1. **Solution 1** (Leo Li, year 10, Christ Church Grammar School, WA.)

*Answer:* 19900.

We prove \( a_n = 0 + 1 + \cdots + (n - 1) \) by strong induction.

The base case \( n = 1 \) is given.

For the inductive part, assume \( a_n = 0 + 1 + \cdots + (n - 1) \) for \( n \leq k \).

We know
\[
a_{k+1} = \max_{i=1,\ldots,k} \{ i + a_i + a_{k+1-i} \}. \tag{1}
\]

We claim that the max in (1) occurs when \( i = k \). For this it is sufficient to prove that whenever \( i < k \), we have
\[
k + a_k + a_i > i + a_i + a_{k+1-i}.
\]

But since \( a_1 = 0 \) and \( k > i \), it suffices to prove that
\[
a_k - a_i \geq a_{k+1-i}. \tag{2}
\]

Using the inductive assumption, we have
\[
a_k - a_i = (0 + 1 + \cdots + (k - 1)) - (0 + 1 + \cdots + (i - 1))
\]
\[
= i + (i + 1) + \cdots + (k - 1) \tag{3}
\]

and
\[
a_{k+1-i} = 0 + 1 + \cdots + (k + 1 - i - 1)
\]
\[
= 1 + 2 \cdots + (k - i). \tag{4}
\]

Observe that the right hand sides of (3) and (4) both consist of the sum of \((n - k)\) consecutive integers and that the first term in (3) is at least as big as the first term in (4). This establishes the truth of (2), and hence also our claim.

Therefore, using our claim, we may substitute \( i = k \) in (1) to find
\[
a_{k+1} = k + 0 + 1 + \cdots + (k - 1) + 0 = 0 + 1 + \cdots + k.
\]

This completes the induction.

Finally, the well-known formula for summing the first so many consecutive positive integers may be used to deduce
\[
a_{200} = 1 + 2 + \cdots + 200 = \frac{199 \times 200}{2} = 19900,
\]
as required. \(\square\)
Solution 2 (Kevin Xian, year 10, James Ruse Agricultural High School, NSW.)

We prove \( a_n = \frac{n(n-1)}{2} \) by strong induction.

The base case \( a_1 = 0 \) is already given.

For the inductive part, assume \( a_n = \frac{n(n-1)}{2} \) for \( n \leq k \). For \( n = k + 1 \) we have

\[
a_{k+1} = \max_{i=1,\ldots,k} \left\{ i + a_i + a_{k+1-i} \right\}
\]

\[
= \max_{i=1,\ldots,k} \left\{ i + \frac{i(i-1)}{2} + \frac{(k+1-i)(k-i)}{2} \right\}
\]

\[
= \max_{i=1,\ldots,k} \left\{ i^2 - ki + \frac{k^2 + k}{2} \right\}
\]

\[
= \frac{k^2 + k}{2} + \max_{i=1,\ldots,k} \{i(i-k)\}.
\]

However, \( i(i-k) < 0 \) for \( 1 \leq i \leq k-1 \), while \( i(i-k) = 0 \) for \( i = k \). Therefore,

\[
\max_{i=1,\ldots,k} \{i(i-k)\} = 0,
\]

and so,

\[
a_{k+1} = \frac{k^2 + k}{2}
\]

\[
= \frac{(k+1)((k+1) - 1)}{2}.
\]

This completes the induction. Hence in particular,

\[
a_{200} = \frac{200 \times 199}{2} = 19900,
\]

as desired. □
Solution 3 (Jerry Mao, year 8, Caulfield Grammar School, VIC.)

We prove \( a_n = \frac{n(n-1)}{2} \) by strong induction.

The base case \( n = 1 \) is given.

For the inductive part, assume \( a_n = \frac{n(n-1)}{2} \) for \( n \leq k \). We know

\[
a_{k+1} = \max_{i=1,...,k} \{ i + a_i + a_{k+1-i} \}.
\]

We claim that the max in (1) occurs when \( i = k \).

Assume, for the sake of contradiction, that the max occurs for some integer \( i = r \) satisfying \( 1 \leq r \leq k-1 \).

Case 1. \( r < \frac{k}{2} \).

Consider \( s = k + 1 - i \). Note that \( \frac{k}{2} < s \leq k \). Furthermore,

\[
r + a_r + a_{k+1-r} < s + a_{k+1-r} + a_r = s + a_s + a_{k+1-s},
\]

in contradiction to the assumption that the max in (1) occurs at \( i = r \).

Case 2. \( \frac{k}{2} \leq r \leq k-1 \).

We claim that \( r + a_r + a_{k+1-r} < r + 1 + a_{r+1} + a_{k-(r+1)} \).

Using the inductive assumption we know

\[
r + a_r + a_{k+1-r} = r + \frac{r(r-1)}{2} + \frac{(k+1-r)(k-r)}{2}
\]

\[
= r^2 - rk + \frac{k^2}{2} + \frac{k}{2}
\]

and

\[
r + 1 + a_{r+1} + a_{k-(r+1)} = r + 1 + \frac{(r+1)r}{2} + \frac{(k-r)(k-r-1)}{2}
\]

\[
= r^2 - rk + \frac{k^2}{2} + \frac{k}{2} + 2r + 1 - k.
\]

So our claim is true because \( k \leq 2r \). Since the claim is true we have contradicted the assumption that the max in (1) occurs at \( i = r \).

Since cases 1 and 2 both end up in contradictions, the max must occur at \( i = k \). Substituting \( i = k \) in (1), using the inductive assumption and simplifying yields \( a_{k+1} = \frac{(k+1)k}{2} \). This completes the induction.

Thus we may conclude \( a_{200} = \frac{200 \times 199}{2} = 19900 \), as required. \( \square \)
2. **Solution 1** (Seyoon Ragavan, year 10, Knox Grammar School, NSW.)

Since $MC$ is tangent to circle $k$ at $C$, then by the power of a point theorem we have

$$MC^2 = MD \cdot MA.$$ 

Since $MB = MC$ it follows that

$$MB^2 = MD \cdot MA.$$ 

Hence considering the power of $M$ with respect to circle $ADB$, it follows that $MB$ is tangent to circle $ADB$ at $M$.

In the angle chase that follows, $AST$ is an abbreviation for the alternate segment theorem.

$$\angle BAC = \angle BAM + \angle MAC$$

$$= \angle MBD + \angle DAC \quad \text{(AST circle $ADB$)}$$

$$= \angle CBD + \angle BCD \quad \text{(AST circle $k$)}$$

$$= \angle CDE \quad \text{(exterior angle $\triangle BCD$)}$$

$$= \angle CAE, \quad \text{($AECD$ cyclic)}$$

which is the desired result. \qed
Solution 2 (Hannah Sheng, year 10, Rossmoyne Senior High School, WA.)

Refer to the diagram used in solution 1.

Since $BC$ is tangent to circle $k$ at $C$, we may apply the alternate segment theorem to deduce

$$\angle MAC = \angle DAC = \angle MCD. \quad (1)$$

Furthermore, since $\angle CMA = \angle CMD$, it follows by (AA) that

$$\triangle MCD \sim \triangle MAC.$$

Therefore,

$$\frac{MC}{MA} = \frac{MD}{MC}.$$ 

Since $MC = MB$, we have

$$\frac{MB}{MA} = \frac{MD}{MB}.$$ 

But now $\angle BMA = \angle BMD$, and so by (PAP) we have

$$\triangle MBD \sim \triangle MAB.$$ 

Hence

$$\angle BAM = \angle MBA = \angle CBA. \quad (2)$$

Finally, adding (1) and (2) together yields

$$\angle BAC = \angle MCD + \angle MBD = \angle CDE \quad \text{(exterior angle $\triangle BCD$)}$$

$$= \angle CAE, \quad \text{($AECD$ cyclic)}$$

as required. \qed

Comment Solutions 1 and 2 are essentially the same solution. This is because the similar triangles used in solution 2 are exactly the same similar triangles that are normally used to prove the power of a point theorem that was used in solution 1.
Solution 3 (Yang Song, year 11, James Ruse Agricultural High School, NSW.)

Refer to the diagram used in solution 1.

As in solution 2, we deduce

$$\triangle MBD \sim \triangle MAB.$$ 

It follows that

$$\angle ABC = \angle ABM$$
$$= \angle BDM \quad (\triangle MBD \sim \triangle MAB)$$
$$= \angle ADE$$
$$= \angle ACE. \quad (\text{AECD cyclic})$$

Since also $\angle ACE = \angle AEC$ from the alternate segment theorem applied to circle $k$, it follows by (AA) that

$$\triangle ABC \sim \triangle ACE.$$ 

From this we may immediately conclude that $\angle BAC = \angle CAE$.  □
Solution 4 (Matthew Sun, year 12, Penleigh and Essendon Grammar School, VIC.)

Let $N$ be the midpoint of $CE$.

Since $M$ is the midpoint of $BC$ and $N$ is the midpoint of $EC$, it follows that $MN \parallel BE$. Using this along with the fact that $AECD$ is cyclic, we find

\[ \angle CNM = \angle CEB = \angle CED = \angle CAD = \angle CAM, \]

from which it follows that $CMAN$ is cyclic.

Hence, $\angle CMA = \angle ENA$. By the alternate segment theorem we have $\angle ACM = \angle AEC = \angle AEN$. So by (AA) we have $\triangle ACM \sim \triangle AEN$.

One implication of this is

\[ \angle MAC = \angle NAE. \quad (3) \]

Another implication is

\[ \frac{AM}{AN} = \frac{MC}{NE} = \frac{MB}{NC}, \]

since $MC = MB$ and $NE = NC$. And we have $\angle AMB = \angle ANC$ due to $CMAN$ being cyclic. Thus by (PAP) we have $\triangle AMB \sim \triangle ANC$ and so

\[ \angle BAM = \angle CAN. \quad (4) \]

Adding together (3) and (4) yields the required result. \qed
Solution 5 (Richard Gong, year 9, Sydney Grammar School, NSW.)

Let $D'$ be the point so that $BDCD'$ is a parallelogram. Since the diagonals of a parallelogram bisect each other and $M$ is the midpoint of $BC$, it follows that $M$ is the midpoint of $DD'$. Therefore, $D'$ is collinear with $D$ and $M$.

\[
\triangle BAC \sim \triangle D'CD \text{ by (AA)}.
\]

It follows that
\[
\angle BD'A = \angle D'DC \quad (BD' \parallel DC)
\]
\[
= \angle AEC \quad (AECD \text{ cyclic})
\]
\[
= \angle BCA, \quad \text{(alternate segment theorem)}
\]

and so $ABD'C$ is cyclic.

Therefore, $\angle ABC = \angle AD'C = \angle D'D'C$. Part of the above angle chase yielded $\angle D'DC = \angle BCA$. Hence $\triangle BAC \sim \triangle D'CD$ by (AA).

Therefore,
\[
\angle BAC = \angle D'CD
\]
\[
= \angle CDE \quad (D'C \parallel BD)
\]
\[
= \angle CAE, \quad (AECD \text{ cyclic})
\]

as required. \qed
Solution 6 (Allen Lu, year 11, Sydney Grammar School, NSW.)

Let lines $CD$ and $AB$ intersect at point $X$, and let lines $BE$ and $AC$ intersect at point $Y$.

Since $AM$, $BY$ and $CX$ we may apply Ceva’s theorem to find

\[
\frac{BM}{MC} \cdot \frac{CY}{YA} \cdot \frac{AX}{XB} = 1
\]

\[
\Rightarrow \quad \frac{AX}{XB} = \frac{AY}{YC} \quad \text{(since } BM = MC). \]

It follows that $XY \parallel BC$.

Therefore,

\[
\angle DXY = \angle DCB \quad \text{(} XY \parallel BC) = \angle DAC, \quad \text{(alternate segment theorem)}
\]

from which it follows that $AXDY$ is cyclic.

Therefore,

\[
\angle BAC = \angle EDC \quad \text{(} AXYD \text{ cyclic)} = \angle CAE, \quad \text{(} AECD \text{ cyclic)}
\]

as desired. $\square$

Comment Solutions 5 and 6 are quite similar underneath the surface. See if you can find the connection!
3. **Solution** (Jerry Mao, year 8, Caulfield Grammar School, VIC.)

*Answer:* $e = 6$.

Since the edge labels only depend on the difference of the vertex labels we can assume without loss of generality that vertex with minimal label is labeled with the number 1. The first diagram below shows a graph of the dodecahedron along with a numbering for which $e = 6$.

In the second diagram below, the vertices marked with $A$, $B$ and $C$, require at least 1, 2 and 3 edges, respectively, to reach them from the vertex labeled 1.

If we assume $e \leq 5$, then each $A$-vertex has label at most 6, each $B$-vertex has label at most 11 and each $C$-vertex has label at most 16. Thus all of the 16 vertices including the one labeled with 1 and the 15 marked with $A$, $B$ or $C$, must be labelled with different positive integers less than or equal to 16. Therefore, they have exactly the labels 1, 2, ..., 16 in some order.

Consequently, the four unmarked vertices have labels at least equal to 17. But all six $C$-vertices are adjacent to one of these unmarked vertices. Since $e \leq 5$, the labels of the $C$-vertices must all be at least $17 - e \geq 12$. Thus the six $C$-vertices have labels lying in the range 12 to 16. This is clearly impossible by the pigeonhole principle because all the labels are different. Hence $e \geq 6$. \[\square\]
4. **Solution 1** (Michael Cherryh, year 11, Gungahlin College, ACT.)

*Answer:* \( f(n) = 0 \) for all integers \( n \geq 2 \).

Suppose \( n = 2k \) is even. Then \( f(n) = f(2)f(k) = 0 \).

Our strategy will be to prove that for each odd integer \( n > 1 \), there exists a positive integer \( k \) such that \( n^k \) begins with a 2. Then since \( f(n)^k = f(n^k) = 0 \) it will follow that \( f(n) = 0 \).

By the pigeonhole principle there exist two integers \( a > b \) such that \( n^a \) and \( n^b \) have the same first two digits, which we shall denote by \( x \) and \( y \). Let us write \( n^a \) and \( n^b \) in scientific notation. That is,

\[
    n^a = x.\text{ya}_2a_3 \ldots \times 10^k \quad \text{and} \quad n^b = x.\text{yb}_2b_3 \ldots \times 10^\ell,
\]

for some non-negative integers \( k \geq \ell \). Then

\[
    n^{a-b} = \frac{x.\text{ya}_2a_3 \ldots}{x.\text{yb}_2b_3 \ldots} \times 10^{k-\ell} = r \times 10^{k-\ell}.
\]

Note that \( r \neq 1 \) because \( n \) is odd and \( a > b \).

If we can find a positive integer \( k \) such that \( r^k \) starts with a 2 when written in scientific notation, then we will be done because \( n^{k(a-b)} \) will also start with a 2.

Our idea is as follows. If \( r > 1 \), then the sequence \( r, r^2, r^3, \ldots \) grows arbitrarily large. But since \( r \) is close to 1 we cannot jump from being less than 2 to at least 3. Thus there is a power of \( r \) that lies between 2 and 3. Similarly, if \( r < 1 \), then the sequence \( r, r^2, r^3, \ldots \) converges to 0. But since \( r \) is close to 1 we cannot jump from being at least 0.3 to less than 0.2 and so there is a power of \( r \) lying between 0.2 and 0.3.

If \( r > 1 \), then

\[
    1 < r < \frac{x.y + 0.1}{x.y} = 1 + \frac{0.1}{x.y} \leq 1 + 0.1 = 1.1
\]

Consider the least positive integer \( k \) such that \( r^k \geq 2 \). Then we have \( r^{k-1} < 2 \). Thus \( r^k < 2 \times 1 = 2 \), and so \( r^k \) starts with a 2.

If \( r < 1 \), then

\[
    1 > r > \frac{x.y}{x.y + 0.1} = 1 - \frac{0.1}{x.y + 0.1} \geq 1 - \frac{0.1}{1.1} > 0.9.
\]

Consider the least positive integer \( k \) such that \( r^k < 0.3 \). Then we have \( r^{k-1} \geq 0.3 \). Thus \( r^k > 0.3 \times 0.9 = 0.27 \), and so \( r^k \) starts with a 2.

In both cases we have shown that the required \( k \) exists. \( \square \)
Solution 2 (Mel Shu, year 12, Melbourne Grammar School, VIC.)

We shall prove that $f(n) = 0$ for all integers $n \geq 2$. Since $f$ is completely multiplicative it suffices to show that $f(p) = 0$ for all primes $p$. Since $f(p^k) = f(p)^k$ for any positive integer $k$ it is enough to prove that any prime has a power that contains the digit 2. In fact we shall prove that any prime has a power whose first digit is 2.

Let $p$ be any prime. We seek integers $i > 0$ and $j \geq 0$ such that

$$2 \cdot 10^j \leq p^i < 3 \cdot 10^j$$

where the logarithm is to base 10. It would be a good idea to estimate the sizes of $\log 2$ and $\log 3$. Indeed since $2^9 < 10^3$ and $3^9 > 10^4$ we have $\log 2 < \frac{3}{5}$ and $\log 3 > \frac{4}{5}$. Hence it suffices to find $i$ and $j$ satisfying

$$\frac{3}{5} < \{i\alpha\} < \frac{4}{5}, \quad (\ast)$$

where $\alpha = \log p$, and $\{i\alpha\}$ denotes the fractional part of $i\alpha$.

We claim that $\alpha$, and consequently also $\{i\alpha\}$, are irrational. Indeed, if $\alpha = \frac{a}{b}$ for $a, b \in \mathbb{N}^+$, then $p^b = 10^a$. But then $p^b$ would be divisible by both 2 and 5. This is impossible and so $\alpha \notin \mathbb{Q}$ as claimed.

Consider the ten irrational numbers $\{\alpha\}, \{2\alpha\}, \ldots, \{10\alpha\}$ and the nine open intervals $(0, \frac{1}{5}), (\frac{1}{5}, \frac{2}{5}), \ldots, (\frac{8}{5}, 1)$. By the pigeonhole principle at least one of these intervals contains at least two of the ten values. Suppose that $\{k\alpha\}$ and $\{\ell\alpha\}$, where $1 \leq \ell < k \leq 10$, both lie within one of these nine intervals. Let $\beta = \{k\alpha\} - \{\ell\alpha\}$. Note that $|\beta| < \frac{1}{5}$ and that $\beta \neq 0$, because $\{(k - \ell)\alpha\}$ is irrational.

Case 1: $\beta > 0$.

By adding $\beta$ to itself enough times, we see that there is a positive integer $m$ such that $\frac{3}{5} < m\beta < \frac{4}{5}$. This is because we cannot jump the interval $(\frac{3}{5}, \frac{4}{5})$ just by adding $\beta$.

Case 2: $\beta < 0$.

By adding $\beta$ to itself enough times, we see that there is a positive integer $m$ such that $\frac{3}{5} - 1 < m\beta < \frac{4}{5} - 1$.

In both cases 1 and 2, we can satisfy $\ast$ by taking $i = (k - \ell)m$. This concludes the proof. \qed
Solution 3 (Angelo Di Pasquale, AMOC Senior Problems Committee.)

As in solution 2, it is sufficient to prove that any prime $p$ has a power that contains the digit 2.

We verify directly that $2 = 2^1$ and $5^2 = 25$.

Consider any other prime $p$. It has last digit 1, 3, 7 or 9. Since $1^4 \equiv 3^4 \equiv 7^4 \equiv 9^4 \equiv 1 \pmod{10}$, we have $p^4 = 10x + 1$ for some positive integer $x$. It follows that $p^8 = 100x^2 + 20x + 1$. Hence the last two digits of $p^8$ are 01, 21, 41, 61 or 81.

One can easily check that $41 \rightarrow 81 \rightarrow 61 \rightarrow 21 \pmod{100}$ upon repeated squaring. This shows that if $p^8$ does not end in 01, then $p$ has a power that contains the digit 2 in its second last position.

**Lemma.** Let $m > 1$ be any integer that ends in 01. Let $k \geq 3$ be the integer such that the last $k$ digits of $m$ are $x0\ldots01$ where $x \neq 0$ and there are $k - 2$ zeros. If $x \neq 5$, then there exists a power of $m$ that contains a 2.

**Proof.** The last $k$ digits of $m^2$ are $y0\ldots01$ where $y \equiv 2x \pmod{10}$. Since $x \neq 0, 5$, it follows that $y$ is a nonzero even digit. One may check that the last $k$ digits go as

$$40\ldots01 \rightarrow 80\ldots01 \rightarrow 60\ldots01 \rightarrow 20\ldots01,$$

upon repeated squaring. This establishes that some power of $m$ will contain the digit 2 in the $k$th last position. \hfill \Box

The lemma solves the problem unless the last $k$ digits of $p^8$ are 50\ldots01. In such a case let $w$ be the next digit to the left of the digit 5. Thus the last $k + 1$ digits of $p^8$ are $w50\ldots01$.

If $w = 2$, we are finished.

If $w \neq 2$, we square again and note that the last $k + 1$ digits of $p^{16}$ are $z0\ldots01$ where $z \equiv 2w + 1 \pmod{10}$. Note that $z \neq 0$. If $z \neq 5$, we may use the lemma to solve the problem. If $z = 5$, this corresponds to $w = 2$ or 7. Since $w \neq 2$ we have $w = 7$.

We are now left with the case of when the last $k + 1$ digits of $p^8$ are 750\ldots01. Since $k \geq 3$ there is at least one zero among the last $k + 1$ digits. Cubing 750\ldots01, we find that the last $k + 1$ digits of $p^{24}$ are 250\ldots01. This has a 2 in the $(k + 1)$th place from the right and therefore solves the problem. \hfill \Box
Solution 4 (Alexander Gunning, year 11, Glen Waverley Secondary College, VIC.)

Clearly \( f(2) = 0 \). Also \( f(5)^2 = f(25) = 0 \), which implies \( f(5) = 0 \). Thus if \( n \) is any positive integer that is divisible by 2 or 5, then \( f(n) = 0 \). From here on we assume that \( n \) is not divisible by 2 or 5.

By Fermat’s little theorem we have \( 5^m \parallel n^4 - 1 \). Let \( 5^m \parallel n^4 - 1 \).

Lemma 1. If \( 5^m \parallel n^4 - 1 \), then also \( 5^m \parallel n^{4a} - 1 \) for \( a \in \mathbb{N}^+ \) and \( 5 \nmid a \).

Proof. Write \( n^4 = 5^m k + 1 \) where \( 5 \nmid k \). Then

\[
\begin{align*}
    n^{4a} - 1 &= (5^m k + 1)^a - 1 \\
    &= \sum_{i=1}^{a} \binom{a}{i} (5^m k)^i \\
    &\equiv 5^m ka \pmod{5^m + 1}.
\end{align*}
\]

Since \( 5^m k a \) is divisible by \( 5^m \) but not \( 5^m + 1 \), the lemma is proven. \( \square \)

Lemma 2. For any positive integer \( b \) we have \( n^{2^m b} \equiv 1 \pmod{2^{m+1}} \).

Proof. An application of Euler’s theorem yields \( n^{2^m} \equiv 1 \pmod{2^{m+1}} \). The lemma follows once we raise both sides to the power of \( b \). \( \square \)

Let \( d = \max\{2^m, 4\} = \text{lcm}\{2^m, 4\} \). Then the two lemmas tell us that \( n^d \equiv 1 \pmod{2^{m+1}} \) and \( 5^m \parallel n^d - 1 \). Thus we may write

\[
n^d = 5^m 2^{m+1} c + 1 = 2 c \cdot 10^m + 1,
\]

where \( 5 \nmid c \). Then for any integer \( e \geq 2 \) we have

\[
n^{de} = (2 c \cdot 10^m + 1)^e
\begin{align*}
    &= 1 + \binom{e}{1} 2 c \cdot 10^m + \binom{e}{2} (2 c \cdot 10^m)^2 + \cdots \\
    &\equiv 2 ce \cdot 10^m + 1 \pmod{10^{m+1}}.
\end{align*}
\]

Since \( \gcd(c, 5) = 1 \), we may choose \( e \) so that \( ce \equiv 1 \pmod{5} \). This implies that \( 2 ce \equiv 2 \pmod{10} \) and so we have

\[
n^{de} \equiv 2 \cdot 10^m + 1 \pmod{10^{m+1}}.
\]

This contains the digit 2 in the \((m+1)\)th place from the right. Thus \( f(n)^{de} = f(n^{de}) = 0 \) and so \( f(n) = 0 \). \( \square \)

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1For a prime \( p \) and an integer \( k \), the notation \( p^m \parallel k \) means that \( p^m \mid k \) but \( p^{m+1} \nmid k \).
5. **Solution** (Kevin Xian, year 10, James Ruse Agricultural High School, NSW.)

*Answer: \( x = -\frac{2014}{45} \).

The given equation may be rewritten as

\[
x - \lfloor x \rfloor = 2014 \left( \frac{1}{\lfloor x \rfloor} - \frac{1}{x} \right) = \frac{2014(x - \lfloor x \rfloor)}{x \lfloor x \rfloor}.
\]

Since \( x \) is not an integer it follows that \( x \neq \lfloor x \rfloor \). Hence we may divide both sides by \( x - \lfloor x \rfloor \) and rearrange to find

\[ x \lfloor x \rfloor = 2014. \quad (1) \]

**Case 1.** \( \lfloor x \rfloor \geq 45 \).

Then \( x > 45 \), and so \( x \lfloor x \rfloor > 45^2 = 2025 > 2014 \).

**Case 2.** \( -44 \leq \lfloor x \rfloor \leq 44 \).

Then \( -44 < x < 45 \), and so \( x \lfloor x \rfloor < 44 \times 45 = 1980 < 2014 \).

**Case 3.** \( \lfloor x \rfloor \leq -46 \).

Then \( x < -45 \), and so \( x \lfloor x \rfloor > 45 \times 46 = 2070 > 2014 \).

**Case 4.** \( \lfloor x \rfloor = -45 \).

Then from (1) we derive \( x = -\frac{2014}{45} = -44\frac{34}{45} \).

Checking this in the original equation we have

\[
\text{LHS} = x + \frac{2014}{x} = -\frac{2014}{45} + \frac{2014}{-\frac{2014}{45}} = -\frac{2014}{45} - 45
\]

and

\[
\text{RHS} = \lfloor x \rfloor + \frac{2014}{\lfloor x \rfloor} = -45 + \frac{2014}{-45} = \text{LHS},
\]

as required. \( \square \)
6. **Solution 1** (Seyoon Ragavan, year 10, Knox Grammar School, NSW.)

*Answer: 96433469.*

It is straightforward to verify that 96433469 is in $S$. Assume that $S$ contains a number $N > 96433469$. If $N$ has nine or more digits, let the first nine such digits in order from the left be $a, b, c, d, e, f, g, h, i$. Condition (ii) implies that

$$2b < a + c$$

$$\Rightarrow \quad a > 2b - c$$

$$\Rightarrow \quad a \geq 2b - c + 1, \quad (1)$$

since $a, b, c$ are all integers. Similarly, we deduce the following.

$$b \geq 2c - d + 1 \quad (2)$$

$$c \geq 2d - e + 1 \quad (3)$$

$$d \geq 2e - f + 1 \quad (4)$$

$$e \geq 2f - g + 1 \quad (5)$$

$$f \geq 2g - h + 1 \quad (6)$$

$$g \geq 2h - i + 1 \quad (7)$$

Since $a \leq 9$, we may use successive substitution to find the following.

$$a \geq 2b - c + 1 \quad \text{(using (1))}$$

$$\Rightarrow \quad 8 \geq 2b - c$$

$$\geq 2(2c - d + 1) - c + 1 \quad \text{(using (2))}$$

$$\Rightarrow \quad 6 \geq 3c - 2d$$

$$\geq 3(2d - e + 1) - 2d \quad \text{(using (3))}$$

$$\Rightarrow \quad 3 \geq 4d - 3e$$

$$\geq 4(2e - f + 1) - 3e \quad \text{(using (4))}$$

$$\Rightarrow \quad -1 \geq 5e - 4f$$

$$\geq 5(2f - g + 1) - 4f \quad \text{(using (5))}$$

$$\Rightarrow \quad -6 \geq 6f - 5g$$

$$\geq 6(2g - h + 1) - 5g \quad \text{(using (6))}$$

$$\Rightarrow \quad -12 \geq 7g - 6h$$

$$\geq 7(2h - i + 1) - 6h \quad \text{(using (7))}$$

$$\Rightarrow \quad -19 \geq 8h - 7i \quad \text{(7')}$$
Concentrating on (4') we have $-1 \geq 5e - 4f \geq -4f$. Hence $f \geq 1$.
Substituting successively into (5'), (6') and (7') yields
\[
\begin{align*}
-6 &\geq 6f - 5g \geq 6 - 5g \quad \Rightarrow \quad g \geq 3 \\
-12 &\leq 7g - 6h \geq 21 - 6h \quad \Rightarrow \quad h \geq 6 \\
-19 &\leq 8h - 7i \geq 48 - 7i \quad \Rightarrow \quad i \geq 10.
\end{align*}
\]
However, this is in contradiction with $i$ being a digit. Therefore, $N$ cannot contain more than eight digits.

We are left to deal with the case where $N$ is an eight-digit number. Let the digits of $N$ in order from the left be $a, b, c, d, e, f, g, h$. We deduce inequalities (1)-(6) and (1')-(6') on the previous page in the same way as we did earlier.

Since $h$ is a digit we know that $h \leq 9$. If we substitute successively into (6'), (5'), (4'), (3'), (2') and (1'), we find the following.
\[
\begin{align*}
-12 &\geq 7g - 6h \quad \Rightarrow \quad 7g \leq 6h - 12 \leq 6 \times 9 - 12 = 42 \quad \Rightarrow \quad g \leq 6 \\
-6 &\geq 6f - 5g \quad \Rightarrow \quad 6f \leq 5g - 6 \leq 5 \times 6 - 6 = 24 \quad \Rightarrow \quad f \leq 4 \\
-1 &\geq 5e - 4f \quad \Rightarrow \quad 5e \leq 4f - 1 \leq 4 \times 4 - 1 = 15 \quad \Rightarrow \quad e \leq 3 \\
3 &\geq 4d - 3e \quad \Rightarrow \quad 4d \leq 3e + 3 \leq 3 \times 3 + 3 = 12 \quad \Rightarrow \quad d \leq 3 \\
6 &\geq 3c - 2d \quad \Rightarrow \quad 3c \leq 2d + 6 \leq 2 \times 3 + 6 = 12 \quad \Rightarrow \quad c \leq 4 \\
8 &\geq 2b - c \quad \Rightarrow \quad 2b \leq c + 8 \leq 4 + 8 = 12 \quad \Rightarrow \quad b \leq 6
\end{align*}
\]
We also know that $a \leq 9$ because $a$ is a digit.

Therefore, each digit of $N$ is less than or equal to the corresponding digit of 96433469. It follows that $N \leq 96433469$. This contradicts that $N > 96433469$. Hence 96433460 is the largest number in $S$. □
Solution 2 (Found independently by Norman Do and Ivan Guo, AMOC Senior Problems Committee.)

Consider the differences $b_i = a_{i+1} - a_i$ for $i = 0, 1, 2, \ldots$. Condition (ii) is equivalent to $b_0, b_1, b_2, \ldots$ being a strictly increasing sequence.

Lemma. At most three $b_i$ are strictly positive and at most three $b_i$ are strictly negative.

Proof. Suppose there are four $b_i$ that are strictly positive. If $b_s$ is the smallest such $b_i$, then we have $b_s \geq 1$, $b_{s+1} \geq 2$, $b_{s+2} \geq 3$ and $b_{s+3} \geq 4$. Therefore,

\[ a_{s+4} - a_s = b_s + b_{s+1} + b_{s+2} + b_{s+3} \geq 1 + 2 + 3 + 4 = 10. \]

However, this is a contradiction because $a_{r+4}$ and $a_r$ are single digits and hence differ by at most 9. A similar argument shows that no four $b_i$ are strictly negative. \qed

It follows from the lemma that $n \leq 7$. If $n = 7$, then we must have $b_0 < b_1 < b_2 < 0$, $b_3 = 0$ and $0 < b_4 < b_5 < b_6$. Since the $b_i$ are distinct integers this implies that $b_0 \leq -3$, $b_1 \leq -2$ and $b_2 \leq -1$. Hence we have the following.

\[ a_0 \leq 9 \]
\[ a_1 = a_0 + b_0 \leq 9 - 3 = 6 \]
\[ a_2 = a_1 + b_1 \leq 6 - 2 = 4 \]
\[ a_3 = a_2 + b_2 \leq 4 - 1 = 3 \]

Similarly, coming from the other end we have $b_6 \geq 3$, $b_5 \geq 2$ and $b_4 \geq 1$. Hence we have following.

\[ a_7 \leq 9 \]
\[ a_6 = a_7 - b_6 \leq 9 - 3 = 6 \]
\[ a_5 = a_6 + b_5 \leq 6 - 2 = 4 \]
\[ a_4 = a_5 + b_4 \leq 4 - 1 = 3 \]

It follows that no number in $S$ exceeds 96433469. Since it is readily verified that 96433469 is in $S$, it is the largest number in $S$. \qed
7. The common chord of two intersecting circles is always perpendicular to the line joining their centres. All the solutions we present reduce the matter to proving that \( A \) lies on the common chord of circles \( DEQ \) and \( DFP \). That is, \( A \) is on the radical axis of that pair of circles.

**Solution 1** (Mel Shu, year 12, Melbourne Grammar School, VIC.)

Let the line through \( A \) and \( D \) intersect \( PQ \) at \( K \) and \( BC \) at \( L \).

The parallel lines imply \( \triangle DKE \sim \triangle DLB \) and \( \triangle DKF \sim \triangle DLC \). Therefore,

\[
\frac{KE}{LB} = \frac{DK}{DL} = \frac{KF}{LC} \Rightarrow \frac{KE}{KF} = \frac{LB}{LC}. \tag{1}
\]

We also have \( \triangle AKP \sim \triangle ALB \) and \( \triangle AKQ \sim \triangle ALC \). Therefore,

\[
\frac{KP}{LB} = \frac{AK}{AL} = \frac{KQ}{LC} \Rightarrow \frac{KP}{KQ} = \frac{LB}{LC}. \tag{2}
\]

Comparing (1) and (2) we find

\[
\frac{KE}{KF} = \frac{KP}{KQ} \Rightarrow KE \cdot KF = KQ \cdot KP.
\]

Thus \( K \) has equal power with respect to circles \( DEQ \) and \( DFP \) and so the line \( ADK \) is the radical axis of the two circles. \( \Box \)
Solution 2 (Alexander Gunning, year 11, Glen Waverley Secondary College, VIC.)

Refer to the diagram in solution 1.

The parallel lines imply \( \triangle APQ \sim \triangle ABC \). Thus

\[
\frac{AP}{AB} = \frac{AQ}{AC} \\
\Rightarrow 1 - \frac{AP}{AB} = 1 - \frac{AQ}{AC} \\
\Rightarrow \frac{BP}{AB} = \frac{CQ}{AC}.
\] (3)

Applying Menelaus’ theorem to triangle \( APK \) with transversal \( DEB \) and then again to triangle \( AQK \) with transversal \( DFC \) we have

\[
\frac{KD}{DA} \cdot \frac{AB}{BP} \cdot \frac{PE}{EK} = -1 = \frac{KD}{DA} \cdot \frac{AC}{CQ} \cdot \frac{QF}{FK}.
\]

Using (3) we can cancel most of this down to derive

\[
\frac{PE}{EK} = \frac{QF}{FK} \\
\Rightarrow 1 + \frac{PE}{EK} = 1 + \frac{QF}{FK} \\
\Rightarrow \frac{PK}{EK} = \frac{QK}{FK} \\
\Rightarrow EK \cdot QK = FK \cdot PK.
\]

Thus \( K \) has equal power with respect to circles \( DEQ \) and \( DFP \) and so the line \( ADK \) is the radical axis of the two circles. \( \square \)
Solution 3 (Seyoon Ragavan, year 10, Knox Grammar School, NSW.)

Let circles $DFP$ and $DEQ$ intersect for a second time at point $D'$. Let circle $DFP$ intersect line $AB$ for the second time at point $X$ and let circle $DEQ$ intersect line $AC$ for the second time at point $Y$.

Then

$$\angle DXA = DFP \quad (\text{DXPF cyclic})$$
$$= \angle DCB \quad (PQ \parallel BC).$$

Hence $DXBC$ is cyclic and so $X$ lies on circle $DBC$. Similarly, $Y$ lies on circle $DBC$. Thus $DXBCY$ is a cyclic pentagon.

In particular, $XBCY$ is cyclic. From this we have

$$\angle AYX = \angle ABC \quad (XBCY \text{ cyclic})$$
$$= \angle APQ \quad (PQ \parallel BC).$$

Therefore, $XPQY$ is cyclic.

Applying the radical axis theorem to the circles $DFPX$, $DEQY$ and $XPQY$ we have that $PQ$, $QY$ and $DD'$ are concurrent. Since $PX$ and $QY$ intersect at $A$, we conclude that $A$ lies on the line $DD'$, as required. □
8. **Solution 1** (Jeremy Yip, year 11, Trinity Grammar School, VIC.)

Assume at the beginning that $k$ tiles are coloured and $n^2 - k$ tiles are uncoloured. Then the perimeter $P$ of the coloured tiles is at most $4k$. (An edge counts towards the perimeter if it is adjacent to a coloured and an uncoloured tile.)

Every tile Sally colours in reduces the perimeter by 2 or 4 according to whether the newly coloured tile is adjacent to three or four coloured tiles. Therefore, when all the tiles have been coloured, $P$ has been reduced by at least $2(n^2 - k)$. Thus the final perimeter $P_{\text{end}}$ satisfies

$$P_{\text{end}} \leq 4k - 2(n^2 - k) = 6k - 2n^2.$$

However, if $k \leq \frac{n^2}{3}$, then $P_{\text{end}} \leq 0$. This is a contradiction because $P_{\text{end}} = 4n$. $\square$

**Comment** A careful reading of this solution reveals the stronger result

$$k \geq \frac{n^2 + 2n}{3}.$$
Solution 2 (George Han, year 12, Westlake Boys’ High School, NZ.)

Let there be initially \( k \) coloured tiles and \( n^2 - k \) uncoloured tiles. We start giving money to uncoloured tiles as follows.

(i) For each of the \( k \) coloured tiles we give $1 to each of its uncoloured neighbours.

(ii) If an uncoloured tile amasses $3, we colour it in and give $1 to each of its uncoloured neighbours.

If all the tiles are eventually coloured, then all of the \( n^2 - k \) tiles, which were originally uncoloured, now each have at least $3 in them. Thus \( D \geq 3(n^2 - k) \) where \( D \) is the total amount of dollars at the end.

All dollars in the array come from (i) and (ii). The amount of dollars coming from (i) is at most \( 4k \). The amount of dollars coming from (ii) is at most \( n^2 - k \). Thus \( D \leq 4k + n^2 - k \).

Combining the two inequalities for \( D \) we deduce

\[
4k + n^2 - k \geq 3(n^2 - k)
\]

\[
\Rightarrow \quad k \geq \frac{n^2}{3}.
\]

However, since a corner tile has only two neighbours, at least one of the inequalities for \( D \) is strict. Thus the final inequality is strict. \( \square \)

\(^2\)Equivalent to year 11 in Australia.
Solution 3 (Alexander Babidge, year 12, Sydney Grammar School, NSW.)

It is convenient for us to use some biology language in this solution. Coloured tiles correspond to organisms, which we shall call squarelings. Each unit square of the $n \times n$ array may be occupied by at most one squareling. Furthermore, each squareling has one unit of genes. If $k$ squarelings ($k = 3$ or $4$) are adjacent to a vacant square, they produce a child in the vacant square. The $k$ squarelings are then said to be parents of the child. The child is also a squareling with one unit of genes made up of $\frac{1}{k}$ of a unit of genes from each of its parents.

Each square is adjacent to at most four other squares. Hence at the beginning, before any children are produced, each squareling, which we shall call a founder, has the potential to be a parent to at most four children. Since each parent contributes at most one third of its genes to any child, the total direct gene contribution from any such founder is at most $\frac{4}{3}$.

Consider any squareling that is not a founder. At least three of its neighbouring squares are occupied by its parents. Hence such a squareling has the potential to be the parent of at most one child. Thus the total direct gene contribution from this squareling is at most $\frac{1}{3}$.

Now children can also become parents to other children, but they only pass on genes from their parents. Thus the total gene count from any given founder is at most $1$ from itself, $\frac{4}{3}$ from its children, $\frac{1}{3} \cdot \frac{4}{3}$ from its children’s children, and so on. If the number of generations is $g$, then by summing the geometric series, the total gene count from any given founder is at most

$$1 + \frac{4}{3} + \frac{4}{9} + \cdots + \frac{4}{3^{g-1}} = 1 + \frac{4}{3} \left( \frac{1 - \frac{1}{3^g}}{1 - \frac{1}{3}} \right) < 1 + \frac{4}{3} \left( \frac{1}{1 - \frac{1}{3}} \right) = \frac{3}{3} = 3.$$

If the total number of founders is at most $\frac{n^2}{3}$, then the total gene contribution from these founders is less than $n^2$, which means that not every square of the array has a squareling in it. This contradiction concludes the proof. □
# Australian Mathematical Olympiad Statistics

## Score Distribution/Problem

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