THE 2014 AUSTRALIAN MATHEMATICAL OLYMPIAD

DAY 1
Tuesday, 11 February 2014
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. The sequence \(a_1, a_2, a_3, \ldots\) is defined by \(a_1 = 0\) and, for \(n \geq 2\),
\[
a_n = \max_{i=1, \ldots, n-1} \{i + a_i + a_{n-i}\}.
\]
(For example, \(a_2 = 1\) and \(a_3 = 3\).)
Determine \(a_{200}\).

2. Let \(ABC\) be a triangle with \(\angle BAC < 90^\circ\). Let \(k\) be the circle through \(A\) that is tangent to \(BC\) at \(C\). Let \(M\) be the midpoint of \(BC\), and let \(AM\) intersect \(k\) a second time at \(D\). Finally, let \(BD\) (extended) intersect \(k\) a second time at \(E\).
Prove that \(\angle BAC = \angle CAE\).

3. Consider labelling the twenty vertices of a regular dodecahedron with twenty different integers. Each edge of the dodecahedron can then be labelled with the number \(|a - b|\), where \(a\) and \(b\) are the labels of its endpoints. Let \(e\) be the largest edge label.
What is the smallest possible value of \(e\) over all such vertex labellings?
(A regular dodecahedron is a polyhedron with twelve identical regular pentagonal faces.)

4. Let \(\mathbb{N}^+\) denote the set of positive integers, and let \(\mathbb{R}\) denote the set of real numbers.
Find all functions \(f : \mathbb{N}^+ \to \mathbb{R}\) that satisfy the following three conditions:
(i) \(f(1) = 1\),
(ii) \(f(n) = 0\) if \(n\) contains the digit 2 in its decimal representation,
(iii) \(f(mn) = f(m)f(n)\) for all positive integers \(m, n\).
5. Determine all non-integer real numbers $x$ such that

$$x + \frac{2014}{x} = \lfloor x \rfloor + \frac{2014}{\lfloor x \rfloor}.$$ 

(Note that $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to the real number $x$. For example, $\lfloor 20.14 \rfloor = 20$ and $\lfloor -20.14 \rfloor = -21$.)

6. Let $S$ be the set of all numbers

$$a_0 + 10 a_1 + 10^2 a_2 + \cdots + 10^n a_n \quad (n = 0, 1, 2, \ldots)$$

where

(i) $a_i$ is an integer satisfying $0 \leq a_i \leq 9$ for $i = 0, 1, \ldots, n$ and $a_n \neq 0$, and

(ii) $a_i < \frac{a_{i-1} + a_{i+1}}{2}$ for $i = 1, 2, \ldots, n - 1$.

Determine the largest number in the set $S$.

7. Let $ABC$ be a triangle. Let $P$ and $Q$ be points on the sides $AB$ and $AC$, respectively, such that $BC$ and $PQ$ are parallel. Let $D$ be a point inside triangle $APQ$. Let $E$ and $F$ be the intersections of $PQ$ with $BD$ and $CD$, respectively. Finally, let $O_E$ and $O_F$ be the circumcentres of triangle $DEQ$ and triangle $DFP$, respectively.

Prove that $O_E O_F$ is perpendicular to $AD$.

8. An $n \times n$ square is tiled with $1 \times 1$ tiles, some of which are coloured. Sally is allowed to colour in any uncoloured tile that shares edges with at least three coloured tiles. She discovers that by repeating this process all tiles will eventually be coloured.

Show that initially there must have been more than $\frac{n^2}{3}$ coloured tiles.

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