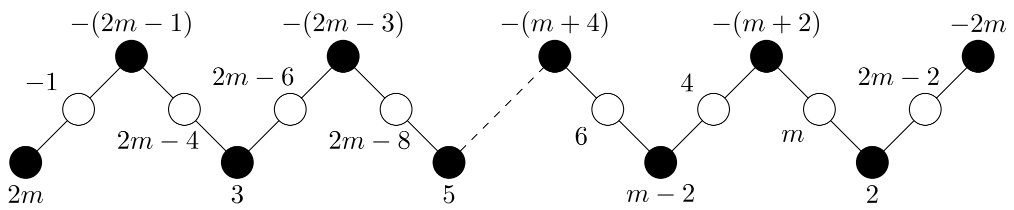
$m = 3$



$m = 5$



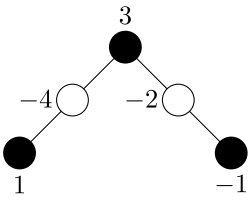
General odd m.



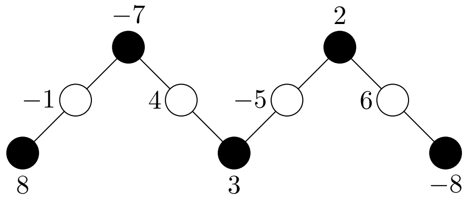
$\boxed{\text{bonus 1}}$

Case 2. m even.

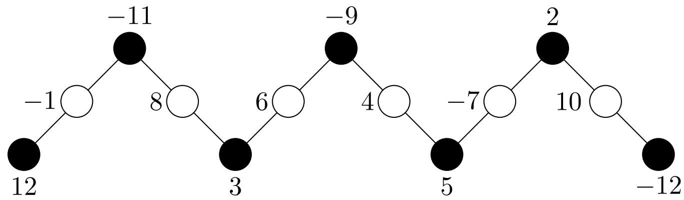
$m = 2$



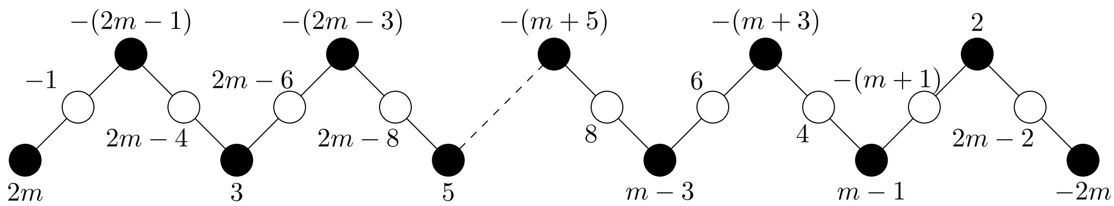
$m = 4$



$m = 6$



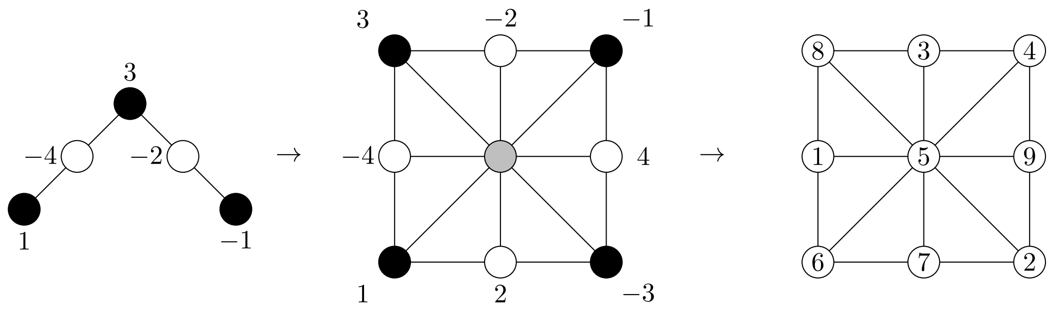
General even m.



Thus the required labelling exists if and only if n is even. $\boxed{\text{bonus 1}}$

Comments

1. The special case $m = 2$ gives the classical magic square:



2. It is easy to check that, except for rotations and reflections, there is only one labelling for $m = 2$. Are the general labellings given above unique for all m?

3. Method 2 shows that the conclusion of the Problem 10 also holds for non-integer labels provided their sum is 0.

Marking Scheme

1. Establishing 3 divides y or 12 divides y^2. $\boxed{1}$
Correct answer (75). $\boxed{1}$

2. Establishing the 100s digit is 3 and the 10s digit is 4. $\boxed{1}$
Correct answer (179). $\boxed{1}$

3. Establishing the interior angle of the ring is 162°. $\boxed{1}$
Further progress. $\boxed{1}$
Correct answer (10). $\boxed{1}$

4. A recurring sequence of last digits. $\boxed{1}$
Correct position of 200th last digit in the repetend. $\boxed{1}$
Correct answer (4). $\boxed{1}$

5. A useful approach. $\boxed{1}$
Establishing 787 as an upper bound. $\boxed{1}$
Establishing 787 as the maximum. $\boxed{1}$

6. A useful diagram. $\boxed{1}$
A useful equation. $\boxed{1}$
Another useful equation. $\boxed{1}$
Correct answer (23). $\boxed{1}$

7. Establishing a relevant upper bound. $\boxed{1}$
Establishing 503^2 as an upper bound for n. $\boxed{1}$
Establishing 503^2 as the maximum for n. $\boxed{1}$
Correct answer (9). $\boxed{1}$

8. Establishing a formula for the number of sweets if Ann has last turn. $\boxed{1}$
Establishing a formula for the number of sweets if Bob has last turn. $\boxed{1}$
Establishing 840 as a possible number of sweets. $\boxed{1}$
Establishing 840 as the maximum number of sweets. $\boxed{1}$

9. Useful diagram and notation. $\boxed{1}$
Some progress. $\boxed{1}$
Further progress. $\boxed{1}$
Substantial progress. $\boxed{1}$
Correct conclusion. $\boxed{1}$

10. A relevant equation. $\boxed{1}$
A second relevant equation. $\boxed{1}$
A third relevant equation. $\boxed{1}$
Further progress. $\boxed{1}$
Correct conclusion. $\boxed{1}$

Investigation:

Establishing n is not odd. $\boxed{\text{bonus } 1}$
Establishing a labelling for $n = 2m$ with m odd. $\boxed{\text{bonus } 1}$
Establishing a labelling for $n = 2m$ with m even. $\boxed{\text{bonus } 1}$

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