MATHEMATICS CONTESTS
THE AUSTRALIAN SCENE 2016

PART 1: MATHEMATICS CHALLENGE FOR YOUNG AUSTRALIANS

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Australian Mathematical Olympiad Committee

A department of the Australian Mathematics Trust

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The Australian Mathematical Olympiad Committee (AMOC) also acknowledges the significant financial support it has received from the Australian Government towards the training of our Olympiad candidates and the participation of our team at the International Mathematical Olympiad (IMO).

The views expressed here are those of the authors and do not necessarily represent the views of the government.

Special thanks

With special thanks to the Australian Mathematical Society, the Australian Association of Mathematics Teachers and all those schools, societies, families and friends who have contributed to the expense of sending the 2016 IMO team to Hong Kong.

ACKNOWLEDGEMENTS

The Australian Mathematical Olympiad Committee (AMOC) sincerely thanks all sponsors, teachers, mathematicians and others who have contributed in one way or another to the continued success of its activities. The editors sincerely thank those who have assisted in the compilation of this book, in particular the students who have provided solutions to the 2016 IMO. Thanks also to members of AMOC and Challenge Problems Committees, Adjunct Professor Mike Clapper, staff of the Australian Mathematics Trust and others who are acknowledged elsewhere in the book.

PREFACE

This year has seen some remarkable progress in the Mathematics Challenge for Young Australians (MCYA) program. We had a record number of entries for the Challenge stage, and the Enrichment stage entries were also very strong, with just over 4300 entries. The new *Ramanujan* book was well received as was the revised *Polya* book. A number of schools have made a commitment to use Challenge and/or Enrichment with whole year levels and are reporting back very positively on the effect of this on problem-solving capacity in their students. The rise in Australian Intermediate Mathematics Olympiad (AIMO) entries continues, with 1829 entries this year, including over 400 from Vietnam. We have more than doubled our numbers in this competition over the last three years. Fifteen students obtained perfect scores, including five Australian students.

In the Olympiad program, the 2016 Australian Mathematical Olympiad (AMO) proved fairly demanding, with just three perfect scores. The International Mathematical Olympiad (IMO) also proved quite challenging for Australia, after three very strong years in a team led by Alex Gunning. Commendably, this year's team all obtained medals (two Silver and four Bronze) but we could not quite maintain our position of the last few years, finishing 25th out of 109 teams. Two of the team members became dual Olympians, while Seyoon Ragavan notched up his 4th IMO. All of the team were Year 12 students, so there will be a challenge ahead to build a new team for 2017. In the Mathematics Ashes we performed above expectations, but still lost in a close contest, to a very strong and experienced British team.

The start for selection of the 2017 team begins with the AMOC Senior contest, held in August, and it was encouraging to see five perfect scores in this competition, including one from Hadyn Tang, a year 7 student from Trinity Grammar School in Melbourne. Director of Training and IMO Team Leader Angelo Di Pasquale, along with Deputy Team Leader Andrew Elvey Price and a dedicated team of tutors, continue to innovate and have made some changes to the structure of the December School of Excellence.

I would particularly wish to thank all the dedicated volunteers without whom this program would not exist. These include the Director of Training and the ex-Olympians who train the students at camps, the various state directors, the Challenge Director, Dr Kevin McAvaney and the various members of his Problems Committee which develop such original problems each year and Dr Norm Do, and his senior problems committee (who do likewise).

Our support from the Australian Government for the AMOC program continues, though from June this year, this is provided through the Department of Science and Industry, rather than the Department of Education. We are most grateful for this support.

Once again, the Australian Scene is produced in electronic form only. Whilst the whole book can be downloaded as a pdf, it is available on our website in two sections, one containing the MCYA reports and papers and the other containing the Olympiad reports and papers.

Mike Clapper
December 2016

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MATHEMATICS CHALLENGE FOR YOUNG AUSTRALIANS

The Mathematics Challenge for Young Australians (MCYA) started on a national scale in 1992. It was set up to cater for the needs of the top 10 percent of secondary students in Years 7–10, especially in country schools and schools where the number of students may be quite small. Teachers with a handful of talented students spread over a number of classes and working in isolation can find it very difficult to cater for the needs of these students. The MCYA provides materials and an organised structure designed to enable teachers to help talented students reach their potential. At the same time, teachers in larger schools, where there are more of these students, are able to use the materials to better assist the students in their care.

The aims of the Mathematics Challenge for Young Australians include:

encouraging and fostering

– a greater interest in and awareness of the power of mathematics

– a desire to succeed in solving interesting mathematical problems

– the discovery of the joy of solving problems in mathematics

identifying talented young Australians, recognising their achievements nationally and providing support that will enable them to reach their own levels of excellence

providing teachers with

– interesting and accessible problems and solutions as well as detailed and motivating teaching discussion and extension materials

– comprehensive Australia-wide statistics of students' achievements in the Challenge.

There are three independent stages in the Mathematics Challenge for Young Australians:

- Challenge (three weeks during the period March–June)
- Enrichment (April–September)
- Australian Intermediate Mathematics Olympiad (September).

Challenge

Challenge consists of four levels. Middle Primary (Years 3–4) and Upper Primary (Years 5–6) present students with four problems each to be attempted over three weeks, students are allowed to work on the problems in groups of up to three participants, but each must write their solutions individually. The Junior (Years 7–8) and Intermediate (Years 9–10) levels present students with six problems to be attempted over three weeks, students are allowed to work on the problems with a partner but each must write their solutions individually.

There were 13461 submissions (1472 Middle Primary, 3472 Upper Primary, 5640 Junior, 2877 Intermediate) for the Challenge in 2016. The 2016 problems and solutions for the Challenge, together with some statistics, appear later in this book.

Enrichment

This is a six-month program running from April to September, which consists of seven different parallel stages of comprehensive student and teacher support notes. Each student participates in only one of these stages.

The materials for all stages are designed to be a systematic structured course over a flexible 12–14 week period between April and September. This enables schools to timetable the program at convenient times during their school year.

Enrichment is completely independent of the earlier Challenge; however, they have the common feature of providing challenging mathematics problems for students, as well as accessible support materials for teachers.

Ramanujan (years 4–5) includes estimation, special numbers, counting techniques, fractions, clock arithmetic, ratio, colouring problems, and some problem-solving techniques. There were 186 entries in 2016.

Newton (years 5–6) includes polyominoes, fast arithmetic, polyhedra, pre-algebra concepts, patterns, divisibility and specific problem-solving techniques. There were 593 entries in 2016.

Dirichlet (years 6–7) includes mathematics concerned with tessellations, arithmetic in other bases, time/distance/speed, patterns, recurring decimals and specific problem-solving techniques. There were 784 entries in 2016.

Euler (years 7–8) includes primes and composites, least common multiples, highest common factors, arithmetic sequences, figurate numbers, congruence, properties of angles and pigeonhole principle. There were 1350 entries in 2016.

Gauss (years 8–9) includes parallels, similarity, Pythagoras' Theorem, using spreadsheets, Diophantine equations, counting techniques and congruence. Gauss builds on the Euler program. There were 702 entries in 2016.

Noether (top 10% years 9–10) includes expansion and factorisation, inequalities, sequences and series, number bases, methods of proof, congruence, circles and tangents. There were 565 entries in 2016.

Polya (top 10% year 10) Topics include angle chasing, combinatorics, number theory, graph theory and symmetric polynomials. There were 200 entries in 2016.

Australian Intermediate Mathematics Olympiad

This four-hour competition for students up to Year 10 offers a range of challenging and interesting questions. It is suitable for students who have performed well in the AMC (Distinction and above), and is designed as an endpoint for students who have completed the Gauss or Noether stage. There were 1829 entries for 2016 and 15 perfect scores.

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Enrichment

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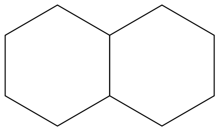
CHALLENGE PROBLEMS – MIDDLE PRIMARY

Students may work on each of these four problems in groups of up to three, but must write their solutions individually.

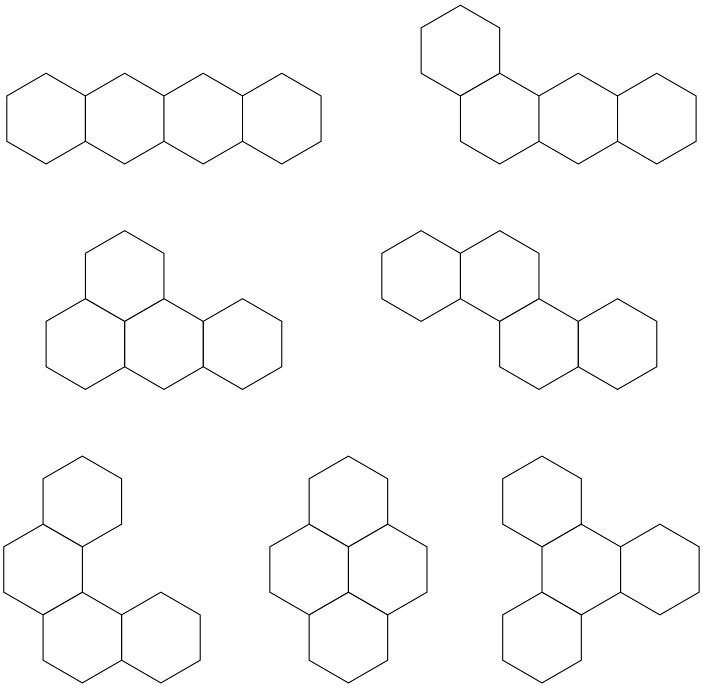
MP1 Hexos

If two identical regular hexagons are joined side-to-side, that is, a side of one meets a side of the other exactly, then only one shape can be made:

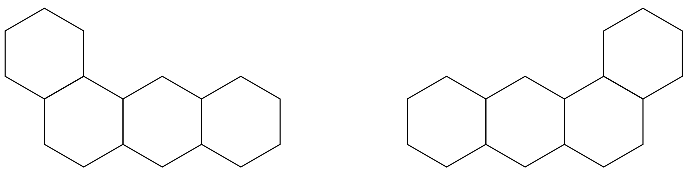


Counting the number of sides of this shape, we find that its perimeter is 10.

A shape made by joining four identical regular hexagons side-to-side is called a *hexo*. There are seven different hexos:



Two shapes are not different if one fits exactly over the other (after turning or flipping if necessary). For example, these two hexos are the same:

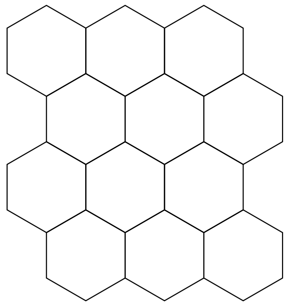


Notice that this hexo has perimeter 18.

a Only five of the seven different hexos have perimeter 18. On the worksheet provided, draw the other two hexos and state the perimeter of each.

b On the worksheet provided, draw two different hexos that are joined side-to-side to make a shape of perimeter 26.

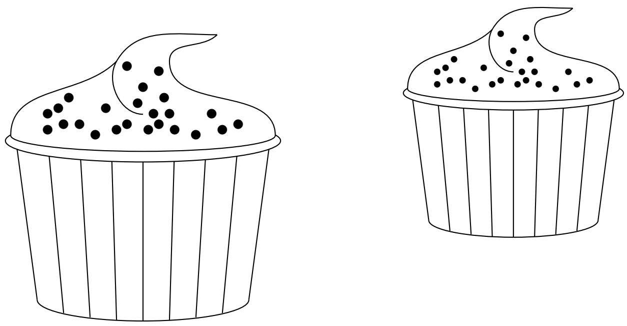
c Here is a shape made by joining three different hexos side-to-side. Colour this shape with three colours to show clearly three different hexos.



d What is the maximum perimeter for a shape that is formed by joining two different hexos side-to-side? Explain how to make such a shape and why no larger perimeter is possible.

MP2 Cupcakes

Bruno's Bakery is famous for its cupcakes.



a On Monday, Bruno baked 3 dozen cupcakes. Maria, the first customer, bought $\frac{1}{4}$ of them. The next customer, Niko, bought $\frac{1}{3}$ of the cupcakes that were left. How many cupcakes did Bruno have left after Niko bought his?

b On Tuesday Eleanor bought 12 cupcakes, which were $\frac{1}{4}$ of the cupcakes baked that day. How many cupcakes were baked on Tuesday?

c On Wednesday, the first customer bought $\frac{1}{2}$ of the cupcakes that Bruno baked that day. The second customer bought $\frac{1}{2}$ of what remained. Then the third customer bought $\frac{1}{2}$ of what remained. Finally the fourth customer bought $\frac{1}{2}$ of what remained. That left just one cupcake, which Bruno ate himself. How many cupcakes did Bruno bake on Wednesday?

d On Thursday, Bruno baked 18 cupcakes. Maryam bought some of these and shared them equally among her 3 children. Thierry bought the rest of the cupcakes and shared them equally between his 2 children. Each child received a whole number of cupcakes. Find all possible fractions of the 18 cupcakes that Maryam could have bought.

MP3 Kimmi Dolls

Suriya decides to give her collection of 33 Kimmi dolls to her four little sisters.

a Show how Suriya could give out the dolls so that the girl who receives the most has only one more than anyone else.

Suriya decides to give out the dolls so that the difference between the largest number of dolls any girl receives and the smallest number of dolls any girl receives is at most 4.

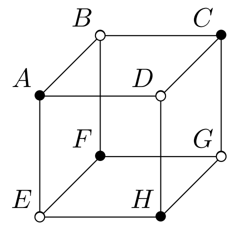
b Show how this could be done if one girl receives 11 dolls.

c Explain why no girl can receive more than 11 dolls.

d What is the smallest number of dolls any girl can receive? Explain why a smaller number is not possible.

MP4 Cube Trails

Label the vertices on the top face of a cube as $ABCD$ clockwise, and the corresponding vertices on the bottom face of the cube $EFGH$. Note that A and G are diagonally opposite in the cube.



We want to consider the possible routes along the edges of the cube from one vertex to another.

A *trail* is a route that does not repeat any edge, but it may repeat vertices. For example, $ABFE$ and $ABCDAE$ are trails, but $ABCB$ is not a trail.

The number of edges in a trail is called its *length*.

a Which vertices can be reached from A with a trail of length 2?

b List all the different trails of length 3 from A to G.

c Find a trail of length 7 from A to G.

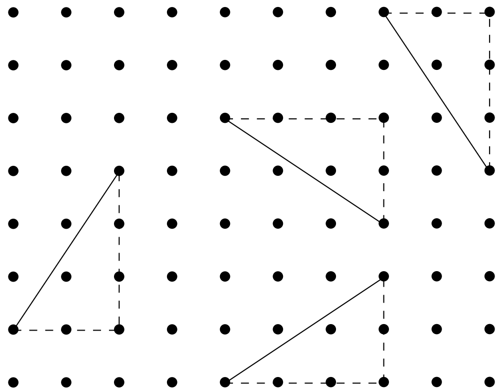
d Which vertices can be reached from A with a trail of length 4?

CHALLENGE PROBLEMS – UPPER PRIMARY

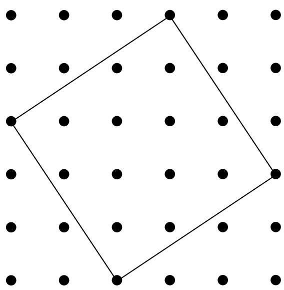
Students may work on each of these four problems in groups of up to three, but must write their solutions individually.

UP1 Knightlines

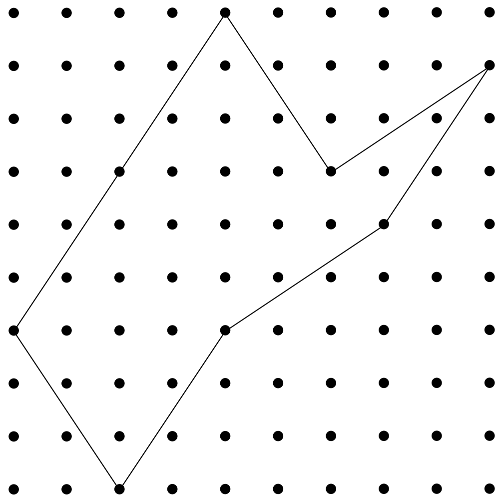
A *knightline* is drawn on 1 cm square dot paper. It is a line segment that starts at a dot and ends at a dot that is 3 cm away in one direction (horizontal or vertical) and 2 cm away in the other direction. Here are some knightlines:



April used knightlines to draw polygons. Here are two that she drew.



A quadrilateral using 4 knightlines



A heptagon using 8 knightlines

Two shapes are identical if one can be cut out and fitted exactly over the other one, flipping if necessary. If two shapes are not identical, they are different.

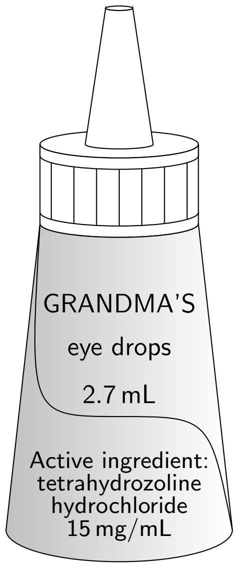
a Draw two different quadrilaterals, each using 4 knightlines and each different from April's quadrilateral above.

b Draw three different quadrilaterals, each using 6 knightlines.

c Draw six different simple hexagons, each using 6 knightlines. (A hexagon is simple if no two sides cross.)

UP2 Grandma's Eye Drops

Grandma's eye drops come in a small bottle. The label says there are 2.7 millilitres (2.7 mL) of solution in the bottle and 15 milligrams per millilitre (15 mg/mL) of active ingredient in the solution.



a How many milligrams of active ingredient are there in the bottle?

b The manufacturer produces the eye drops in batches of 10 litres. How many grams of active ingredient do they put in each batch?

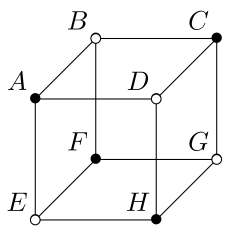
The instructions on the bottle say to put one drop in each eye at night. One 2.7 mL bottle lasts 30 days.

c How many millilitres of solution are there in each drop?

d A different bottle contains 63 mg of active ingredient in 4.5 mL of solution. Is this solution stronger or weaker than the solution in the 2.7 mL bottle? Explain why.

UP3 Cube Trails

Label the vertices on the top face of a cube as $ABCD$ clockwise, and the corresponding vertices on the bottom face of the cube $EFGH$. Note that A and G are diagonally opposite on the cube.



We want to consider the possible routes along the edges of the cube from one vertex to another.

A *trail* is a route that does not repeat any edge, but it may repeat vertices. For example, $ABFE$ and $ABCDAE$ are trails, but $ABCB$ is not a trail.

The number of edges in a trail is called its *length*.

a List all the different trails of length 3 from A to G.

b Find a trail of length 9 that starts at A and finishes at D.

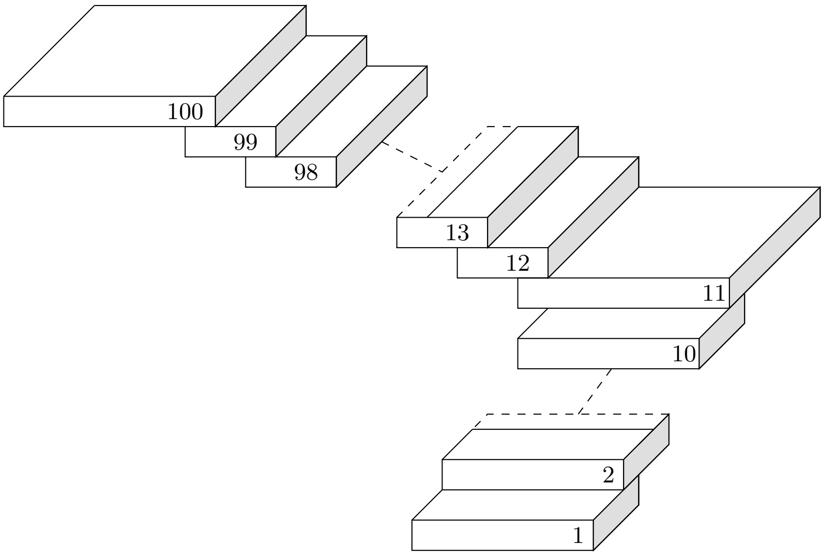
c List the vertices that can be reached from A with:

(1) a trail of length 2
(2) a trail of length 3
(3) a trail of length 4.

d Explain why there is no trail of even length from A to E.

UP4 Magic Staircase

The steps on the staircase that leads to the Mathemagician's Castle are numbered from 1 at the bottom to 100 at the top. Step 11 is a landing on which every climber is stopped by a guard.



When you arrive at the landing, the guard rolls a 20-faced die to give you a number from 1 to 20. From then on you have to take that number of steps at a time to get as close as possible to the castle courtyard, which is on step 100. Only on your final step up to the courtyard may you take less than your given number of steps. For example, if your given number is 15, then from step 11 you move to steps 26, 41, 56, 71, 86, and then 100. If you make a mistake the courtyard guard at the top will send you back to the landing and make you do it again (instead of giving you a cup of hot chocolate, which is how you are welcomed if you climb the steps correctly)!

a If you take 5 steps at a time, what is the number of the last step you would land on before reaching the courtyard?

b Once when I climbed the stairs, the last step I landed on before the courtyard was 89. How many steps was I taking at a time from the landing? Explain your answer.

c Make a list of all the last steps you could have come from to land on step 95.

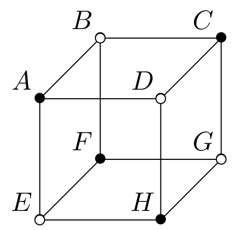
d What is the number of the lowest step that you could land on immediately before landing on the courtyard? Explain your answer.

CHALLENGE PROBLEMS – JUNIOR

Students may work on each of these six problems with a partner but each must write their solutions individually.

J1 Cube Trails

Label the vertices on the top face of a cube as $ABCD$ clockwise, and the corresponding vertices on the bottom face of the cube $EFGH$. Note that A and G are diagonally opposite on the cube.



We want to consider the possible routes along the edges of the cube from one vertex to another. A *trail* is a route that does not repeat any edge, but it may repeat vertices. For example, $ABFE$ and $ABCDAE$ are trails, but $ABCB$ is not a trail. The number of edges in a trail is called its *length*.

a List all the different trails of length 3 from A to G.

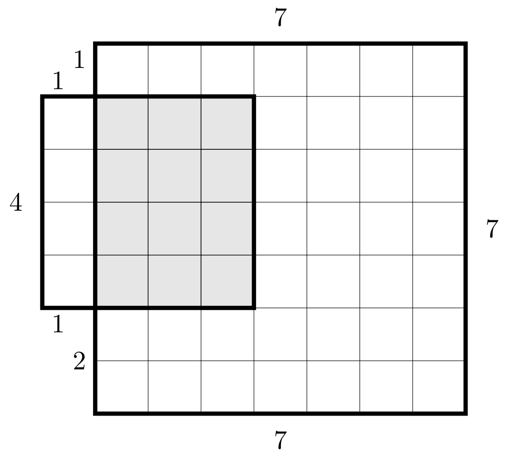
b List the vertices that can be reached from A with:
(1) a trail of length 2
(2) a trail of length 3
(3) a trail of length 4.

c Explain why there is no trail of even length from A to E.

d Determine how many trails of length 3 there are in the cube. Here a trail between two vertices is regarded as the same in either direction.

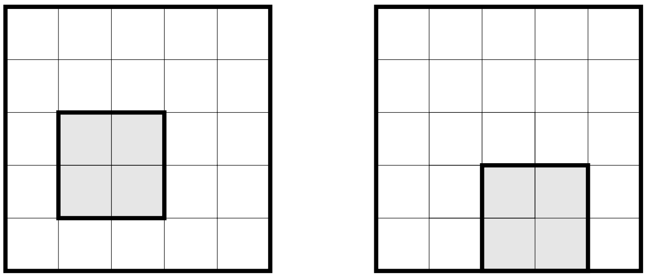
J2 Overlaps

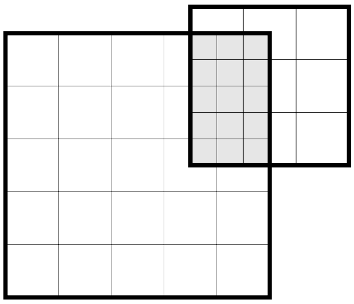
Cutting along the lines on 1 cm square grid paper, Tia produced many squares. The sides of her squares were all longer than 1 cm. She then *partially* overlapped various pairs of squares and worked out the perimeter of the final (combined) shape. For example, here is a 7×7 square overlapped with a 4×4 square with the overlap shaded.



$$\text{Perimeter} = 7 + 7 + 7 + 1 + 1 + 4 + 1 + 2 = 30\,\text{cm}$$

Note that one square must not be wholly on top of another and that the grid lines of both squares must coincide. Thus the following three shapes are not allowed.





a Show how to overlap a 6×6 square and a 7×7 square so that the perimeter of the final shape is 48 cm.

b Show how to overlap a 6×6 square and a 7×7 square so that the perimeter of the final shape is 30 cm.

c The overlap of two squares has area 1 cm^2. The perimeter of the final shape is 32 cm. Find all possible sizes of the two squares.

d The overlap of two squares has area 12 cm^2. The perimeter of the final shape is 30 cm. Find all possible sizes of the two squares.

J3 Stocking Farms

The number of stock that can be grazed on a paddock is given in terms of DSEs (Dry Sheep Equivalents). So if a particular area of land has a stocking capacity of 12 DSEs per hectare, this means that the maximum number of dry sheep (that is, sheep that eat dry food, not milk) that can be grazed on each hectare of land is 12. So a 5 hectare farm with a stocking capacity of 12 DSEs per hectare could be stocked with a maximum of 60 dry sheep. We call this a fully stocked farm.

For other animals, a conversion needs to be made. For dairy cattle the DSEs are as follows.

- One adult (milking) cow: 15 DSEs
- One yearling (a cow not yet milking but eating dry food): 6 DSEs
- One calf (6 to 12 months old): 4 DSEs

Farmers Green, White, Black, Grey and Brown all have farms in the Adelaide Hills, where the maximum stocking capacity of a farm is 12 DSEs per hectare.

a Farmer Green has a 100 hectare farm on which there are 60 adult cows, 40 yearlings and 15 calves. Show that the farm is fully stocked.

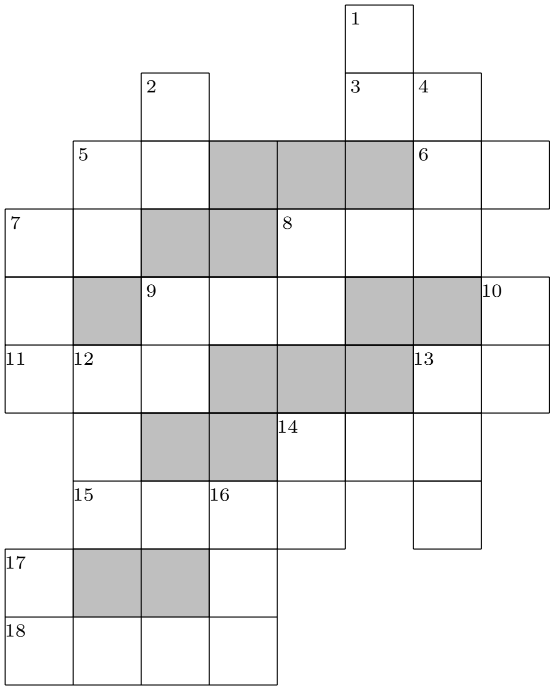
b Farmer White has an 80 hectare farm on which there are 50 adult cows. There are also some yearlings and some calves. Give two possible combinations of yearlings and calves that would fully stock the property.

c Grey's farm has 12 adult cows, 18 yearlings, and 4 calves. Brown's farm has 10 adult cows, 4 yearlings, and 6 calves. Both farms are fully stocked. To specialise they agree to exchange stock so that Grey has only adult cows and Brown has only yearlings and calves. Any stock that can't be accommodated will be sold at the market. What stock has to be sold?

d Farmer Black has a fully stocked farm. She sells some of her adult cows and restocks with 12 yearlings and some calves, so that her farm is again fully stocked. What is the smallest number of adult cows she could have sold?

J4 Cross Number

Each number in the following cross number puzzle is a 2-, 3- or 4-digit factor of 2016.
No number starts with 0.
No number is a 2-digit square.
No number is repeated.



a Explain why 16 down is 168.

b Which factor is 7 down? Explain why.

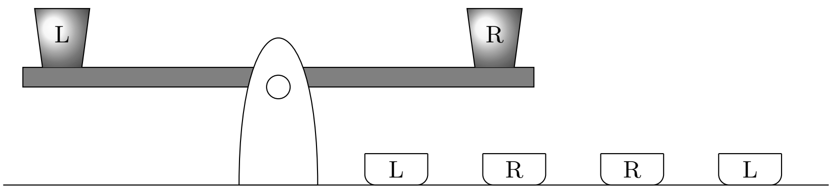
c Fill in the remaining 3-digit factors.

d Complete the puzzle.

J5 Tipping Points

Two empty buckets are placed on a balance beam, one at each end. Balls of the same weight are placed in the buckets one at a time. If the number of balls is the same in each bucket, the beam stays horizontal. If there is a difference of only one ball between the buckets, the beam moves a little but the buckets and balls remain in place. However, if the difference between the number of balls is two or more, the beam tips all the way, the buckets fall off, and all the balls fall out.

There are several bowls, each containing some of the balls and each labelled L or R. If a ball is taken from a bowl labelled L, the ball is placed in the left bucket on the beam. If a ball is taken from a bowl labelled R, the ball is placed in the right bucket.



a Julie arranges six labelled bowls in a row. She takes a ball from each bowl in turn from left to right, and places it in the appropriate bucket. List all sequences of six bowls which do *not* result in the beam tipping.

b Julie starts again with both buckets empty and with six bowls in a row. As before, she takes a ball from each bowl in turn, places it in the appropriate bucket, and the beam does not tip. She then empties both buckets and takes a ball from the 2nd, 4th, and 6th bowl in turn and places it in the appropriate bucket. Again the beam does not tip. Once more she empties both buckets but this time takes a ball from the 3rd and 6th bowl in turn and places it in the appropriate bucket. Yet again the beam does not tip. List all possible orders in which the six bowls could have been arranged.

For a large number of bowls, a ball could be taken from every bowl, or every second bowl, or every third bowl, and so on. If a ball is taken from bowl m, followed by bowl $2m$, then bowl $3m$, and so on (every mth bowl), we say an *m-selection* was used. For example, in Part **b**, Julie used a 1-selection, then a 2-selection, and finally a 3-selection.

c Find all sequences of 11 bowls for which the beam does not tip no matter what m-selection is used.

d Show that it is impossible to have a sequence of 12 bowls so that every m-selection is non-tipping.

J6 Tossing Counters

Ms Smartie told all the students in her class to write four different integers from 1 to 9 on the four faces of two counters, with one number on each face. She then asked them to toss both counters simultaneously many times and write down the sum of the numbers that appeared on the upper faces each time.

a Emily put a 1 on one face of one counter and a 7 on one face of the other counter. The only sums that Emily was able to get were 8, 9, 10 and 11. List the two possible combinations for the four numbers on her counters.

b The only sums that Jack was able to get were 7, 8, 9 and 10. List all four combinations for the four numbers on his counters.

c Jill wrote 4 and 5 on opposite sides of one counter. The only sums she was able to get were three consecutive integers. Find all possible ways the second counter could be numbered.

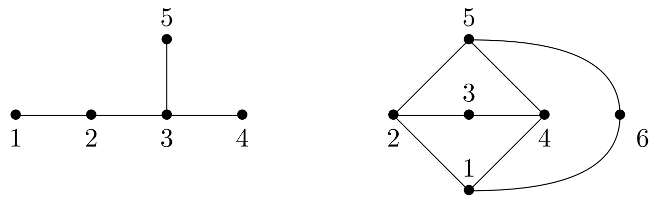
d Ben was only able to get sums that were four consecutive numbers. Show that either one or three of the numbers he wrote on the counters were even.

CHALLENGE PROBLEMS – INTERMEDIATE

Students may work on each of these six problems with a partner but each must write their solutions individually.

I1 COP-graphs

The Roads Department in the state of Graphium uses graphs consisting of dots (called *vertices*) that represent towns, and lines (called *edges*) that represent roads between towns. To keep track of things, they label the vertices with consecutive positive integers so that if two towns are joined by a road, then the numbers on the corresponding vertices are *coprime*, that is, their only common factor is 1. Such a labelled graph is called a *COP-graph*. Note that if two towns do not have a road joining them, then their labels may or may not be coprime. Here are two examples of COP-graphs, one with labels 1 to 5, the other with labels 1 to 6.



COP-graph A COP-graph B

a Show that the labels on COP-graphs A and B could be replaced with integers 2 to 6 and 2 to 7 respectively so they remain COP-graphs.

b Explain why COP-graph B could not remain a COP-graph if its labels were replaced with integers 5 to 10.

c Explain why COP-graph A remains a COP-graph no matter what set of five consecutive positive integers label the vertices.

d If a COP-graph has six vertices and labels 1 to 6, what is the maximum number of edges it can have? Explain.

I2 Tipping Points

See Junior Problem 5.

I3 Tossing Counters

Ms Smartie told all the students in her class to write four different integers from 1 to 9 on the four faces of two counters, with one number on each face. She then asked them to toss both counters simultaneously many times and write down the sum of the numbers that appeared on the upper faces each time.

a The only sums that Jack was able to get were 8, 9, 10, and 11. Find all five possible combinations of four numbers on the counters.

b Jill wrote 4 and 5 on opposite sides of one counter. The only sums she was able to get were three consecutive integers. Find all possible ways the second counter could be numbered.

c Ben was only able to get sums that were four consecutive numbers. Show that either one or three of the numbers he wrote on the counters were even.

d Show that it is possible to number four counters with 8 different positive integers less than 20, one number on each face, so that the sums that appear are 16 consecutive numbers.

I4 Ionofs

The *Ionof* (integer on number of factors) of an integer is the integer divided by the number of factors it has. For example, 18 has 6 factors so Ionof(18) = 18/6 = 3, and 27 has 4 factors so Ionof(27) = 27/4 = 6.75.

a Find Ionof(36).

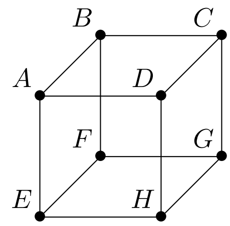
b Explain why Ionof(pq) is not an integer if p and q are distinct primes.

c If p and q are distinct primes, find all numbers of the form pq^4 whose Ionof is an integer.

d Show that the square of any prime number is the Ionof of some integer.

I5 Cube Trails

Label the vertices on the top face of a cube as $ABCD$ clockwise, and the corresponding vertices on the bottom face of the cube $EFGH$. Note that A and G are diagonally opposite on the cube.



A robot travels along the edges of this cube always starting at A and never repeating an edge. This defines a *trail* of edges. For example, $ABFE$ and $ABCDAE$ are trails, but $ABCB$ is not a trail. The number of edges in a trail is called its *length*.

At each vertex, the robot must proceed along one of the edges that has not yet been traced, if there is one. If there is a choice of untraced edges, the following probabilities for taking each of them apply.

If only one edge at a vertex has been traced and that edge is vertical, then the probability of the robot taking each horizontal edge is 1/2.

If only one edge at a vertex has been traced and that edge is horizontal, then the probability of the robot taking the vertical edge is 2/3 and the probability of the robot taking the horizontal edge is 1/3.

If no edge at a vertex has been traced, then the probability of the robot taking the vertical edge is 2/3 and the probability of the robot taking each of the horizontal edges is 1/6.

In your solutions to the following problems use *exact fractions* not decimals.

a If the robot moves from A to D, what is the probability it will then move to H? If the robot moves from A to E, what is the probability it will then move to H?

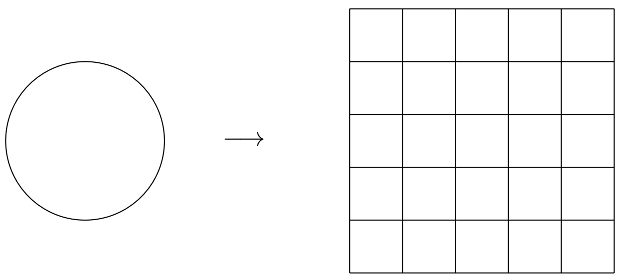
b What is the probability the robot takes the trail $ABCG$?

c Find two trails of length 3 from A to G that have probabilities of being traced by the robot that are different to each other and to the probability for the trail $ABCG$.

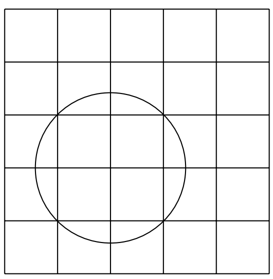
d What is the probability that the robot will trace a trail of length 3 from A to G?

I6 Coverem

A disc is placed on a grid composed of small 1×1 squares.

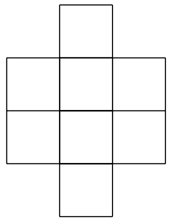


If the disc has diameter $2\sqrt{2}$ and its centre is at a grid point, then it completely covers four grid squares.



In all of the following questions use exact surds in your calculations, not decimal approximations.

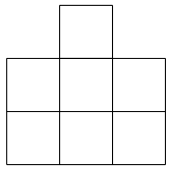
a Find the smallest diameter for a disc that will completely cover all 8 grid squares shown here.



b Find the smallest diameter for a disc that can cover 5 grid squares if they are in some suitable configuration.

c Find the minimum and maximum number of grid squares that can be completely covered by a disc of diameter 3.

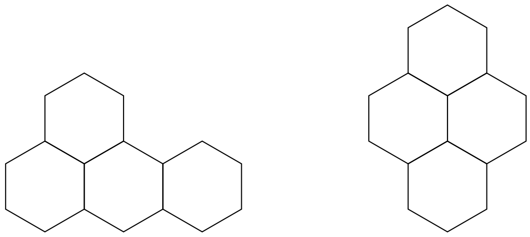
d Find the smallest radius for a disc that will completely cover all 7 grid squares shown here.



CHALLENGE SOLUTIONS – MIDDLE PRIMARY

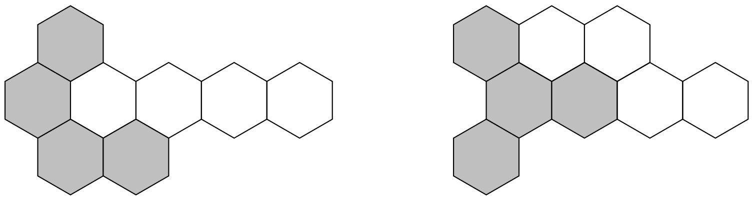
MP1 Hexos

a

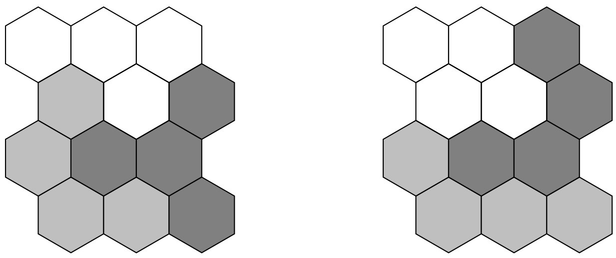


Perimeter = 16 Perimeter = 14

b Here are two shapes with perimeter 26. There are others.



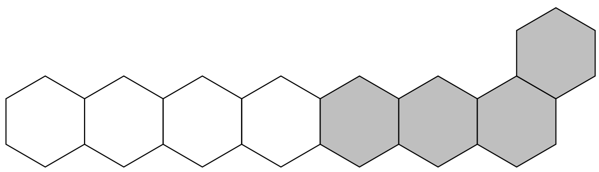
c Here are two colourings showing three different hexos. There are others.



d The perimeter of the combined shape is the sum of the perimeters of the separate hexos minus twice the number of common sides where they meet. There must be at least one common side.

The maximum perimeter of a hexo is 18. Hence the perimeter of the combined shape can't be more than $18+18-2 = 34$.

The following shape has perimeter 34.



So the maximum perimeter is 34.

MP2 Cupcakes

a **Alternative i**

The number of cupcakes baked on Monday is $3 \times 12 = 36$.

The number of cupcakes bought by Maria is $\frac{1}{4} \times 36 = 9$.

The number of cupcakes left is $36 - 9 = 27$.

The number of cupcakes bought by Niko is $\frac{1}{3} \times 27 = 9$.

So the number of cupcakes Bruno had left is $27 - 9 = 18$.

Alternative ii

The number of cupcakes baked on Monday is $3 \times 12 = 36$.

The number of cupcakes left after Maria is $\frac{3}{4} \times 36 = 27$.

The number of cupcakes left after Niko is $\frac{2}{3} \times 27 = 18$.

b Eleanor bought 12 cupcakes baked on Tuesday.

This was $\frac{1}{4}$ of all cupcakes baked on Tuesday.

So the number of cupcakes baked on Tuesday is $4 \times 12 = 48$.

c **Alternative i**

Work backwards.

Only 1 cupcake remained after the fourth customer. So the fourth customer bought exactly 1 cupcake.

Hence the third customer left exactly 2 cupcakes. So the third customer bought exactly 2 cupcakes.

Hence the second customer left exactly 4 cupcakes. So the second customer bought exactly 4 cupcakes.

Hence the first customer left exactly 8 cupcakes. So the first customer bought exactly 8 cupcakes.

Hence there were originally 16 cupcakes.

Alternative ii

The first customer bought $\frac{1}{2}$ of the cupcakes baked that day. So the fraction left by the first customer was $1-\frac{1}{2} = \frac{1}{2}$.

The second customer bought $\frac{1}{2}$ of those left, that is, $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. So the fraction left by the second customer was $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

The third customer bought $\frac{1}{2}$ of those left, that is, $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$. So the fraction left by the third customer was $\frac{1}{4} - \frac{1}{8} = \frac{1}{8}$.

The fourth customer bought $\frac{1}{2}$ of those left, that is, $\frac{1}{2} \times \frac{1}{8} = \frac{1}{16}$. So the fraction left by the fourth customer was $\frac{1}{8} - \frac{1}{16} = \frac{1}{16}$.

Since only one was left, there must have been 16 baked.

d The number of cupcakes that Maryam bought is 3, 6, 9, 12, or 15. The number of cupcakes that Thierry bought is 2, 4, 6, 8, 10, 12, 14, or 16.

The only combinations that total 18 are 6 for Maryam and 12 for Thierry, and 12 for Maryam and 6 for Thierry.

So the only fractions of the 18 cupcakes that Maryam could have bought were $\frac{6}{18} = \frac{1}{3}$ and $\frac{12}{18} = \frac{2}{3}$.

MP3 Kimmi Dolls

a **Alternative i**

Since $33 \div 4 = 8$ with a remainder of 1, three girls could get 8 dolls each and the fourth girl 9 dolls.

Alternative ii

If one girl gets 7 or fewer dolls, then each of the other three girls gets at most 8 dolls. This means the total number of dolls can't be any more than $7 + 8 + 8 + 8 = 31$, which is not enough.

So every girl must get at least 8 dolls. Since $4 \times 8 = 32$, which is 1 less than 33, Suriya must give 8 dolls to each of 3 sisters and 9 dolls to the fourth sister.

b If one girl gets 11 dolls, each of the other three girls must get at least 7 dolls. Since $11 + 7 + 7 + 7 = 32$, which is 1 less than 33, two girls get 7 dolls each and one girl gets 8.

c **Alternative i**

If one girl gets 12 or more dolls, the other three must get at least 8 each. Then the total number of dolls would have to be at least $12 + 8 + 8 + 8 = 36$, which is too many. So no girl can get 12 or more dolls.

Alternative ii

If one girl gets 12 or more dolls, then there are at most 21 dolls for the other three girls. This means that one girl gets at most 7 dolls, which is 5 less than 12. So no girl can get 12 or more dolls.

d In Part **b** we saw how it's possible for at least one girl to get 7 dolls. So we need to see if we can give one girl less than 7 dolls.

Suppose one girl gets 6 dolls. This leaves 27 dolls for the other three girls. If each of them gets 9, all the dolls are used up and Suriya's rule is satisfied.

If one girl gets 5 or fewer dolls, then each of the other girls gets at most 9. This means the total number of dolls can't be any more than $5 + 9 + 9 + 9 = 32$, which is not enough. So no girl can get 5 dolls.

(Alternatively, if one girl gets 5 or fewer dolls, then there are at least 28 dolls for the other three girls. This means that one girl gets at least 10 dolls, which is 5 more than 5. So no girl can get 5 dolls.)

Therefore 6 is the least number of dolls that any girl can get.

MP4 Cube Trails

a Trails of length 2 from A are:

ABC	ADC	AEF
ABF	ADH	AEH

So the vertices that can be reached from A with a trail of length 2 are C, F, H.

b Trails of length 3 from A to G are the trails of length 2 in Part **a** followed by G:

$ABCG$, $ABFG$, $ADCG$, $ADHG$, $AEFG$, $AEHG$.

c Here is one trail of length 7: $ABFEHDCG$. There are others.

d **Alternative i**

We list, in alphabetical order, all trails of length 4 from A:

$ABCDA$	$ADCBA$	$AEFBA$
$ABCDH$	$ADCBF$	$AEFBC$
$ABCGF$	$ADCGF$	$AEFGC$
$ABCGH$	$ADCGH$	$AEFGH$
$ABFEA$	$ADHEA$	$AEHDA$
$ABFEH$	$ADHEF$	$AEHDC$
$ABFGC$	$ADHGC$	$AEHGC$
$ABFGH$	$ADHGF$	$AEHGF$

Hence trails of length 4 from A reach only vertices A, C, F, H.

Alternative ii

We can reach vertices A, C, F, H from A with trails of length 4. For example: $ABCDA$, $AEHGC$, $ADHEF$, $ABCGH$.

We now show that no other vertices can be reached from A with a trail of length 4.

Any trail of length 4 can be divided into 2 trails of length 2.

From Part **a**, any trail of length 2 that starts at A must finish at one of C, F, H. Notice that C, F, H are the diagonal opposites of A on the 3 cube faces that have A as a vertex. From the symmetry of the cube, any trail of length 2 that starts at one of A, C, F, H must finish at one of the other three vertices.

So each of the trails of length 2 starts at one of A, C, F, H and finishes at one of A, C, F, H. Hence any trail of length 4 that starts at A must finish at one of A, C, F, H.

Alternative iii

We can reach vertices A, C, F, H from A with trails of length 4. For example: $ABCDA$, $AEHGC$, $ADHEF$, $ABCGH$.

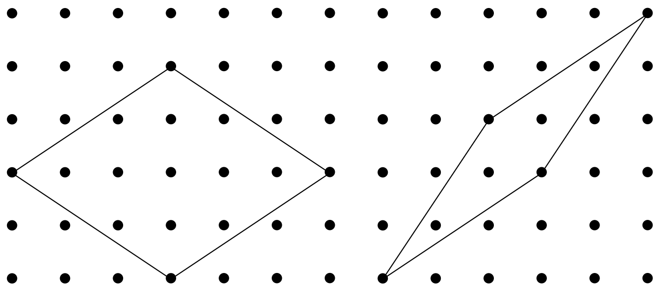
We now show that no other vertices can be reached from A with a trail of length 4.

Notice that the vertices A, C, F, H are black and the vertices B, D, E, G are white. Also each edge in the cube joins a black vertex to a white vertex. So the vertices in any trail of length 4 that starts at A must alternate black and white, starting with black and finishing with black. Hence any trail of length 4 that starts at A must finish at one of A, C, F, H.

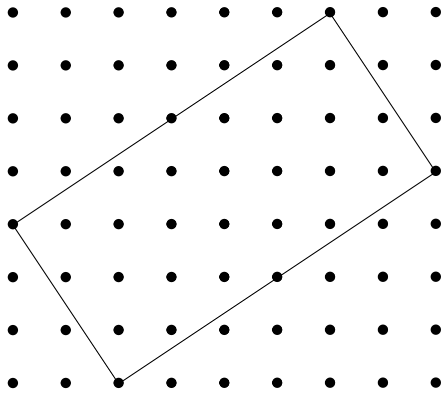
CHALLENGE SOLUTIONS – UPPER PRIMARY

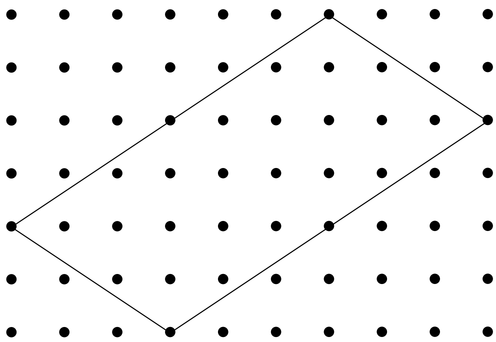
UP1 Knightlines

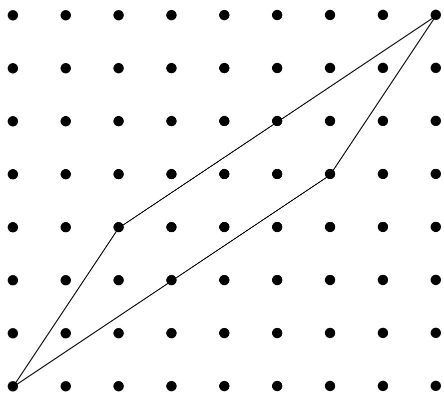
a



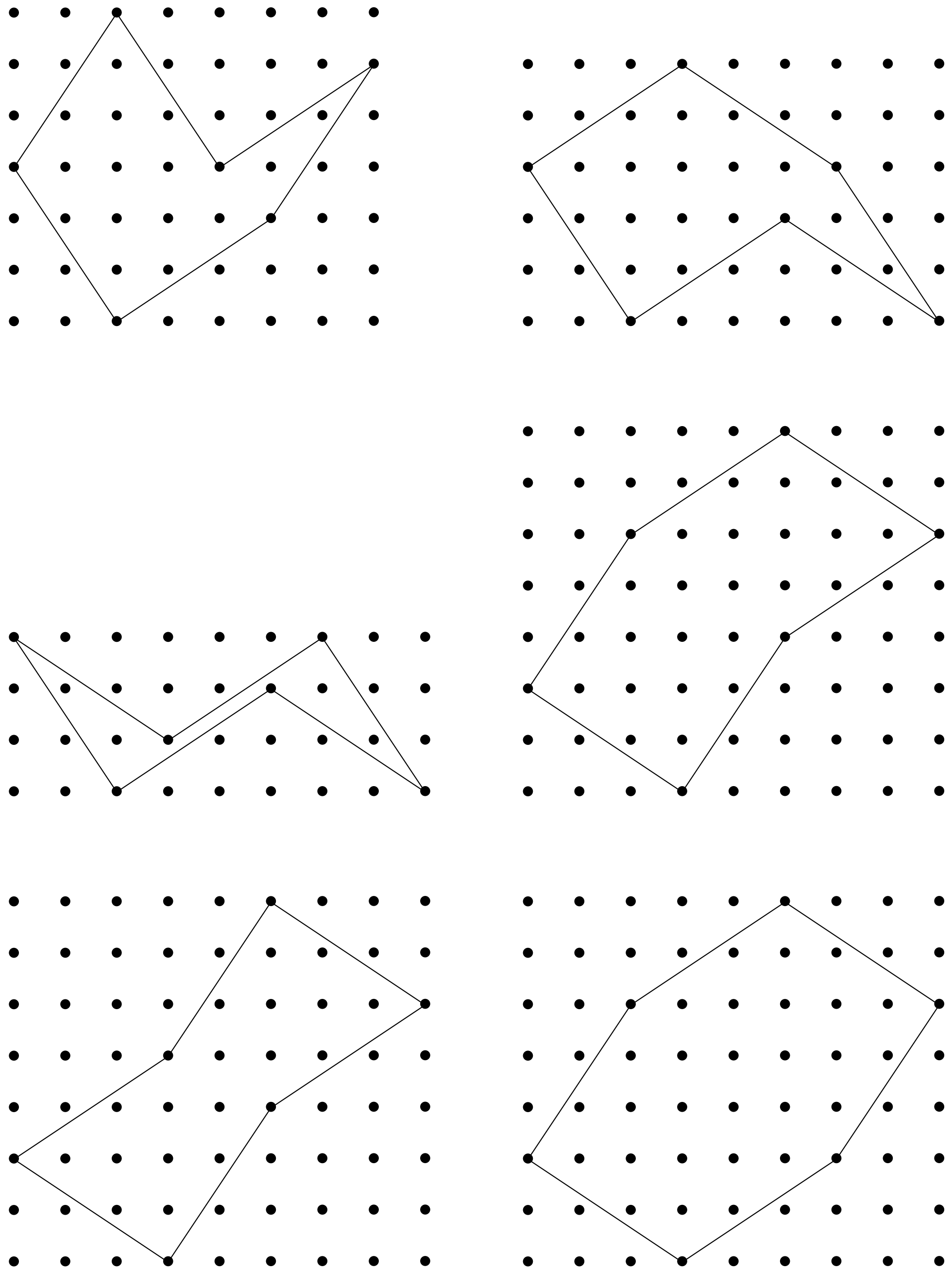
b







c



UP2 Grandma's Eye Drops

a The amount of active ingredient in the bottle is $15 \times 2.7 = 40.5\,\text{mg}$.

b Since $1000\,\text{mL} = 1\,\text{L}$ and each mL of solution has 15 mg of active ingredient, 1 L of solution has 15 000 mg of active ingredient. Since $1000\,\text{mg} = 1\,\text{g}$, 1 L of solution has 15 g of active ingredient. Hence the amount of active ingredient in each batch of 10 L is 150 g.

c The bottle lasts 30 days and each day 2 drops are used. Hence there are 60 drops in each 2.7 mL bottle. So the amount of solution in each drop is

$$\frac{2.7}{60} = \frac{27}{600} = \frac{9}{200} = \frac{4.5}{100} = 0.045\,\text{mL}.$$

d The number of milligrams of active ingredient per millilitre of solution in the 4.5 mL bottle is

$$\frac{63}{4.5} = \frac{630}{45} = \frac{126}{9} = 14.$$

This is less than 15 mg/mL. So the solution in the larger bottle is weaker.

UP3 Cube Trails

a $ABCG$, $ABFG$, $ADCG$, $ADHG$, $AEFG$, $AEHG$.

b Here is one trail: $ADHEABFGCD$. There are others.

c **Alternative i**

(1) The trails of length 2 from A are:

$$\begin{array}{ccc} ABC & ADC & AEF \\ ABF & ADH & AEH \end{array}$$

So trails of length 2 from A reach only vertices C, F, H.

(2) The trails of length 3 from A are:

$$\begin{array}{ccc} ABCD & ADCB & AEFB \\ ABCG & ADCG & AEFG \\ \\ ABFE & ADHE & AEHD \\ ABFG & ADHG & AEHG \end{array}$$

So trails of length 3 from A reach only vertices B, D, E, G.

(3) The trails of length 4 from A are:

$$\begin{array}{ccc} ABCDA & ADCBA & AEFBA \\ ABCDH & ADCBF & AEFBC \\ \\ ABCGF & ADCGF & AEFGC \\ ABCGH & ADCGH & AEFGH \\ \\ ABFEA & ADHEA & AEHDA \\ ABFEH & ADHEF & AEHDC \\ \\ ABFGC & ADHGC & AEHGC \\ ABFGH & ADHGF & AEHGF \end{array}$$

So trails of length 4 from A reach only vertices A, C, F, H.

Alternative ii

(1) We can reach vertices C, F, H from A with trails of length 2. For example: ABC, ABF, ADH.

(2) We can reach vertices B, D, E, G from A with trails of length 3. For example: $ABCD$, $ABCG$, $ABFE$, $ADCB$.

(3) We can reach vertices A, C, F, H from A with trails of length 4. For example: $ABCDA$, $ABCDH$, $ABCGF$, $ABFGC$.

We now show that, in each case, no other vertices can be reached from A. Notice that the vertices A, C, F, H are black and the vertices B, D, E, G are white. Also each edge in the cube joins a black vertex to a white vertex. So the vertices in any trail that starts at A must alternate black and white, starting with black.

Hence, for any trail that starts at A, (1) if it has length 2 then it must finish at one of the black vertices C, F, H (A is excluded because it would repeat an edge), (2) if it has length 3 then it must finish at one of the white vertices B, D, E, G, (3) if it has length 4 then it must finish at one of the black vertices A, C, F, H.

d **Alternative i**

Any trail of even length can be divided into trails of length 2.

From Part **c**, any trail of length 2 that starts at A must finish at one of C, F, H. Notice that C, F, H are the diagonal opposites of A on the 3 cube faces that have A as a vertex. From the symmetry of the cube, any trail of length 2 that starts at one of A, C, F, H must finish at one of the other three vertices.

So each of the trails of length 2 starts at one of A, C, F, H and finishes at one of A, C, F, H. Hence any trail of even length that starts at A must finish at one of A, C, F, H. This excludes E.

Alternative ii

Notice that vertices A, C, F, H are black and vertices B, D, E, G are white. Also each edge in the cube joins a black vertex to a white vertex.

So the vertices in any trail from A to E must alternate black and white, starting with black and finishing with white. Therefore any trail from A to E must have odd length.

UP4 Magic Staircase

a You would start from the landing like this: 11, 16, 21, 26, ... Each step you land on before the courtyard has a number ending in 1 or 6. So the number of the last step you land on before the courtyard is 96.

b Since $100 - 89 = 11$, the number of steps you were taking at a time was at least 11. The number of steps from the landing at step 11 to step 89 is $89 - 11 = 78$. So the number of steps you were taking at a time was also a factor of 78. The factors of 78 are 1, 2, 3, 6, 13, 26, 39 and 78. The only one of these factors that is greater than or equal to 11 and less than or equal to 20 is 13.

c Since $95 - 11 = 84$, the number of steps you were taking at a time was a factor of 84. The factors of 84 are 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 84. If you were taking 1 step at a time you came from 94. If 2, 93. If 3, 92. If 4, 91. If 6, 89. If 7, 88. If 12, 83. If 14, 81. The other factors of 84 are greater than 20 so could not have been rolled with the die.

d **Alternative i**

Since the number of steps you take at a time is less than or equal to 20, your last step cannot be lower than 80.

If your last step is 80, then you must take 20 steps at a time to reach the courtyard. Since $80 - 11 = 69$, the number of steps you take at a time must also be a factor of 69. This is impossible. So the last step was not 80.

Suppose your last step is 81. Since $81 - 11 = 70$, the number of steps you take at a time must be a factor of 70 that is less than or equal to 20. These factors are 1, 2, 5, 7, 10, 14. In each case at least one more step could be reached before 100. So the last step was not 81.

Suppose your last step is 82. Since $82 - 11 = 71$, the number of steps you take at a time must be a factor of 71 that is less than or equal to 20. The only such factor is 1. Since $82 + 1$ is less than 100, the last step was not 82.

Suppose your last step is 83. Since $83 - 11 = 72$, the number of steps you take at a time must be a factor of 72 less than or equal to 20. Since $100 - 83 = 17$, the factor must be greater than 17. Since 18 is such a factor, the last step could be 83.

Thus the number of the lowest step that you could land on before stepping onto the courtyard is 83.

Alternative ii

The table shows, for each number n from 1 to 20, the largest number m less than 100 that is 11 plus a multiple of n.

n	m	n	m
1	$99 = 11 + 88 \times 1$	11	$99 = 11 + 8 \times 11$
2	$99 = 11 + 44 \times 2$	12	$95 = 11 + 7 \times 12$
3	$98 = 11 + 29 \times 3$	13	$89 = 11 + 6 \times 13$
4	$99 = 11 + 22 \times 4$	14	$95 = 11 + 6 \times 14$
5	$96 = 11 + 17 \times 5$	15	$86 = 11 + 5 \times 15$
6	$95 = 11 + 14 \times 6$	16	$91 = 11 + 5 \times 16$
7	$95 = 11 + 12 \times 7$	17	$96 = 11 + 5 \times 17$
8	$99 = 11 + 11 \times 8$	18	$83 = 11 + 4 \times 18$
9	$92 = 11 + 9 \times 9$	19	$87 = 11 + 4 \times 19$
10	$91 = 11 + 8 \times 10$	20	$91 = 11 + 4 \times 20$

Thus the number of the lowest step that you could land on before stepping onto the courtyard is 83.

Alternative iii

If your step size is 20, then the last step number before the courtyard will be $11 + 4 \times 20 = 91$.
If your step size is 19, then the last step number before the courtyard will be $11 + 4 \times 19 = 87$.
If your step size is 18, then the last step number before the courtyard will be $11 + 4 \times 18 = 83$.
If your step size is 17 or smaller, then the last step number before the courtyard will be at least $100 - 17 = 83$.

Thus the number of the lowest step that you could land on before stepping onto the courtyard is 83.

CHALLENGE SOLUTIONS – JUNIOR

J1 Cube Trails

a $ABCG$, $ABFG$, $ADCG$, $ADHG$, $AEFG$, $AEHG$.

b Alternative i

(1) The trails of length 2 from A are:

$$\begin{array}{ccc} ABC & ADC & AEF \\ ABF & ADH & AEH \end{array}$$

So trails of length 2 from A can reach vertices C, F, H.

(2) The trails of length 3 from A are:

$$\begin{array}{ccc} ABCD & ADCB & AEFB \\ ABCG & ADCG & AEFG \\ \\ ABFE & ADHE & AEHD \\ ABFG & ADHG & AEHG \end{array}$$

So trails of length 3 from A can reach vertices B, D, E, G.

(3) The trails of length 4 from A are:

$$\begin{array}{ccc} ABCDA & ADCBA & AEFBA \\ ABCDH & ADCBF & AEFBC \\ \\ ABCGF & ADCGF & AEFGC \\ ABCGH & ADCGH & AEFGH \\ \\ ABFEA & ADHEA & AEHDA \\ ABFEH & ADHEF & AEHDC \\ \\ ABFGC & ADHGC & AEHGC \\ ABFGH & ADHGF & AEHGF \end{array}$$

So trails of length 4 from A can reach vertices A, C, F, H.

Alternative ii

(1) We can reach vertices C, F, H from A with trails of length 2. For example: ABC, ABF, ADH.

(2) We can reach vertices B, D, E, G from A with trails of length 3. For example: $ABCD$, $ABCG$, $ABFE$, $ADCB$.

(3) We can reach vertices A, C, F, H from A with trails of length 4. For example: $ABCDA$, $ABCDH$, $ABCGF$, $ABFGC$.

We now show that, in each case, no other vertices can be reached from A. Notice that the vertices A, C, F, H are black and the vertices B, D, E, G are white. Also each edge in the cube joins a black vertex to a white vertex. So the vertices in any trail that starts at A must alternate black and white, starting with black.

Hence, for any trail that starts at A, (1) if it has length 2 then it must finish at one of the black vertices C, F, H (A is excluded because it would repeat an edge), (2) if it has length 3 then it must finish at one of the white vertices B, D, E, G, (3) if it has length 4 then it must finish at one of the black vertices A, C, F, H.

c Alternative i

Any trail of even length can be divided into trails of length 2.

From Part **b**, any trail of length 2 that starts at A must finish at one of C, F, H. Notice that C, F, H are the diagonal opposites of A on the 3 cube faces that have A as a vertex. From the symmetry of the cube, any trail of length 2 that starts at one of A, C, F, H must finish at one of the other three vertices.

So each of the trails of length 2 starts at one of A, C, F, H and finishes at one of A, C, F, H. Hence any trail of even length that starts at A must finish at one of A, C, F, H. This excludes E.

Alternative ii

Notice that vertices A, C, F, H are black and vertices B, D, E, G are white. Also each edge in the cube joins a black vertex to a white vertex.

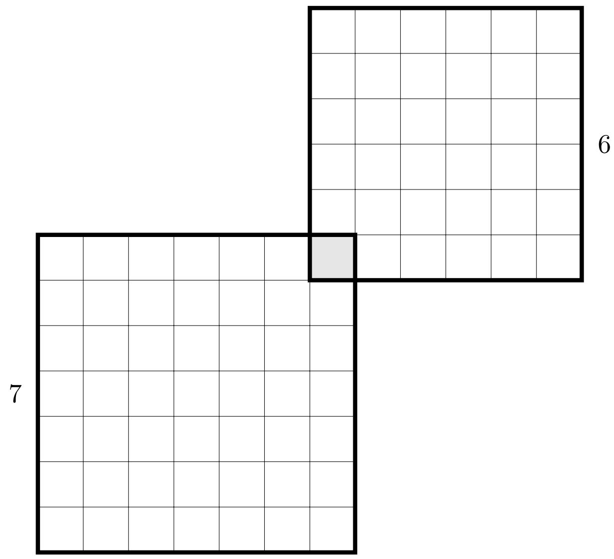
So the vertices in any trail from A to E must alternate black and white, starting with black and finishing with white. Therefore any trail from A to E must have odd length.

d From the first solution to Part **b**, there are 12 trails of length 3 starting at A. From the symmetry of the cube there are 12 trails of length 3 starting at each of the 8 vertices. Listing these gives 96 trails.

No trail of length 3 starts and finishes at the same vertex. So in our list of 96 trails, each trail appears exactly twice: once in one direction and once in the opposite direction. Hence the number of trails of length 3 in the cube is $96/2 = 48$.

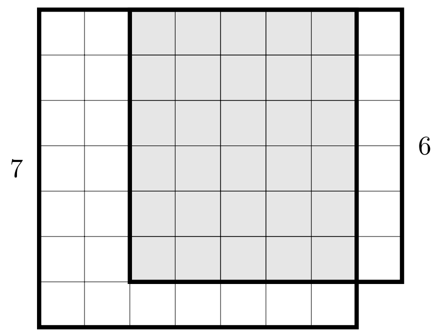
J2 Overlaps

a



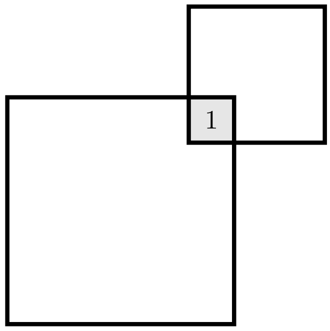
Perimeter is $(2 \times 7) + (2 \times 6) + (2 \times 6) + (2 \times 5) = 14 + 12 + 12 + 10 = 48\,\text{cm}$, or perimeter is $(4 \times 7) + (4 \times 6) - 4 = 28 + 24 - 4 = 48\,\text{cm}$.

b



Perimeter is $(2 \times 7) + 8 + 6 + 1 + 1 = 14 + 16 = 30\,\text{cm}$, or perimeter is $(4 \times 7) + (4 \times 6) - (2 \times 6) - (2 \times 5) = 28 + 24 - 12 - 10 = 30\,\text{cm}$.

c Since the overlap has area $1\,\text{cm}^2$, it must be a grid square in the corner of each overlapping square.



Since the perimeter of the final shape is 32 cm and the perimeter of the overlapping square is 4 cm, the sum of the perimeters of the original two squares is 36 cm.

Alternative i

Since the sides of a square are at least 2 cm, its perimeter is at least 8 cm. So the perimeter of the other square is at most 28 cm. Since the perimeter of a square is a multiple of 4, one square has perimeter 28, 24, 20 and the other has perimeter 8, 12, 16 respectively. So the squares are 7×7 and 2×2, or 6×6 and 3×3, or 5×5 and 4×4.

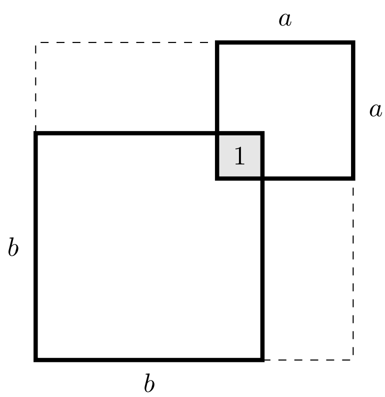
Alternative ii

Since the perimeter of a square is 4 times its side length, the sum of the perimeters of the two overlapping squares is 4 times the sum of the lengths of one side from each square. So the sum of the lengths of one side from each square is $36/4 = 9$ cm. Hence the only possible side lengths for the two overlapping squares are 2 and 7, 3 and 6, and 4 and 5 cm.

Alternative iii

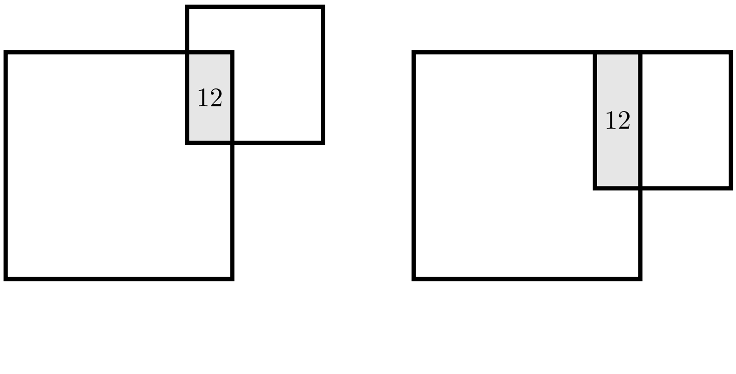
Let the squares be $a \times a$ and $b \times b$ with $a \leq b$.

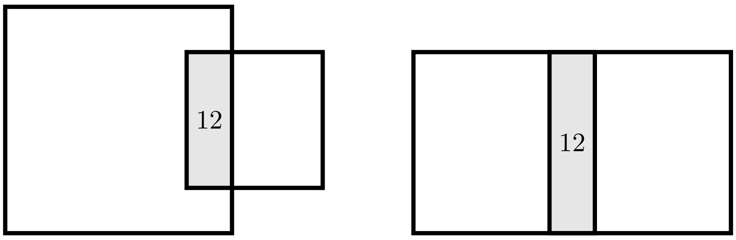
As shown in the next diagram, a shape formed from the two overlapping squares has the same perimeter as the smallest square that encloses the shape.



Hence $4(a + b - 1) = 32$. So $a + b - 1 = 8$ and $a + b = 9$. Thus $a = 2$ and $b = 7$, or $a = 3$ and $b = 6$, or $a = 4$ and $b = 5$.

d The overlap is a rectangle which is wholly inside each of the overlapping squares and along at least one side of each. Disregarding rotations and reflections, there are four cases as indicated.





Note that the perimeter of the final shape is the sum of the perimeters of the two overlapping squares minus the perimeter of the overlap.

Since the area of the overlap is 12 cm^2, the overlap in each of the diagrams above is one of the rectangles 12×1, 6×2, or 4×3.

Alternative i

If the overlap is a 12×1 rectangle, then the side length of each overlapping square is at least 12 cm. Then the perimeter of the final shape is at least $8 \times 12 - 2 \times (12+1) = 96 - 26 = 70$ cm. Since $70 > 30$, the overlap rectangle is not 12×1.

If the overlap is a 6×2 rectangle, then the side length of each overlapping square is at least 6 cm. Then the perimeter of the final shape is at least $8 \times 6 - 2 \times (6+2) = 48 - 16 = 32$ cm. Since $32 > 30$, the overlap rectangle is not 6×2.

If the overlap is a 4×3 rectangle, the sum of the perimeters of the overlapping squares is $30 + 2 \times (4+3) = 30 + 14 = 44$ cm. Hence the sum of the lengths of one side from each square is $44/4 = 11$ cm. Also the side length of each overlapping square is at least 4 cm. So the only possibilities are a 4×4 square overlapping a 7×7 square, and a 5×5 square overlapping a 6×6 square.

Alternative ii

Let the squares be $a \times a$ and $b \times b$ with $a \leq b$.

If the overlap is a 12×1 rectangle, the perimeter of the final shape is $4a + 4b - 26 = 30$ cm. So $4a + 4b = 56$ and $a + b = 14$. This is impossible since each of a and b must be at least 12 cm.

If the overlap is a 6×2 rectangle, the perimeter of the final shape is $4a + 4b - 16 = 30$ cm. So $4a + 4b = 46$, which is impossible because 4 does not divide 46.

If the overlap is a 4×3 rectangle, the perimeter of the final shape is $4a + 4b - 14 = 30$ cm. So $4a + 4b = 44$ and $a + b = 11$. Since a must be at least 4, we have $a = 4$ and $b = 7$ or $a = 5$ and $b = 6$.

So the two squares must be 4×4 and 7×7, or 5×5 and 6×6.

J3 Stocking Farms

a The capacity of Farmer Green's 100 hectare farm is $100 \times 12 = 1200$ DSEs. It is stocked with 60 adult cows or $60 \times 15 = 900$ DSEs, 40 yearling or $40 \times 6 = 240$ DSEs, and 15 calves or $15 \times 4 = 60$ DSEs. Since $900+240+60 = 1200$, Green's farm is fully stocked.

b The capacity of Farmer White's 80 hectare farm is $80 \times 12 = 960$ DSEs. The farm is carrying 50 adult cows which is $50 \times 15 = 750$ DSEs. So, to fully stock the farm, the yearlings and calves must total $960 - 750 = 210$ DSEs.

If the number of yearlings is y and the number of calves is c, then $6y + 4c = 210$ or $3y + 2c = 105$. So y must be odd. There are many possible combinations:

Yearlings	1	3	5	7	9	11	13	15	17
Calves	51	48	45	42	39	36	33	30	27

Yearlings	19	21	23	25	27	29	31	33
Calves	24	21	18	15	12	9	6	3

c The table shows the number of DSEs for the various combinations of animals.

Farm	Adult cows	Yearlings and calves
Grey's	$12 \times 15 = 180$	$18 \times 6 + 4 \times 4 = 124$
Brown's	$10 \times 15 = 150$	$4 \times 6 + 6 \times 4 = 48$

If Brown took away all his adult cows (150 DSEs), then Grey could move all her yearlings and calves (124 DSEs) to his farm. Since $124 = (8 \times 15) + 4$, Grey's farm can then accommodate only 8 of Brown's adult cows. Hence Brown must sell 2 adult cows at the market.

d If Black's farm is to remain fully stocked the number of adult cows must have the same DSE as the total DSE for the yearlings and calves. So, if the number of adult cows she sold is x and the number of calves she bought is y, we have $15x = 12 \times 6 + 4y = 72 + 4y$. Thus x is even and as small as possible. If $x = 2$ or 4, then y is negative. If $x = 6$, then y is not an integer. If $x = 8$, then $y = 12$. So the minimum number of adult cows Farmer Black could have sold is 8.

J4 Cross Number

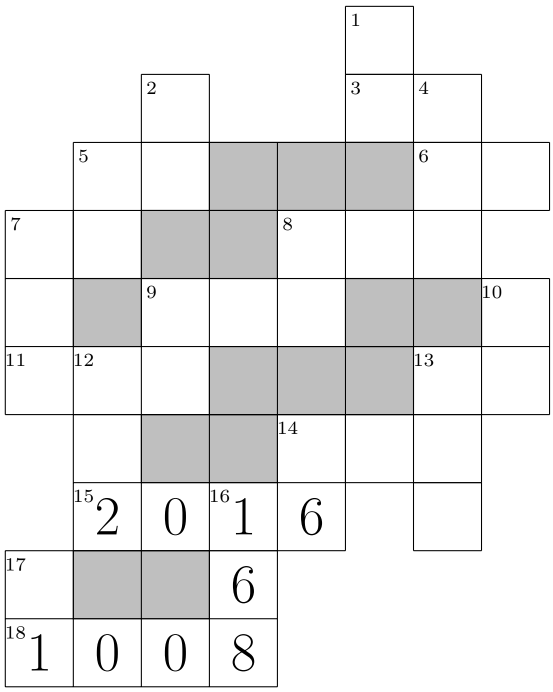
The only available numbers to fill the puzzle are:

12, 14, 18, 21, 24, 28, 32, 42, 48, 56, 63, 72, 84, 96,
112, 126, 144, 168, 224, 252, 288, 336, 504, 672,
1008, 2016.

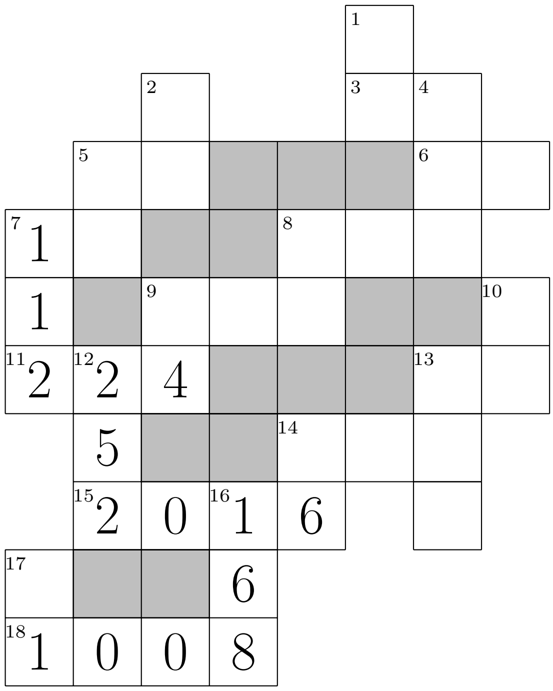
In the following, A means across and D means down.

a Number 15A must be 2016 or 1008. Since 16D cannot start with 0, 15A is 2016. So 18A is 1008. Then 168 is the only possible factor for 16D.

b From Part **a**, we have:



Now 12D is 112, 252 or 672. If it is 112, then 11A is also 112, which is not allowed. If 12D is 672, then 11A is 168, which is already used. So 12D must be 252. Hence 11A is 126 or 224. Since 7D cannot end in 1, 11A must be 224. Then 7D is 112 or 672. If 7D is 672, then 7A is 63, leaving no factor for 5D. So 7D is 112 and we have:

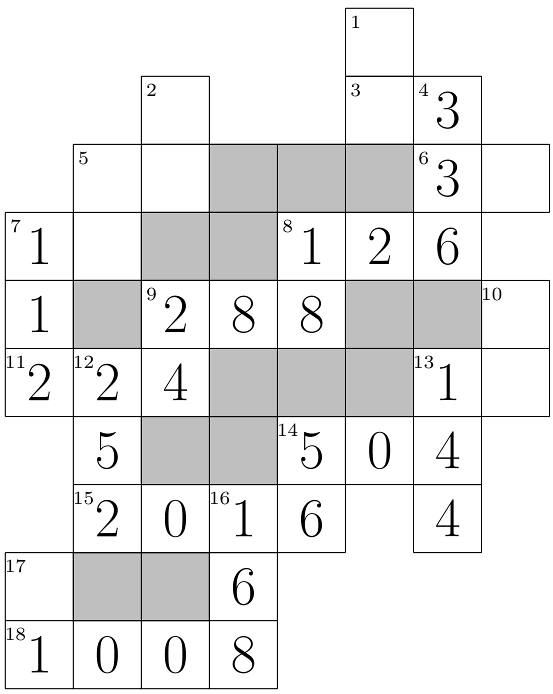


c The remaining 3-digit factors are: 126, 144, 288, 336, 504, 672.

Now 14D is 56 or 96. Since no 3-digit factor starts with 9, 14D is 56. Hence 14A is 504, the only 3-digit factor starting with 5. Then 13D is 144, the only 3-digit factor with middle digit 4.

The remaining 3-digit factors are now: 126, 288, 336, 672.

Since 8A and 4D end in the same digit, they must be 126 and 336 in some order. Hence 9A is 288 or 672. If 9A is 672, then 9D is 64 which is not a factor of 2016. So 9A is 288. Since 8A is 126 or 336 and 38 is not a factor of 2016, 8D is 18. So 8A is 126, 4D is 336, and we have:

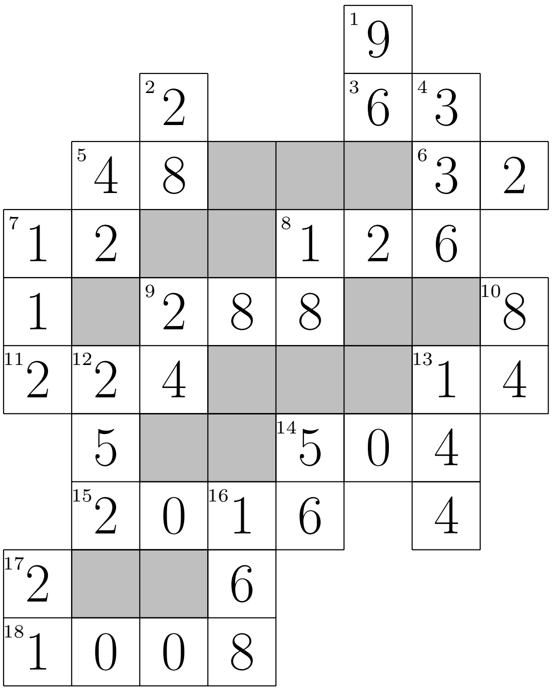


d The remaining 2-digit factors are:
12, 14, 21, 28, 32, 42, 48, 63, 72, 84, 96.

Then 17D is 21 (the only 2-digit factor ending in 1), 6A is 32 (the only 2-digit factor starting with 3), 3A is 63 (the only 2-digit factor ending in 3), and 1D is 96 (the only 2-digit factor ending in 6).

The remaining 2-digit factors are now: 12, 14, 28, 42, 48, 72, 84.

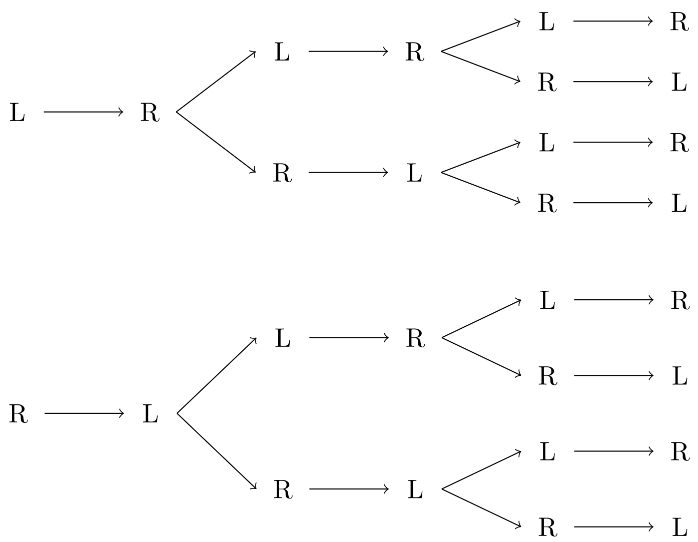
Since 5A and 5D start with the same digit, that digit is 1 or 4. Since 7A starts with 1, 5A and 5D are 42 and 48 in some order. Since 7A is 12 or 14, 5D is 42. So 7A is 12 and 5A is 48. Hence 2D is 28, 13A is 14, and 10D is 84. So we have:



J5 Tipping Points

a Alternative i

Moving through the sequence of bowls from the first to the last, the beam will tip if and only if the difference in the number of Ls and Rs is at any stage greater than 1. The following tree diagrams show the possible sequences, from left to right, of 6 bowls that avoid the beam tipping.



So the only sequences of bowls for which the beam does not tip are:

LRLRLR, LRLRRL, LRRLLR, LRRLRL,
RLLRLR, RLLRRL, RLRLLR, RLRLRL.

Alternative ii

To avoid tipping the beam, the first two bowls in the sequence must be LR or RL. Either way, the beam remains perfectly balanced. So the next two bowls in the sequence must be LR or RL. Again, either way, the beam remains perfectly balanced. So the last two bowls in the sequence must be LR or RL. Hence the only sequences of bowls for which the beam does not tip are:

LRLRLR, LRLRRL, LRRLLR, LRRLRL,
RLLRLR, RLLRRL, RLRLLR, RLRLRL.

b Since Julie uses all bowls and does not tip the beam, the bowls must be in one of the eight sequences found in Part **a**:

LRLRLR, LRLRRL, LRRLLR, LRRLRL,
RLLRLR, RLLRRL, RLRLLR, RLRLRL.

Julie uses bowls 2, 4, 6 without tipping the beam. This eliminates the sequences LRLRLR, LRLRRL, RLRLLR, RLRLRL. So she is left with the sequences LRRLLR, LRRLRL, RLLRLR, RLLRRL.

Julie uses bowls 3 and 6 without tipping the beam. This eliminates the sequences LRRLLR and RLLRRL.

So the only sequences that work for all three procedures are:
LRRLRL and RLLRLR.

c Alternative i

The beam will not tip for any m-selection with $m \geq 6$ since, in those cases, a ball is drawn from only one bowl. For each m-selection with $m \leq 5$, the first 2 bowls must be RL or LR. For $m = 1$, let the first 2 bowls be LR.

Then, for $m = 2$, bowl 4 must be L. Hence bowl 3 is R.

Bowl	1	2	3	4	5	6	7	8	9	10	11
Letter	L	R	R	L							

For $m = 3$, bowl 6 must be L. Hence bowl 5 is R.

Bowl	1	2	3	4	5	6	7	8	9	10	11
Letter	L	R	R	L	R	L					

For $m = 2$, bowl 8 must be R. Hence bowl 7 is L.

Bowl	1	2	3	4	5	6	7	8	9	10	11
Letter	L	R	R	L	R	L	L	R			

For $m = 5$, bowl 10 must be L. Hence bowl 9 is R.

Bowl	1	2	3	4	5	6	7	8	9	10	11
Letter	L	R	R	L	R	L	L	R	R	L	

Bowl 11 can be L or R. Thus we have two sequences starting with LR such that no m-selection causes the beam to tip.

Similarly, there are two sequences starting with RL such that no m-selection causes the beam to tip.

So there are four sequences of 11 bowls such that no m-selection causes the beam to tip:

LRRLRLLRRLL, LRRLRLLRRLR,
RLLRLRRLLRR, RLLRLRRLLRL.

Alternative ii

As in the second solution to Part **a**, since the first two letters in any non-tipping sequence must be different, the next two letters in the sequence must be different, then the next two and so on.

If a sequence of 11 bowls is non-tipping for every m-selection, then for $m = 1$, the 5th and 6th bowls are different and the 7th and 8th bowls are different. So these four bowls are LRLR, LRRL, RLLR, or RLRL. For $m = 2$, the 6th and 8th bowls must be different. This eliminates LRLR and RLRL, and leaves us with:

Bowl	1	2	3	4	5	6	7	8	9	10	11
Seq. 1					L	R	R	L			
Seq. 2					R	L	L	R			

For $m = 3$, the 3rd and 6th letters must be different. For $m = 4$, the 4th and 8th letters must be different. So we have:

Bowl	1	2	3	4	5	6	7	8	9	10	11
Seq. 1			L	R	L	R	R	L			
Seq. 2			R	L	R	L	L	R			

For $m = 2$, the 2nd and 4th letters must be different. For $m = 1$, the 1st and 2nd letters must be different. So we have:

Bowl	1	2	3	4	5	6	7	8	9	10	11
Seq. 1	R	L	L	R	L	R	R	L			
Seq. 2	L	R	R	L	R	L	L	R			

For $m = 5$, the 5th and 10th letters must be different. For $m = 1$, the 9th and 10th letters must be different. So we have:

Bowl	1	2	3	4	5	6	7	8	9	10	11
Seq. 1	R	L	L	R	L	R	R	L	L	R	
Seq. 2	L	R	R	L	R	L	L	R	R	L	

Finally, the 11th bowl could be either L or R. So there are four sequences of 11 bowls for which every m-selection is non-tipping:

RLLRLRRLLRR, RLLRLRRLLRL,
LRRLRLLRRLL, LRRLRLLRRLR.

d Alternative i

Suppose we have a sequence of 12 bowls for which every m-selection is non-tipping. Then, as in Part **c**, up to bowl 10 we have only two possible sequences:

Bowl	1	2	3	4	5	6	7	8	9	10	11	12
Letter	L	R	R	L	R	L	L	R	R	L		

Bowl	1	2	3	4	5	6	7	8	9	10	11	12
Letter	R	L	L	R	L	R	R	L	L	R		

For $m = 6$, bowl 12 must be R and L respectively. However, for $m = 3$, bowl 12 must be L and R respectively. So there is no sequence of 12 bowls for which every m-selection is non-tipping.

Alternative ii

As in the second solution to Part **a**, since the first two letters in any non-tipping sequence must be different, the next two letters in the sequence must be different, then the next two and so on.

Suppose we have a sequence of 12 bowls for which every m-selection is non-tipping. Then for $m = 1$, the 9th and 10th bowls are different and the 11th and 12th bowls are different. So the last 4 letters in the sequence are LRLR, LRRL, RLLR, or RLRL.

For $m = 2$, the 10th and 12th letters must be different. This eliminates LRLR and RLRL. For $m = 3$, the 9th and 12th letters must be different. This eliminates LRRL and RLLR. So there is no sequence of 12 bowls for which every m-selection is non-tipping.

J6 Tossing Counters

a Alternative i

The smallest sum, 8, is the sum of the smaller numbers on the two counters. Since $1 + 7 = 8$, the smaller numbers on the two counters are 1 and 7.

Since the sum of the larger numbers on the two counters is 11, the sum of 7 and the larger number on the first counter is 9 or 10. So the larger number on the first counter is 2 or 3.

If the first counter is 1/2, then the second counter must be 7/9. If the first counter is 1/3, then the second counter must be 7/8.

Alternative ii

The smallest sum, 8, is the sum of the smaller numbers on the two counters. The largest sum, 11, is the sum of the larger numbers on the two counters. So we have either of the following addition tables for the four numbers on the counters.

+	1	?
7	8	9
?	10	11

+	1	?
7	8	10
?	9	11

There is only one way to complete each table:

+	1	2
7	8	9
9	10	11

+	1	3
7	8	10
8	9	11

Thus the two counters are 1/2 and 7/9 or 1/3 and 7/8.

b Alternative i

The smallest sum, 7, is the sum of the smaller numbers on the two counters. The largest sum, 10, is the sum of the larger numbers on the two counters. Since $7 = 1 + 6 = 2 + 5 = 3 + 4$, the two smaller numbers on the counters are 1 and 6 or 2 and 5 or 3 and 4. Since $10 = 1+9 = 2+8 = 3+7 = 4+6$, the two larger numbers on the counters are 1 and 9 or 2 and 8 or 3 and 7 or 4 and 6. The four numbers on the two counters are all different. The table shows the only combinations we need to consider.

Smaller numbers	Larger numbers	Give sums 7, 8, 9, 10?
1 and 6	2 and 8	yes
	3 and 7	yes
2 and 5	1 and 9	no
	3 and 7	yes
	4 and 6	yes
3 and 4	1 and 9	no
	2 and 8	no

So the two counters are:
1/2 and 6/8, 1/3 and 6/7, 2/3 and 5/7, or 2/4 and 5/6.

Alternative ii

Let the numbers on one counter be a and $a + r$.
Let the numbers on the other counter be b and $b + s$.
Then $a + b = 7$ and $a + r + b + s = 10$. So $r + s = 3$.
Hence (a,b) = (1,6), (2,5), (3,4), (4,3), (5,2), or (6,1),
and (r,s) = (1,2) or (2,1).

From symmetry we may assume $r = 1$ and $s = 2$.
So the two counters are:
1/2 and 6/8, 2/3 and 5/7, 3/4 and 4/6 (disallowed), 4/5 and 3/5 (disallowed), 5/6 and 2/4, or 6/7 and 1/3.

Alternative iii

The smallest sum, 7, is the sum of the smaller numbers on the two counters. So these numbers are 1 and 6, 2 and 5, or 3 and 4. The largest sum, 10, is the sum of the larger numbers on the two counters. So we have the following addition tables for the four numbers on the counters.

+	1	?
6	7	8
?	9	10

+	1	?
6	7	9
?	8	10

+	2	?
5	7	8
?	9	10

+	2	?
5	7	9
?	8	10

+	3	?
4	7	8
?	9	10

+	3	?
4	7	9
?	8	10

There is only one way to complete each table:

+	1	2
6	7	8
8	9	10

+	1	3
6	7	9
7	8	10

+	2	3
5	7	8
7	9	10

+	2	4
5	7	9
6	8	10

+	3	4
4	7	8
6	9	10

+	3	5
4	7	9
5	8	10

We must exclude the last two tables because the four numbers on the counters must be different. So the two counters are:
1/2 and 6/8, 1/3 and 6/7, 2/3 and 5/7, or 2/4 and 5/6.

c Alternative i

Suppose the numbers on the second counter are c and d with $c < d$. The minimum sum is $4 + c$ and the maximum sum is $5 + d$. The other two sums, $4 + d$ and $5 + c$, are between these two. If these four sums form three consecutive integers, then $4 + d = 5 + c$ and $d - c = 1$. Since d is less than 10 and the numbers 4 and 5 already appear on the first counter, the second counter is 1/2, 2/3, 6/7, 7/8, or 8/9.

Alternative ii

Suppose the three sums are s, $s + 1$, $s + 2$. The smallest sum, s, is the sum of the smaller numbers on the two counters. The largest sum, $s + 2$, is the sum of the larger numbers on the two counters. Let the smaller number on the second counter be x. Then the addition table for the four numbers on the counters is:

+	4	5
x	s	$s+1$
?	?	$s+2$

So the larger number on the second counter is $x + 1$. Since each of x and $x + 1$ is less than 10 and is neither 4 nor 5, x is one of the numbers 1, 2, 6, 7, 8. So the second counter is 1/2, 2/3, 6/7, 7/8, or 8/9.

d Alternative i

Suppose the numbers on the first counter are a and b with $a < b$, and on the second counter c and d with $c < d$.

The minimum sum is $a + c$ and the maximum sum is $b + d$. The other two sums, $a + d$ and $b + c$, are between these two.

Suppose $a + d < b + c$. Since the four sums are consecutive, we have $a + d = a + c + 1$ and $b + c = a + d + 1$. Hence $d = c + 1$ and $b = a + 2$. So only one of c and d is even and a and b are either both odd or both even.

Similarly, if $b + c < a + d$, then only one of a and b is even and c and d are either both odd or both even.

Thus either one or three of a, b, c, d are even.

Alternative ii

Suppose the numbers on the first counter are a and b with $a < b$, and on the second counter c and d with $c < d$.

The minimum sum is $a+c$ and the maximum sum is $b+d$. Since the four sums are consecutive, we have $b+d = a+c+3$.

If $a + c$ is odd, then $b + d$ is even. So one of a and c is even and neither or both of b and d are even.

If $a + c$ is even, then $b + d$ is odd. So one of b and d is even and neither or both of a and c are even.

In both cases one or three of a, b, c, d are even.

Alternative iii

Suppose the four sums are s, $s + 1$, $s + 2$, $s + 3$. The smallest sum, s, is the sum of the smaller numbers on the two counters. The largest sum, $s + 3$, is the sum of the larger numbers on the two counters. So we can arrange the addition table for the four numbers on the counters as follows:

+	?	?
?	s	$s+1$
?	$s+2$	$s+3$

Since s is either even or odd, we have:

+	x	?
?	even	odd
?	even	odd

+	x	?
?	odd	even
?	odd	even

Since x is either even or odd, we can complete each of these tables in two ways:

+	even	odd
even	even	odd
even	even	odd

+	even	odd
odd	odd	even
odd	odd	even

+	odd	even
odd	even	odd
odd	even	odd

+	odd	even
even	odd	even
even	odd	even

In each case either one or three of the four numbers on the counters are even.

Alternative iv

If all four numbers on the counters were even or all four were odd, then all sums would be even and therefore not consecutive.

Suppose two numbers on the counters were even and the other two odd.

If the two even numbers were on the same counter, then all sums would be odd and therefore not consecutive. So each counter must have an even and an odd number.

If the two smaller numbers on the counters were odd, then the two larger numbers would be even. Then the lowest and highest sums would both be even and the four sums would not be consecutive. Similarly, the two smaller numbers on the counters cannot be even.

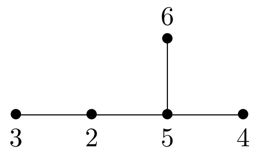
If one of the smaller numbers on the counters was odd and the other smaller number was even, then one of the larger numbers would be even and the other odd. Then the lowest and highest sums would both be odd and the four sums would not be consecutive.

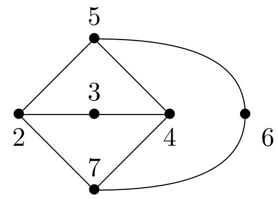
So either one or three of the four numbers on the counters are even.

CHALLENGE SOLUTIONS – INTERMEDIATE

I1 COP-graphs

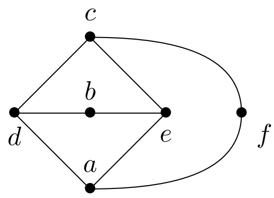
a Here are two COP-graphs with the required labels. There are other ways to place the labels.





b Alternative i

Amongst the integers 5, 6, 7, 8, 9, 10, only 5 and 7 are coprime to 6. So the vertex with label 6 must be joined to at most two other vertices. Hence, only vertex b or f can have label 6.



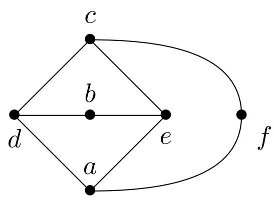
If vertex f has label 6, then vertices a and c have labels 5 and 7. So vertex b has label 10. Then neither of the remaining vertices d and e can have label 8.

If vertex b has label 6, then vertices d and e have labels 5 and 7. So vertex f has label 10. Then neither of the remaining vertices a and c can have label 8.

Hence COP-graph B cannot remain a COP-graph if its labels are replaced with integers 5 to 10.

Alternative ii

Amongst the integers 5, 6, 7, 8, 9, 10, there are three even numbers and three odd numbers. No two vertices with even labels can be joined by an edge. So 6, 8, 10 must be assigned in some order to vertices a, b, c or vertices d, e, f.



If 6, 8, 10 are assigned to vertices a, b, c, then either 5 or 9 is assigned to one of d and e. Since 5 and 10 are not coprime and 9 and 6 are not coprime, the labelling does not give a COP-graph.

If 6, 8, 10 are assigned to vertices d, e, f, then either 5 or 9 is assigned to one of a and c. Again, the labelling does not give a COP-graph.

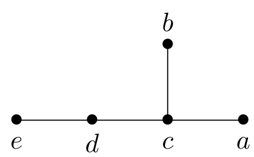
Hence COP-graph B cannot remain a COP-graph if its labels are replaced with integers 5 to 10.

c Let a, b, c, d, e be consecutive positive integers. Any two consecutive integers are coprime and any two consecutive odd integers are coprime.

If a is odd, we have the following table which has each integer in the top row and the integers coprime to it in the bottom row.

a (odd)	b (even)	c (odd)	d (even)	e (odd)
b, c	a, c	a, b, d, e	c, e	c, d

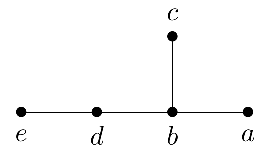
So we have the following COP-graph.



If a is even, we have the following table which has each integer in the top row and the integers coprime to it in the bottom row.

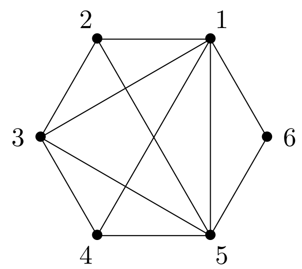
a (even)	b (odd)	c (even)	d (odd)	e (even)
b	a, c, d	b, d	b, c, e	d

So we have the following COP-graph.



d Alternative i

By joining each vertex to every other vertex, we can see that a graph with 6 vertices has at most 15 edges. We cannot have edges {2,4}, {2,6}, {4,6} and {3,6}. That eliminates 4 edges. Here is a COP-graph with 6 vertices, 11 edges, and labels 1 to 6.



(This graph can be drawn in many ways, for example, without edges crossing by placing edge {3,4} inside triangle {1,2,5}.) So the maximum number of edges in a COP-graph that has 6 vertices and labels 1 to 6 is 11.

Alternative ii

The following table lists all the integers from 1 to 6 that are coprime to each integer from 1 to 6.

1	2	3	4	5	6
2, 3, 4, 5, 6	1, 3, 5	1, 2, 4, 5	1, 3, 5	1, 2, 3, 4, 6	1, 5

Thus, amongst the integers 1 to 6 there are exactly 11 pairs that are coprime: {1,2}, {1,3}, {1,4}, {1,5}, {1,6}, {2,3}, {2,5}, {3,4}, {3,5}, {4,5}, {5,6}. The COP-graph in Alternative i displays these edges. So 11 is the maximum number of edges in a COP-graph that has 6 vertices and labels 1 to 6.

I2 Tipping Points

See Junior Problem 5.

I3 Tossing Counters

a Alternative i

The smallest sum, 8, is the sum of the smaller numbers on the two counters. The largest sum, 11, is the sum of the larger numbers on the two counters. Since $8 = 1 + 7 = 2 + 6 = 3 + 5$, the two smaller numbers on the counters are 1 and 7 or 2 and 6 or 3 and 5. Since $11 = 2+9 = 3+8 = 4+7 = 5+6$, the two larger numbers on the counters are 2 and 9 or 3 and 8 or 4 and 7 or 5 and 6. The four numbers on the two counters are all different. The table shows the only combinations we need to consider.

Smaller numbers	Larger numbers	Give sums 8, 9, 10, 11?
1 and 7	2 and 9	yes
	3 and 8	yes
	5 and 6	no
2 and 6	3 and 8	yes
	4 and 7	yes
3 and 5	2 and 9	no
	4 and 7	yes

So the two counters are:
1/2 and 7/9, 1/3 and 7/8, 2/3 and 6/8, 2/4 and 6/7, or 3/4 and 5/7.

Alternative ii

Let the numbers on one counter be a and $a + r$.
Let the numbers on the other counter be b and $b + s$.
Then $a + b = 8$ and $a + r + b + s = 11$. So $r + s = 3$.
Hence (a,b) = (1,7), (2,6), (3,5), (5,3), (6,2), or (7,1),
and (r,s) = (1,2) or (2,1).

From symmetry we may assume $r = 1$ and $s = 2$.
So the two counters are:
1/2 and 7/9, 2/3 and 6/8, 3/4 and 5/7, 5/6 and 3/5 (disallowed), 6/7 and 2/4, or 7/8 and 1/3.

Alternative iii

The smallest sum, 8, is the sum of the smaller numbers on the two counters. So these numbers are 1 and 7, 2 and 6, or 3 and 5. The largest sum, 11, is the sum of the larger numbers on the two counters. So we have the following addition tables for the four numbers on the counters.

+	1	?
7	8	9
?	10	11

+	1	?
7	8	10
?	9	11

+	2	?
6	8	9
?	10	11

+	2	?
6	8	10
?	9	11

+	3	?
5	8	9
?	10	11

+	3	?
5	8	10
?	9	11

There is only one way to complete each table:

+	1	2
7	8	9
9	10	11

+	1	3
7	8	10
8	9	11

+	2	3
6	8	9
8	10	11

+	2	4
6	8	10
7	9	11

+	3	4
5	8	9
7	10	11

+	3	5
5	8	10
6	9	11

We must exclude the last table because the four numbers on the counters must be different. So the two counters are 1/2 and 7/9, 1/3 and 7/8, 2/3 and 6/8, 2/4 and 6/7, or 3/4 and 5/7.

b Alternative i

Suppose the numbers on the second counter are c and d with $c < d$. The minimum sum is $4 + c$ and the maximum sum is $5 + d$. The other two sums, $4 + d$ and $5 + c$, are between these two. If these four sums form three consecutive integers, then $4 + d = 5 + c$ and $d - c = 1$. Since d is less than 10 and the numbers 4 and 5 already appear on the first counter, the second counter is 1/2, 2/3, 6/7, 7/8, or 8/9.

Alternative ii

Suppose the three sums are s, $s + 1$, $s + 2$. The smallest sum, s, is the sum of the smaller numbers on the two counters. The largest sum, $s + 2$, is the sum of the larger numbers on the two counters. Let the smaller number on the second counter be x. Then the addition table for the four numbers on the counters is:

+	4	5
x	s	$s+1$
?	?	$s+2$

So the larger number on the second counter is $x + 1$. Since each of x and $x + 1$ is less than 10 and is neither 4 nor 5, x is one of the numbers 1, 2, 6, 7, 8. So the second counter is 1/2, 2/3, 6/7, 7/8, or 8/9.

c Alternative i

Suppose the numbers on the first counter are a and b with $a < b$, and on the second counter c and d with $c < d$.

The minimum sum is $a + c$ and the maximum sum is $b + d$. The other two sums, $a + d$ and $b + c$, are between these two.

Suppose $a + d < b + c$. Since the four sums are consecutive, we have $a + d = a + c + 1$ and $b + c = a + d + 1$. Hence $d = c + 1$ and $b = a + 2$. So only one of c and d is even and a and b are either both odd or both even.

Similarly, if $b + c < a + d$, then only one of a and b is even and c and d are either both odd or both even.

Thus either one or three of a, b, c, d are even.

Alternative ii

Suppose the numbers on the first counter are a and b with $a < b$, and on the second counter c and d with $c < d$.

The minimum sum is $a+c$ and the maximum sum is $b+d$. Since the four sums are consecutive, we have $b+d = a+c+3$.

If $a + c$ is odd, then $b + d$ is even. So one of a and c is even and neither or both of b and d are even.

If $a + c$ is even, then $b + d$ is odd. So one of b and d is even and neither or both of a and c are even.

In both cases one or three of a, b, c, d are even.

Alternative iii

Suppose the four sums are s, $s + 1$, $s + 2$, $s + 3$. The smallest sum, s, is the sum of the smaller numbers on the two counters. The largest sum, $s + 3$, is the sum of the larger numbers on the two counters. So we can arrange the addition table for the four numbers on the counters as follows:

+	?	?
?	s	$s+1$
?	$s+2$	$s+3$

Since s is either even or odd, we have:

+	x	?
?	even	odd
?	even	odd

+	x	?
?	odd	even
?	odd	even

Since x is either even or odd, we can complete each of these tables in two ways:

+	even	odd
even	even	odd
even	even	odd

+	even	odd
odd	odd	even
odd	odd	even

+	odd	even
odd	even	odd
odd	even	odd

+	odd	even
even	odd	even
even	odd	even

In each case either one or three of the four numbers on the counters are even.

Alternative iv

If all four numbers on the counters were even or all four were odd, then all sums would be even and therefore not consecutive.

Suppose two numbers on the counters were even and the other two odd.

If the two even numbers were on the same counter, then all sums would be odd and therefore not consecutive. So each counter must have an even and an odd number.

If the two smaller numbers on the counters were odd, then the two larger numbers would be even. Then the lowest and highest sums would both be even and the four sums would not be consecutive. Similarly, the two smaller numbers on the counters cannot be even.

If one of the smaller numbers on the counters was odd and the other smaller number was even, then one of the larger numbers would be even and the other odd. Then the lowest and highest sums would both be odd and the four sums would not be consecutive.

So either one or three of the four numbers on the counters are even.

d In Part **a** we found that to get the specified four consecutive sums from two counters, the difference of the numbers on one counter was 1 and the difference for the other counter was 2.

If the counters are so numbered, then they will always produce four consecutive sums. To see why, suppose counter 1 is numbered a and $a+1$ and counter 2 is numbered b and $b+2$. Then the sums will be $a+b$, $a+b+1$, $a+b+2$, $a+b+3$.

If a third counter is introduced and its two numbers differ by 4, say c and $c+4$, then the three counters will produce 8 consecutive sums: $a+b+c$ to $a+b+c+7$.

If a fourth counter is introduced and its two numbers differ by 8, say d and $d+8$, then the four counters will produce 16 consecutive sums: $a+b+c+d$ to $a+b+c+d+15$.

For example, number counter 1 with 1/2, counter 2 with 3/5, counter 3 with 4/8, and counter 4 with 6/14.

I4 Ionofs

a The factors of 36 are: 1, 2, 3, 4, 6, 9, 12, 18, 36.
Hence Ionof(36) $= 36/9 = 4$.

b The factors of pq are: 1, p, q, pq.
Hence Ionof(pq) $= pq/4$.

The only even prime is 2. Hence at most one of p and q is 2 and the other is odd. Therefore 4 does not divide pq and Ionof(pq) is not an integer.

c There are 10 factors of pq^4: 1, p, q, q^2, q^3, q^4, pq, pq^2, pq^3, pq^4.
So Ionof(pq^4)$=pq^4/10$.

Since the only prime factors of 10 are 2 and 5 and we want 10 to divide pq^4, we must have $p=2$ and $q=5$ or $p=5$ and $q=2$.

So $pq^4 = 2\times 5^4 = 1250$ or $pq^4 = 5\times 2^4 = 80$.

d Suppose p is prime, and $p^2 = \text{Ionof}(m)$ for some integer m. Let k be the number of factors of m. Then $m = kp^2$. Since 1, p, p^2 are factors of m, we know $k \geq 3$.

In the following table, we check integers of the form kp^2 to see if $\text{Ionof}(kp^2) = p^2$. To simplify calculations, for each value of k we choose a prime p that is not a factor of k.

m	factors of m	Ionof(m)
$3p^2$	1, 3, p, $3p$, p^2, $3p^2$	$3p^2/6 = p^2/2$
$4p^2$	1, 2, 4, p, $2p$, $4p$, p^2, $2p^2$, $4p^2$	$3p^2/9 = p^2/3$
$5p^2$	1, 5, p, $5p$, p^2, $5p^2$	$5p^2/6$
$6p^2$	1, 2, 3, 6, p, $2p$, $3p$, $6p$, p^2, $2p^2$, $3p^2$, $6p^2$	$6p^2/12 = p^2/2$
$7p^2$	1, 7, p, $7p$, p^2, $7p^2$	$7p^2/6$
$8p^2$	1, 2, 4, 8, p, $2p$, $4p$, $8p$, p^2, $2p^2$, $4p^2$, $8p^2$	$8p^2/12 = 2p^2/3$
$9p^2$	1, 3, 9, p, $3p$, $9p$, p^2, $3p^2$, $9p^2$	$9p^2/9 = p^2$

Thus, if p is a prime and not 3, then $\text{Ionof}(9p^2) = p^2$.

We now want $m = k3^2$ with k equal to the number of factors of m. Again, $k \geq 3$. The next table is a check list for possible k.

k	prime factorisation of $m = k9$	f = number of factors of m	$f = k$?
3	3^3	4	no
4	$2^2 \times 3^2$	9	no
5	5×3^2	6	no
6	2×3^3	8	no
7	7×3^2	6	no
8	$2^3 \times 3^2$	12	no
9	3^4	5	no
10	$2 \times 5 \times 3^2$	12	no
11	11×3^2	6	no
12	$2^2 \times 3^3$	12	yes

So $\text{Ionof}(12 \times 3^2) = 3^2$.

Thus the square of any prime number is the Ionof of some integer.

I5 Cube Trails

a At D the choices are the horizontal edge DC and the vertical edge DH. So the probability of choosing DH is $\frac{2}{3}$.

At E the choices are the horizontal edge EF and the horizontal edge EH. So the probability of choosing EH is $\frac{1}{2}$.

b Starting at A there is a choice of vertical edge AE and horizontal edges AB and AD. So the probability of choosing AB is $\frac{1}{6}$.

Then at B the choices are vertical edge BF and horizontal edge BC. So the probability of choosing BC is $\frac{1}{3}$.

Then at C the choices are vertical edge CG and horizontal edge CD. So the probability of choosing CG is $\frac{2}{3}$.

So the probability the robot traces the trail $ABCG$ is $\frac{1}{6} \times \frac{1}{3} \times \frac{2}{3} = \frac{1}{27}$.

c There are six trails of length 3 from A to G:
$ABCG$ and $ADCG$, which both have the edge sequence horizontal, horizontal, vertical;
$ABFG$ and $ADHG$, which both have the edge sequence horizontal, vertical, horizontal;
$AEFG$ and $AEHG$, which both have the edge sequence vertical, horizontal, horizontal.

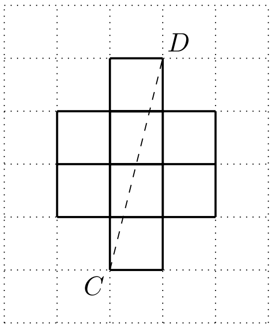
The probability that the robot traces trail $ABFG$ is $\frac{1}{6} \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{18}$.

The probability that the robot traces trail $AEFG$ is $\frac{2}{3} \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{9}$.

d From the solutions to Parts **b** and **c**, the probability of the robot tracing a path of length 3 to G is $2 \times (\frac{1}{27} + \frac{1}{18} + \frac{1}{9}) = 2 \times (\frac{2}{54} + \frac{3}{54} + \frac{6}{54}) = 2 \times \frac{11}{54} = \frac{11}{27}$.

I6 Coverem

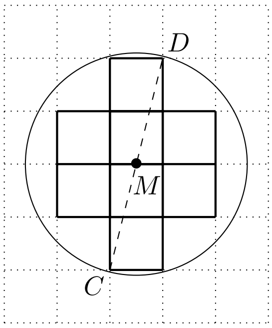
a The disc must cover the line segment CD.



From Pythagoras' theorem, $CD^2 = 1^2 + 4^2 = 17$. So the diameter of the disc must be at least $\sqrt{17}$.

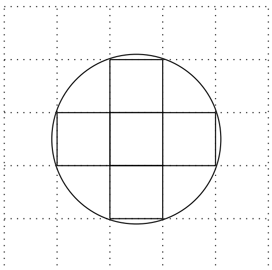
Now we ask: is there a disc with diameter $\sqrt{17}$ that will cover all 8 grid squares?

A disc with its centre at the midpoint M of CD and diameter $\sqrt{17}$ will cover all 8 grid squares, as this diagram shows.



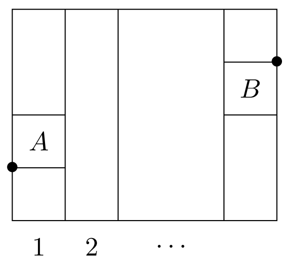
So the smallest diameter for a disc that will completely cover all 8 grid squares is $\sqrt{17}$.

b All five grid squares in the following group can be covered by a disc of diameter $\sqrt{1+3^2} = \sqrt{10}$.



Now we ask: is there some other group of five grid squares that can be covered by a disc with diameter smaller than $\sqrt{10}$?

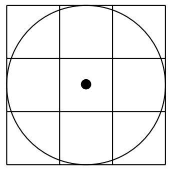
Any set of 5 grid squares can be enclosed in a grid rectangle of sufficient size. Reducing the rectangle if necessary, we may assume that the first and last grid rows and columns of the rectangle each contain at least one of the grid squares. The rectangle must have at least 3 columns or 3 rows. Rotating the grid if necessary, we may assume that the rectangle has at least 3 columns. This is shown in the next diagram, where A and B are two of the grid squares that have to be covered by the disc.



Since both dots must be covered by the disc, the disc diameter is at least $\sqrt{1+3^2} = \sqrt{10}$.

So the smallest diameter for a disc that can cover a group of 5 grid squares is $\sqrt{10}$.

c If the centre of a disc of diameter 3 is at the centre of a grid square, then it is the only grid square completely covered by the disc as this diagram shows.

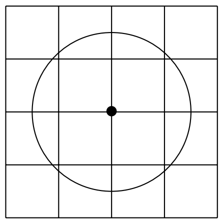


Now we ask: is it possible to locate a disc of diameter 3 so that no grid square is completely covered by the disc.

The centre of the disc lies somewhere on a grid square. The longest line in a square is its diagonal, which has length $\sqrt{1^2+1^2}=\sqrt{2}$. So the distance from the disc centre to any point on the perimeter of the square is at most $\sqrt{2}<1.5$. Hence the disc will cover that square. So a disc of diameter 3 always covers at least one grid square.

Therefore the minimum number of grid squares that can be covered by a disc of diameter 3 is 1.

Suppose the centre of the disc is at a grid point. The diagonal of a grid square is $\sqrt{1^2+1^2}=\sqrt{2}<1.5$. Hence the disc covers 4 grid squares as this diagram shows.

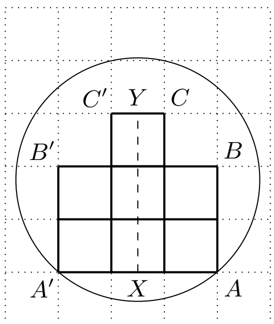


From Part **b**, the diameter of a disc that covers five grid squares is at least $\sqrt{10}$. Since $\sqrt{10}>3$, five grid squares cannot be covered by a disc of diameter 3.

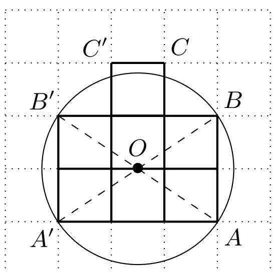
Hence 4 is the maximum number of grid squares that can be covered by a disc of diameter 3.

d Alternative i

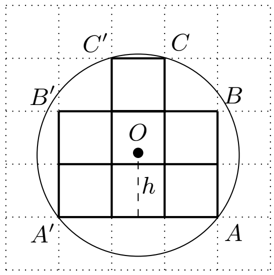
Consider the smallest disc that covers the given figure and has its centre on XY, the vertical line of symmetry of the figure. Moving the disc if necessary while keeping its centre on XY, we may assume that the rim of the disc covers vertices A and A'.



Then reducing the disc radius if necessary while ensuring its rim continues to cover A and A', we may assume the rim of the disc also covers B and B' or C and C'. If the rim of the disc covers B and B', then its centre O is located as shown.



Then its radius is $\sqrt{1^2+(3/2)^2}=\sqrt{3.25}$ and $OC=\sqrt{(1/2)^2+2^2}=\sqrt{4.25}$, which places C outside the disc. So the rim of the disc covers A, A', C and C'. Let the distance from O to AA' be h.



Since $OA = OC$ we have

$$h^2 + (3/2)^2 = (3 - h)^2 + (1/2)^2 = h^2 - 6h + 9 + 1/4.$$

So $6h = 9 - 2 = 7$ and $h = 7/6$. The radius of that disc is $OA = \sqrt{(7/6)^2 + (3/2)^2} = \sqrt{(49/36) + (9/4)} = \sqrt{130/36} = \sqrt{130}/6$.

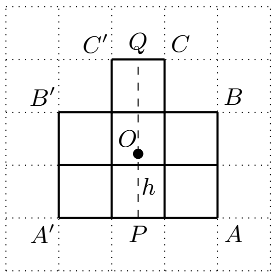
Now we ask: is there a disc whose radius is less than $\sqrt{130}/6$ that covers all seven grid squares.

If there is, than the disc has its centre off the symmetry line XY, say to the left. If A lies under this disc, then its centre must be below the horizontal line through O. If C lies under this disc, then its centre must be above the horizontal line through O. This is a contradiction. Hence there is no disc whose radius is less than $\sqrt{130}/6$ that covers all seven grid squares.

So the smallest disc that covers the given figure has its centre on XY, its rim covers A, A', C, C', and it has radius $\sqrt{130}/6$.

Alternative ii

Let PQ be the vertical line of symmetry of the figure. Let O be a point on PQ and $PO = h$.



If $OA = OC$, then

$$h^2 + (3/2)^2 = (3 - h)^2 + (1/2)^2 = h^2 - 6h + 9 + 1/4.$$

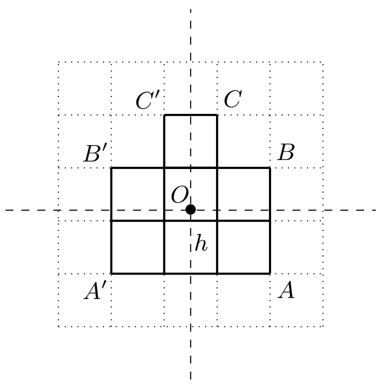
So $6h = 9 - 2 = 7$, $h = 7/6$, and

$$OA = \sqrt{(7/6)^2 + (3/2)^2} = \sqrt{(49/36) + (9/4)} = \sqrt{130/36} = \sqrt{130}/6.$$

Since $OB < OA$, the disc with centre O and radius $\sqrt{130}/6$ covers the seven grid squares.

Now we ask: is there a disc with radius less than $\sqrt{130}/6$ that covers all seven grid squares?

Draw cartesian axes on the figure with the origin at O.



Suppose there is a disc that covers the seven grid squares and has centre O' different from O.

If O' is on the horizontal axis, then either $O'A' > OA'$ or $O'A > OA$.
If O' is on the vertical axis, then either $O'A > OA$ or $O'C > OC$.

If O' is in the first (top-right) quadrant, then $O'A' > OA'$.
If O' is in the second (top-left) quadrant, then $O'A > OA$.
If O' is in the third (bottom-left) quadrant, then $O'C > OC$.
If O' is in the fourth (bottom-right) quadrant, then $O'C' > OC'$.

In each case the radius of the disc must be greater than $\sqrt{130}/6$.

So the smallest radius for a disc that covers the seven grid squares is $\sqrt{130}/6$.

CHALLENGE STATISTICS – MIDDLE PRIMARY

Mean Score/School Year/Problem

Year	Number of Students	Mean				
		Overall	Problem			
			1	2	3	4
3	537	8.8	2.5	2.5	2.1	2.2
4	927	10.9	3.0	2.9	2.5	2.7
*ALL YEARS	1472	10.1	2.8	2.8	2.4	2.5

Please note:* This total includes students who did not provide their school year.

Score Distribution %/Problem

Score	Challenge Problem			
	1 Hexos	2 Cupcakes	3 Kimmi Dolls	4 Cube Trails
Did not attempt	2%	1%	4%	6%
0	5%	5%	9%	11%
1	14%	10%	16%	11%
2	16%	20%	24%	18%
3	26%	34%	28%	26%
4	38%	30%	20%	27%
Mean	2.8	2.8	2.4	2.5
Discrimination Factor	0.6	0.6	0.7	0.8

Please note:

The discrimination factor for a particular problem is calculated as follows:

(1) The students are ranked in regard to their overall scores.

(2) The mean score for the top 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the ‘mean top score’.

(3) The mean score for the bottom 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the ‘mean bottom score’.

(4) The discrimination factor = $\frac{\text{mean top score} - \text{mean bottom score}}{4}$

Thus the discrimination factor ranges from 1 to –1. A problem with a discrimination factor of 0.4 or higher is considered to be a good discriminator.

CHALLENGE STATISTICS – UPPER PRIMARY

Mean Score/School Year/Problem

Year	Number of Students	Mean				
		Overall	Problem			
			1	2	3	4
5	1401	9.0	2.0	2.3	2.6	2.3
6	1943	10.2	2.3	2.6	2.8	2.6
7	109	12.0	2.9	3.2	3.2	2.9
*ALL YEARS	3472	9.8	2.2	2.5	2.8	2.5

Please note:* This total includes students who did not provide their school year.

Score Distribution %/Problem

Score	Challenge Problem			
	1 Knightlines	2 Grandma's Eye Drops	3 Cube Trails	4 Magic Staircase
Did not attempt	2%	1%	2%	3%
0	20%	7%	5%	6%
1	17%	15%	10%	14%
2	13%	22%	19%	28%
3	19%	26%	33%	28%
4	29%	28%	31%	21%
Mean	2.2	2.5	2.8	2.5
Discrimination Factor	0.8	0.6	0.6	0.6

Please note:

The discrimination factor for a particular problem is calculated as follows:

(1) The students are ranked in regard to their overall scores.

(2) The mean score for the top 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the 'mean top score'.

(3) The mean score for the bottom 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the 'mean bottom score'.

(4) The discrimination factor = $\frac{\text{mean top score} - \text{mean bottom score}}{4}$

Thus the discrimination factor ranges from 1 to –1. A problem with a discrimination factor of 0.4 or higher is considered to be a good discriminator.

CHALLENGE STATISTICS – JUNIOR

Mean Score/School Year/Problem

Year	Number of Students	Mean						
		Overall	Problem					
			1	2	3	4	5	6
7	2978	12.6	2.3	2.6	2.6	2.6	1.8	2.0
8	2638	14.6	2.7	2.8	2.9	2.9	2.2	2.3
*ALL YEARS	5640	13.5	2.5	2.7	2.7	2.8	2.0	2.1

Please note:* This total includes students who did not provide their school year.

Score Distribution %/Problem

Score	Challenge Problem					
	1 Cube Trails	2 Overlaps	3 Stocking Farms	4 Cross Number	5 Tipping Points	6 Tossing Counters
Did not attempt	2%	4%	8%	11%	16%	15%
0	4%	4%	6%	8%	16%	15%
1	15%	6%	9%	9%	15%	14%
2	28%	31%	19%	14%	21%	17%
3	32%	26%	27%	22%	16%	20%
4	20%	28%	31%	36%	16%	18%
Mean	2.5	2.7	2.7	2.8	2.0	2.1
Discrimination Factor	0.5	0.5	0.7	0.8	0.8	0.8

Please note:

The discrimination factor for a particular problem is calculated as follows:

(1) The students are ranked in regard to their overall scores.

(2) The mean score for the top 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the 'mean top score'.

(3) The mean score for the bottom 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the 'mean bottom score'.

(4) The discrimination factor = $\dfrac{\text{mean top score} - \text{mean bottom score}}{4}$

Thus the discrimination factor ranges from 1 to –1. A problem with a discrimination factor of 0.4 or higher is considered to be a good discriminator.

CHALLENGE STATISTICS – INTERMEDIATE

Mean Score/School Year/Problem

Year	Number of Students	Mean						
		Overall	Problem					
			1	2	3	4	5	6
9	1787	12.9	2.8	2.6	2.2	2.2	2.6	1.6
10	1061	14.5	2.9	2.8	2.4	2.4	2.9	2.0
*ALL YEARS	2877	13.6	2.9	2.7	2.3	2.3	2.7	1.8

Please note:* This total includes students who did not provide their school year.

Score Distribution %/Problem

Score	Challenge Problem					
	1 Indim Integers	2 Digital Sums	3 Coin Flips	4 Jogging	5 Folding Fractions	6 Crumbling Cubes
Did not attempt	2%	5%	6%	5%	10%	16%
0	4%	9%	13%	5%	9%	23%
1	11%	9%	13%	22%	11%	13%
2	18%	20%	22%	28%	11%	18%
3	28%	20%	25%	23%	25%	19%
4	37%	37%	21%	17%	34%	10%
Mean	2.9	2.7	2.3	2.3	2.7	1.8
Discrimination Factor	0.6	0.7	0.7	0.6	0.8	0.7

Please note:

The discrimination factor for a particular problem is calculated as follows:

(1) The students are ranked in regard to their overall scores.

(2) The mean score for the top 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the 'mean top score'.

(3) The mean score for the bottom 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the 'mean bottom score'.

(4) The discrimination factor = $\frac{\text{mean top score} - \text{mean bottom score}}{4}$

Thus the discrimination factor ranges from 1 to –1. A problem with a discrimination factor of 0.4 or higher is considered to be a good discriminator.

AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD

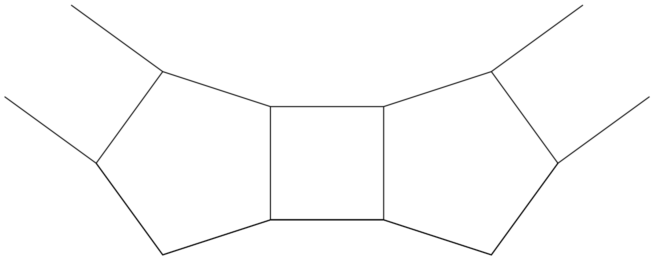
Time allowed: 4 hours. NO calculators are to be used.

Questions 1 to 8 only require their numerical answers all of which are non-negative integers less than 1000.
Questions 9 and 10 require written solutions which may include proofs.
The bonus marks for the Investigation in Question 10 may be used to determine prize winners.

1. Find the smallest positive integer x such that $12x = 25y^2$, where y is a positive integer. [**2 marks**]

2. A 3-digit number in base 7 is also a 3-digit number when written in base 6, but each digit has increased by 1. What is the largest value which this number can have when written in base 10? [**2 marks**]

3. A ring of alternating regular pentagons and squares is constructed by continuing this pattern.



How many pentagons will there be in the completed ring? [**3 marks**]

4. A sequence is formed by the following rules: $s_1 = 1, s_2 = 2$ and $s_{n+2} = s_n^2 + s_{n+1}^2$ for all $n \geq 1$.

What is the last digit of the term s_{200}? [**3 marks**]

5. Sebastien starts with an 11×38 grid of white squares and colours some of them black. In each white square, Sebastien writes down the number of black squares that share an edge with it.

Determine the maximum sum of the numbers that Sebastien could write down. [**3 marks**]

6. A circle has centre O. A line PQ is tangent to the circle at A with A between P and Q. The line PO is extended to meet the circle at B so that O is between P and B. $\angle APB = x^\circ$ where x is a positive integer. $\angle BAQ = kx^\circ$ where k is a positive integer. What is the maximum value of k? [**4 marks**]

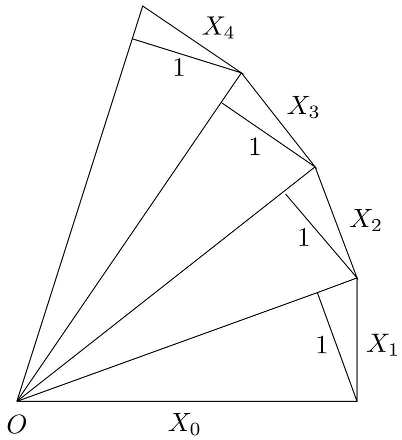
7. Let n be the largest positive integer such that $n^2 + 2016n$ is a perfect square. Determine the remainder when n is divided by 1000. [**4 marks**]

8. Ann and Bob have a large number of sweets which they agree to share according to the following rules. Ann will take one sweet, then Bob will take two sweets and then, taking turns, each person takes one more sweet than what the other person just took. When the number of sweets remaining is less than the number that would be taken on that turn, the last person takes all that are left. To their amazement, when they finish, they each have the same number of sweets.

They decide to do the sharing again, but this time, they first divide the sweets into two equal piles and then they repeat the process above with each pile, Ann going first both times. They still finish with the same number of sweets each.

What is the maximum number of sweets less than 1000 they could have started with? [**4 marks**]

9. All triangles in the spiral below are right-angled. The spiral is continued anticlockwise.



Prove that $X_0^2 + X_1^2 + X_2^2 + \cdots + X_n^2 = X_0^2 \times X_1^2 \times X_2^2 \times \cdots \times X_n^2$. [**5 marks**]

10. For $n \geq 3$, consider $2n$ points spaced regularly on a circle with alternate points black and white and a point placed at the centre of the circle.

The points are labelled $-n$, $-n+1$, ... , $n-1$, n so that:
(a) the sum of the labels on each diameter through three of the points is a constant s, and
(b) the sum of the labels on each black-white-black triple of consecutive points on the circle is also s.

Show that the label on the central point is 0 and $s = 0$. [**5 marks**]

Investigation

Show that such a labelling exists if and only if n is even. [**3 bonus marks**]

AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD SOLUTIONS

1. *Method 1*

We have $2^2 \times 3x = 5^2y^2$ where x and y are integers. So 3 divides y^2.
Since 3 is prime, 3 divides y.

Hence 3 divides x. Also 25 divides x. So the smallest value of x is $3 \times 25 = \mathbf{75}$.

Method 2

The smallest value of x will occur with the smallest value of y.
Since 12 and 25 are relatively prime, 12 divides y^2.

The smallest value of y for which this is possible is $y = 6$.
So the smallest value of x is $(25 \times 36)/12 = \mathbf{75}$.

2. $abc_7 = (a+1)(b+1)(c+1)_6$.

This gives $49a + 7b + c = 36(a+1) + 6(b+1) + c + 1$. Simplifying, we get $13a + b = 43$. Remembering that $a+1$ and $b+1$ are less than 6, and therefore a and b are less than 5, the only solution of this equation is $a = 3$, $b = 4$.

Hence the number is $34c_7$ or $45(c+1)_6$. But $c + 1 \leq 5$ so, for the largest such number, $c = 4$.

Hence the number is $344_7 = \mathbf{179}$.

3. *Method 1*

The interior angle of a regular pentagon is 108°. So the angle inside the ring between a square and a pentagon is $360° - 108° - 90° = 162°$. Thus on the inside of the completed ring we have a regular polygon with n sides whose interior angle is 162°.

The interior angle of a regular polygon with n sides is $180°(n-2)/n$.
So $162n = 180(n-2) = 180n - 360$. Then $18n = 360$ and $n = 20$.

Since half of these sides are from pentagons, the number of pentagons in the completed ring is **10**.

Method 2

The interior angle of a regular pentagon is 108°. So the angle inside the ring between a square and a pentagon is $360° - 108° - 90° = 162°$.

Thus on the inside of the completed ring we have a regular polygon with n sides whose exterior angle is $180° - 162° = 18°$. Hence $18n = 360$ and $n = 20$.

Since half of these sides are from pentagons, the number of pentagons in the completed ring is **10**.

Method 3

The interior angle of a regular pentagon is 108°. So the angle inside the ring between a square and a pentagon is $360° - 108° - 90° = 162°$. Thus on the inside of the completed ring we have a regular polygon whose interior angle is 162°.

The bisectors of these interior angles form congruent isosceles triangles on the sides of this polygon. So all these bisectors meet at a point, O say.

The angle at O in each of these triangles is $180° - 162° = 18°$. If n is the number of pentagons in the ring, then $18n = 360/2 = 180$. So $n = \mathbf{10}$.

4. Working modulo 10, we can make a sequence of last digits as follows:

$$1, 2, 5, 9, 6, 7, 5, 4, 1, 7, 0, 9, 1, 2, \ldots$$

Thus the last digits repeat after every 12 terms. Now $200 = 16 \times 12 + 8$. Hence the 200th last digit will the same as the 8th last digit.

So the last digit of s_{200} is **4**.

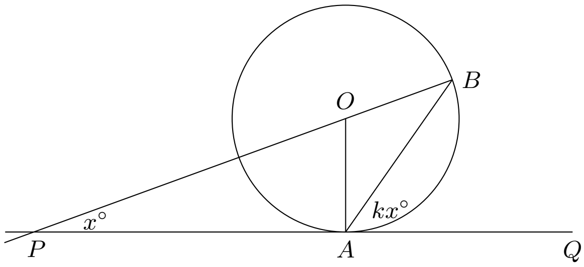
5. For each white square, colour in red the edges that are adjacent to black squares. Observe that the sum of the numbers that Sebastien writes down is the number of red edges.

The number of red edges is bounded above by the number of edges in the 11×38 grid that do not lie on the boundary of the grid. The number of such horizontal edges is 11×37, while the number of such vertical edges is 10×38. Therefore, the sum of the numbers that Sebastien writes down is bounded above by $11 \times 37 + 10 \times 38 = 787$.

Now note that this upper bound is obtained by the usual chessboard colouring of the grid. So the maximum sum of the numbers that Sebastien writes down is **787**.

6. *Method 1*

Draw OA.



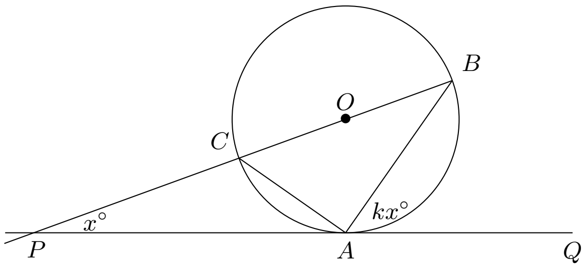
Since OA is perpendicular to PQ, $\angle OAB = 90° - kx°$.
Since $OA = OB$ (radii), $\angle OBA = 90° - kx°$.

Since $\angle QAB$ is an exterior angle of $\triangle PAB$, $kx = x + (90 - kx)$.
Rearranging gives $(2k - 1)x = 90$.

For maximum k we want $2k - 1$ to be the largest odd factor of 90.
Then $2k - 1 = 45$ and $k = \mathbf{23}$.

Method 2

Let C be the other point of intersection of the line PB with the circle.



By the Tangent-Chord theorem, $\angle ACB = \angle QAB = kx$. Since BC is a diameter, $\angle CAB = 90°$. By the Tangent-Chord theorem, $\angle PAC = \angle ABC = 180 - 90 - kx = 90 - kx$.

Since $\angle ACB$ is an exterior angle of $\triangle PAC$, $kx = x + 90 - kx$.
Rearranging gives $(2k - 1)x = 90$.

For maximum k we want $2k - 1$ to be the largest odd factor of 90.
Then $2k - 1 = 45$ and $k = \mathbf{23}$.

7. *Method 1*

If $n^2 + 2016n = m^2$, where n and m are positive integers, then $m = n + k$ for some positive integer k. Then $n^2 + 2016n = (n + k)^2$. So $2016n = 2nk + k^2$, or $n = k^2/(2016 - 2k)$. Since both n and k^2 are positive, we must have $2016 - 2k > 0$, or $2k < 2016$. Thus $1 \leq k \leq 1007$.

As k increases from 1 to 1007, k^2 increases and $2016 - 2k$ decreases, so n increases. Conversely, as k decreases from 1007 to 1, k^2 decreases and $2016 - 2k$ increases, so n decreases. If we take $k = 1007$, then $n = 1007^2/2$, which is not an integer. If we take $k = 1006$, then $n = 1006^2/4 = 503^2$. So $n \leq 503^2$.

If $k = 1006$ and $n = 503^2$, then $(n + k)^2 = (503^2 + 1006)^2 = (503^2 + 2 \times 503)^2 = 503^2(503 + 2)^2 = 503^2(503^2 + 4 \times 503 + 4) = 503^2(503^2 + 2016) = n^2 + 2016n$. So $n^2 + 2016n$ is indeed a perfect square. Thus 503^2 is the largest value of n such that $n^2 + 2016n$ is a perfect square.

Since $503^2 = (500 + 3)^2 = 500^2 + 2 \times 500 \times 3 + 3^2 = 250000 + 3000 + 9 = 253009$, the remainder when n is divided by 1000 is **9**.

Method 2

If $n^2 + 2016n = m^2$, where n and m are positive integers, then $m^2 = (n + 1008)^2 - 1008^2$.
So $1008^2 = (n + 1008 + m)(n + 1008 - m)$ and both factors are even and positive.
Hence $n + 1008 + m = 1008^2/(n + 1008 - m) \leq 1008^2/2$.

Since m increases with n, maximum n occurs when $n + 1008 + m$ is maximum. If $n + 1008 + m = 1008^2/2$, then $n + 1008 - m = 2$. Adding these two equations and dividing by 2 gives $n + 1008 = 504^2 + 1$ and $n = 504^2 - 1008 + 1 = (504 - 1)^2 = 503^2$.

If $n = 503^2$, then $n^2 + 2016n = 503^2(503^2 + 2016)$. Now $503^2 + 2016 = (504 - 1)^2 + 2016 = 504^2 + 1008 + 1 = (504 + 1)^2 = 505^2$. So $n^2 + 2016n$ is indeed a perfect square. Thus 503^2 is the largest value of n such that $n^2 + 2016n$ is a perfect square.

Since $503^2 = (500 + 3)^2 = 500^2 + 2 \times 500 \times 3 + 3^2 = 250000 + 3000 + 9 = 253009$, the remainder when n is divided by 1000 is **9**.

Method 3

If $n^2 + 2016n = m^2$, where n and m are positive integers, then solving the quadratic for n gives $n = (-2016 + \sqrt{2016^2 + 4m^2})/2 = \sqrt{1008^2 + m^2} - 1008$. So $1008^2 + m^2 = k^2$ for some positive integer k. Hence $(k - m)(k + m) = 1008^2$ and both factors are even and positive. Hence $k + m = 1008^2/(k - m) \leq 1008^2/2$.

Since m, n, k increase together, maximum n occurs when $m + k$ is maximum. If $k + m = 1008^2/2$, then $k - m = 2$. Subtracting these two equations and dividing by 2 gives $m = 504^2 - 1$ and $1008^2 + m^2 = 1008^2 + (504^2 - 1)^2 = 4 \times 504^2 + 504^4 - 2 \times 504^2 + 1 = 504^4 + 2 \times 504^2 + 1 = (504^2 + 1)^2$. So $n = 504^2 + 1 - 2 \times 504 = (504 - 1)^2 = 503^2$.

If $n = 503^2$, then $n^2 + 2016n = 503^2(503^2 + 2016)$. Now $503^2 + 2016 = (504 - 1)^2 + 2016 = 504^2 + 1008 + 1 = (504 + 1)^2 = 505^2$. So $n^2 + 2016n$ is indeed a perfect square. Thus 503^2 is the largest value of n such that $n^2 + 2016n$ is a perfect square.

Since $503^2 = (500 + 3)^2 = 500^2 + 2 \times 500 \times 3 + 3^2 = 250000 + 3000 + 9 = 253009$, the remainder when n is divided by 1000 is **9**.

8. Suppose Ann has the last turn. Let n be the number of turns that Bob has. Then the number of sweets that he takes is $2+4+6+\cdots+2n = 2(1+2+\cdots+n) = n(n+1)$. So the total number of sweets is $2n(n+1)$.

Suppose Bob has the last turn. Let n be the number of turns that Ann has. Then the number of sweets that she takes is $1+3+5+\cdots+(2n-1) = n^2$. So the total number of sweets is $2n^2$.

So half the number of sweets is $n(n+1)$ or n^2. Applying the same sharing procedure to half the sweets gives, for some integer m, one of the following four cases:

1. $n(n+1) = 2m(m+1)$
2. $n(n+1) = 2m^2$
3. $n^2 = 2m(m+1)$
4. $n^2 = 2m^2$.

In the first two cases we want n such that $n(n+1) < 500$. So $n \leq 21$.

In the first case, since 2 divides m or $m+1$, we also want 4 to divide $n(n+1)$. So $n \leq 20$.
Since $20 \times 21 = 420 = 2 \times 14 \times 15$, the total number of sweets could be $2 \times 420 = 840$.

In the second case $\frac{1}{2}n(n+1)$ is a perfect square. So $n < 20$.

In the last two cases we look for n so that $n^2 > 840/2 = 420$.
We also want n even and $n^2 < 500$. So $n = 22$.

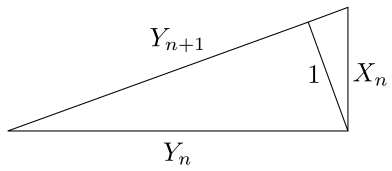
In the third case, $m(m+1) = \frac{1}{2} \times 22^2 = 242$ but $15 \times 16 = 240$ while $16 \times 17 = 272$.

In the fourth case, $m^2 = 242$ but 242 is not a perfect square.

So the maximum total number of sweets is **840**.

9. *Method 1*

For each large triangle, one leg is X_n. Let Y_n be the other leg and let Y_{n+1} be the hypotenuse. Note that $Y_1 = X_0$.



By Pythagoras,

$$\begin{aligned}
Y_{n+1}^2 &= X_n^2 + Y_n^2 \\
&= X_n^2 + X_{n-1}^2 + Y_{n-1}^2 \\
&= X_n^2 + X_{n-1}^2 + X_{n-2}^2 + Y_{n-2}^2 \\
&= X_n^2 + X_{n-1}^2 + X_{n-2}^2 + \cdots + X_1^2 + Y_1^2 \\
&= X_n^2 + X_{n-1}^2 + X_{n-2}^2 + \cdots + X_1^2 + X_0^2
\end{aligned}$$

The area of the triangle shown is given by $\dfrac{1}{2}Y_{n+1}$ and by $\dfrac{1}{2}X_nY_n$. Using this or similar triangles we have

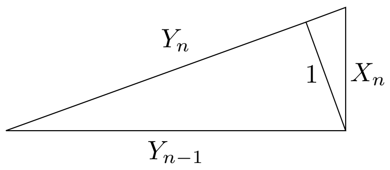
$$\begin{aligned}
Y_{n+1} &= X_n \times Y_n \\
&= X_n \times X_{n-1} \times Y_{n-1} \\
&= X_n \times X_{n-1} \times X_{n-2} \times Y_{n-2} \\
&= X_n \times X_{n-1} \times X_{n-2} \times \cdots \times X_1 \times Y_1 \\
&= X_n \times X_{n-1} \times X_{n-2} \times \cdots \times X_1 \times X_0
\end{aligned}$$

So

$$X_0^2 + X_1^2 + X_2^2 + \cdots + X_n^2 = X_0^2 \times X_1^2 \times X_2^2 \times \cdots \times X_n^2$$

Method 2

For each large triangle, one leg is X_n. Let Y_{n-1} be the other leg and let Y_n be the hypotenuse. Note that $Y_0 = X_0$.



From similar triangles we have $Y_1/X_1 = X_0/1$. So $Y_1 = X_0 \times X_1$.
By Pythagoras, $Y_1^2 = X_0^2 + X_1^2$. So $X_0^2 + X_1^2 = Y_1^2 = X_0^2 \times X_1^2$.
Assume for some $k \geq 1$

$$Y_k^2 = X_0^2 + X_1^2 + X_2^2 + \cdots + X_k^2 = X_0^2 \times X_1^2 \times X_2^2 \times \cdots \times X_k^2$$

From similar triangles we have $Y_{k+1}/X_{k+1} = Y_k/1$. So $Y_{k+1} = Y_k \times X_{k+1}$.
By Pythagoras, $Y_{k+1}^2 = X_{k+1}^2 + Y_k^2$. So $X_{k+1}^2 + Y_k^2 = Y_{k+1}^2 = Y_k^2 \times X_{k+1}^2$. Hence

$$X_0^2 + X_1^2 + X_2^2 + \cdots + X_k^2 + X_{k+1}^2 = X_0^2 \times X_1^2 \times X_2^2 \times \cdots \times X_k^2 \times X_{k+1}^2$$

By induction,

$$X_0^2 + X_1^2 + X_2^2 + \cdots + X_n^2 = X_0^2 \times X_1^2 \times X_2^2 \times \cdots \times X_n^2$$

for all $n \geq 1$.

10. *Method 1*

Let b and w denote the sum of the labels on all black and white vertices respectively. Let c be the label on the central vertex. Then

$$b + w + c = 0 \qquad (1)$$

Summing the labels over all diameters gives

$$b + w + nc = ns \qquad (2)$$

Summing the labels over all black-white-black arcs gives

$$2b + w = ns \qquad (3)$$

From (1) and (2),

$$(n-1)c = ns \qquad (4)$$

Hence n divides c. Since $-n \leq c \leq n$, $c = 0$, $-n$, or n.

Suppose $c = \pm n$. From (2) and (3), $b = nc = \pm n^2$.
Since $|b| \leq 1 + 2 + \cdots + n < n^2$, we have a contradiction.
So $c = 0$. From (4), $s = 0$.

Method 2

Case 1. n is even.

For any label x not at the centre, let x' denote the label diametrically opposite x. Let the centre have label c. Then

$$x + c + x' = s.$$

If x, y, z are any three consecutive labels where x and z are on black points, then we have

$$x + c + x' = y + c + y' = z + c + z' = s.$$

Adding these yields

$$x + y + z + 3c + x' + y' + z' = 3s.$$

Since n is even, diametrically opposite points have the same colour. So

$$x + y + z = s = x' + y' + z' \quad \text{and} \quad s = 3c.$$

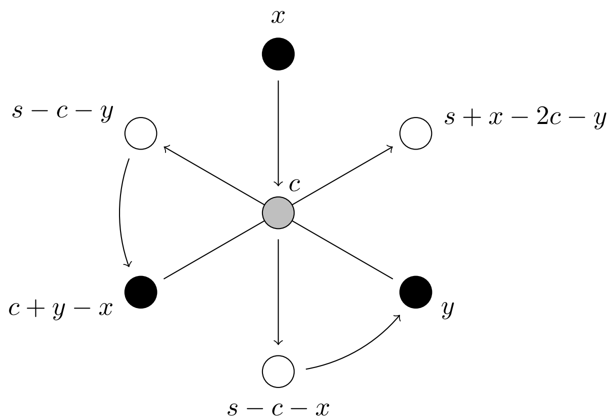
Hence $p + p' = 2c$ for any label p on the circle. Since there are n such diametrically opposite pairs, the sum of all labels on the circle is $2nc$.

Since the sum of all the labels is zero, we have $0 = 2nc + c = c(2n+1)$. Thus $c = 0$, and $s = 3c = 0$.

Case 2. n is odd.

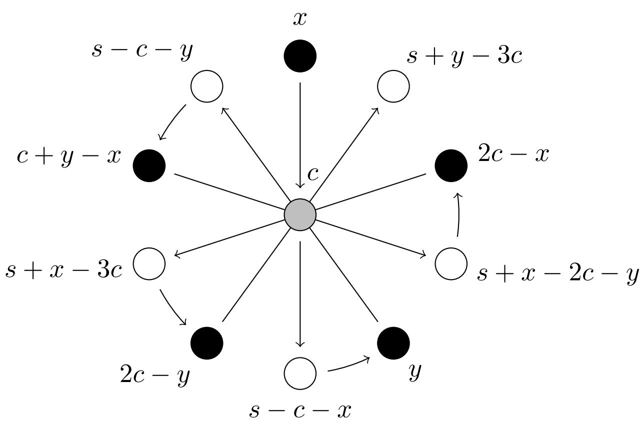
We show that the required labelling is impossible. In each of the following diagrams, the arrows indicate the order in which the labels are either arbitrarily prescribed (x, c, y) or dictated by the given conditions.

If $n = 3$, we have:



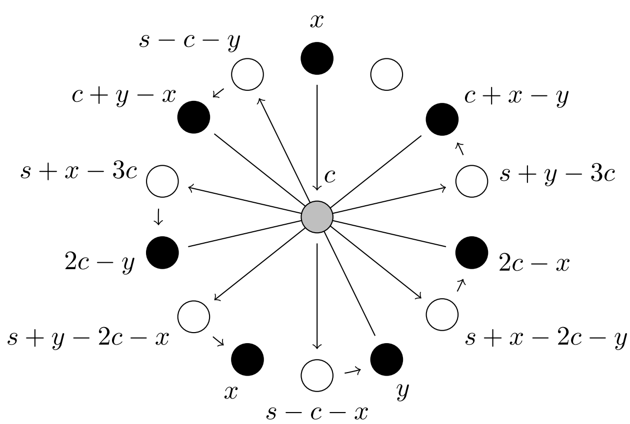
From the first, last, and fourth last labels, $s = x + (s + x - 2c - y) + y = s + 2x - 2c$. Hence $x = c$, which is disallowed.

If $n = 5$, we have:



From the first, last, and fourth last labels, $s = x + (s + y - 3c) + (2c - x) = s + y - c$. Hence $y = c$, which is disallowed.

If $n \geq 7$, we have:



The first and last labels are the same, which is disallowed.

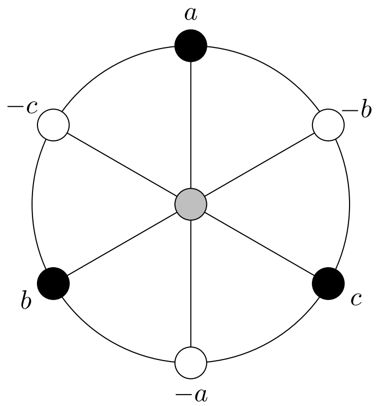
Investigation

Since $c = 0 = s$, for each diameter, the label at one end is the negative of the label at the other end.

Let n be an odd number.

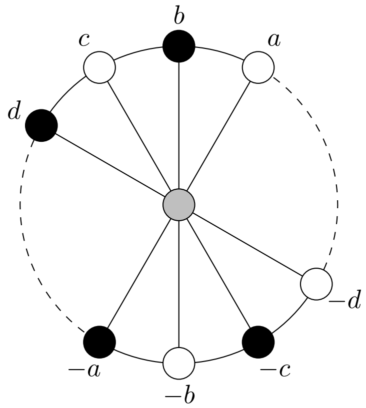
Each diameter is from a black point to a white point.

If $n = 3$, we have:



Hence $a + b - c = 0 = a - b + c$. So $b = c$, which is disallowed.

If $n > 3$, we have:



Hence $b + c + d = 0 = -a - b - c = a + b + c$. So $a = d$, which is disallowed.

So the required labelling does not exist for odd n.

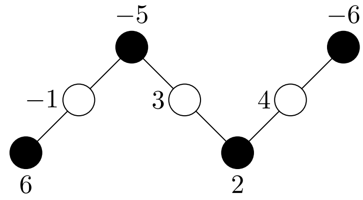
(Alternatively, the argument in Method 2 for odd n could be awarded a bonus mark.)

Now let n be an even number.

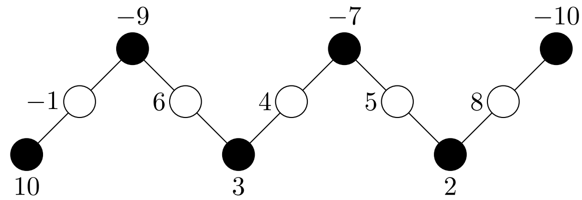
We show that a required labelling does exist for $n = 2m \geq 4$. It is sufficient to show that $n+1$ consecutive points on the circle from a black point to a black point can be assigned labels from $\pm 1, \pm 2, \ldots, \pm n$, so that the absolute values of the labels are distinct except for the two end labels, and the sum of the labels on each black-white-black arc is 0. We demonstrate such labellings with a zigzag pattern for clarity. Essentially, with some adjustments at the ends and in small cases, we try to place the odd labels on the black points, which are at the corners of the zigzag, and the even labels on the white points in between.

Case 1. m odd.

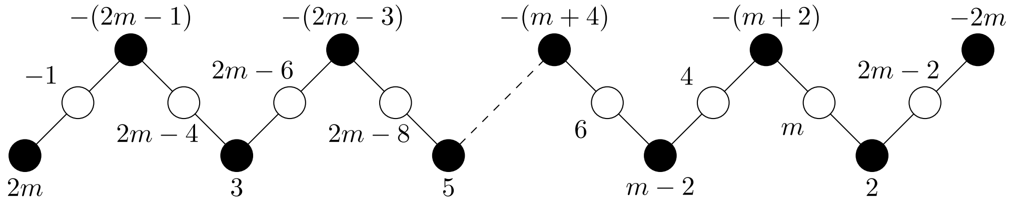
$m = 3$



$m = 5$

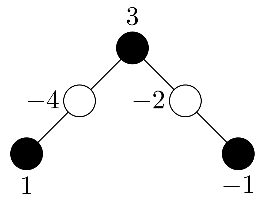


General odd m.

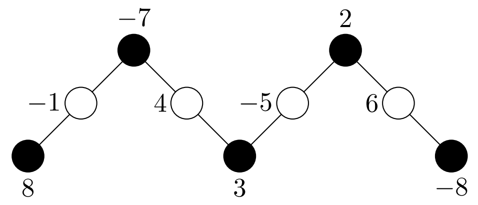


Case 2. m even.

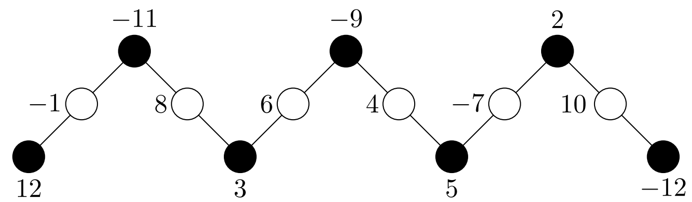
$m = 2$



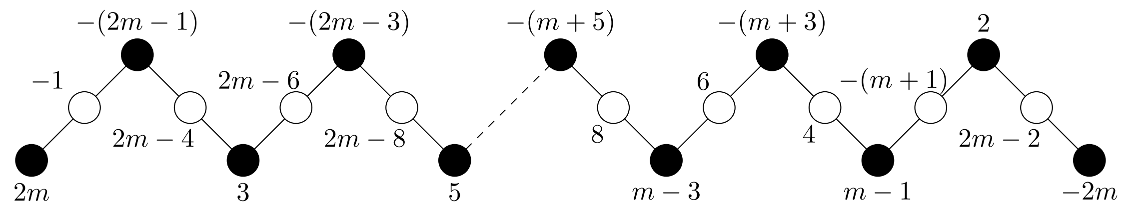
$m = 4$



$m = 6$



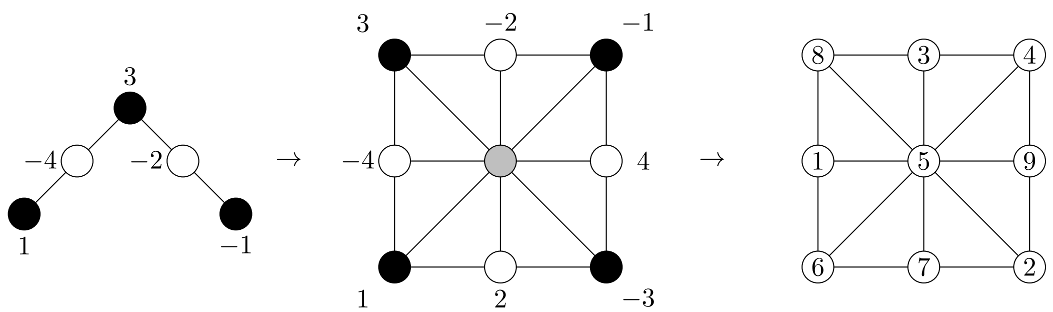
General even m.



Thus the required labelling exists if and only if n is even.

Comments

1. The special case $m = 2$ gives the classical magic square:



2. It is easy to check that, except for rotations and reflections, there is only one labelling for $m = 2$. Are the general labellings given above unique for all m?

3. Method 2 shows that the conclusion of the Problem 10 also holds for non-integer labels provided their sum is 0.

AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD STATISTICS

Distribution of Awards/School Year

Year	Number of Students	Number of Awards				
		Prize	High Distinction	Distinction	Credit	Participation
8	576	4	27	81	157	307
9	481	16	53	80	148	184
10	405	18	61	77	127	122
Other	367	3	5	7	65	287
All Years	1829	41	146	245	497	900

Number of Correct Answers Questions 1–8

Year	Number Correct/Question							
	1	2	3	4	5	6	7	8
8	449	158	344	222	166	86	56	110
9	413	154	345	241	195	138	72	139
10	328	171	320	225	189	156	54	133
Other	255	61	160	92	61	21	18	42
All Years	1445	544	1169	780	611	401	200	424

Mean Score/Question/School Year

Year	Number of Students	Mean Score			Overall Mean
		Question			
		1–8	9	10	
8	576	8.6	1.0	0.3	10.1
9	481	11.5	1.2	0.5	13.1
10	405	12.4	1.3	0.7	14.5
Other	367	6.1	0.2	0.1	6.3
All Years	1829	9.8	1.0	0.4	11.1

AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD RESULTS

NAME	SCHOOL	YEAR	SCORE
Prize and Perfect Score			
Boon Han Nathaniel Ang	Anglo-Chinese School (Independent) Sin	8	35
Haocheng Li	Anglo-Chinese School (Independent) Sin	10	35
Minh Tuan Nguyen	Anglo-Chinese School (Independent) Sin	9	35
William Li	Barker College NSW	10	35
Jackson Liu	Brighton Grammar School Vic	10	35
Jerry Mao	Caulfield Grammar School Wheelers Hill Vic	10	35
Zlatina Mileva	Childrens Academy 21St Century Bul	9	35
Cheng, Edison Shi	Hwa Chong Institution Sin	9	35
An Jun Lim	Hwa Chong Institution Sin	9	35
Linus Cooper	James Ruse Agricultural High School NSW	10	35
Ruixin Shi	Raffles Girls School Sin	9	35
Wang Jianzhi	Raffles Institution Sin	10	35
Low Choo Ray	Raffles Institution Sin	7	35
Hu Yongao	Raffles Institution Sin	9	35
Guowen Zhang	St Joseph's College Qld	10	35
Prize			
Xinyue (Alice) Zhang	A. B. Paterson College Qld	10	34
Zien Lin	Anglo-Chinese School (Independent) Sin	10	34
Charles Li	Camberwell Grammar School Vic	10	34
Pham Huy Giang Nam	Thpt Chuyen Ha Noi - Amsterdam Viet	10	34
Haowen Gao	Knox Grammar School NSW	9	34
Nguyen Manh Quan	Thpt Chuyen Ha Noi - Amsterdam Viet	8	34
Boo Tse Yang Lucas	Raffles Institution Sin	7	34
William Hu	Christ Church Grammar School WA	10	33
Eric Bryan	Anglo-Chinese School (Independent) Sin	9	33
Shen Xin Yi	Raffles Institution Sin	8	33
Hadyn Tang	Trinity Grammar School Vic	7	33
Theodore Leebrant	Anglo-Chinese School (Independent) Sin	10	32
Michael Sui	Caulfield Grammar School Wheelers Hill Vic	10	32
Han Yang	Hwa Chong Institution Sin	10	32
Stanley Zhu	Melbourne Grammar School Vic	10	32
Ryan Stocks	Radford College Act	10	32
Frank Zhao	Geelong Grammar School Vic	9	32
Ning Li Wang	Hwa Chong Institution Sin	9	32
Sharvil Kesarwani	Merewether High School NSW	9	32
Khor Jun Wei	Raffles Institution Sin	9	32

NAME	SCHOOL	YEAR	SCORE
Marcus Rees	Taroona High School Tas	9	32
Nguyen Minh Duc	Thcs Nhan Chinh Viet	9	32
Nguyen Tuan Hoang	Thpt Chuyen Ha Noi - Amsterdam Viet	9	32
Preet Patel	Vermont Secondary College Vic	9	32
Aloysius Ng Yangyi	Raffles Institution Sin	8	32
Hilton Nguyen	Sydney Technical High School NSW	2	32
High Distinction			
James Bang	Baulkham Hills High School NSW	9	31
Arand Bharadwaj	Trinity Grammar School Vic	10	31
Xiang Rong Dai	Raffles Institution Sin	9	31
Tianjie Huang	Hwa Chong Institution Sin	10	31
Le Duc Khoi	Thcs Cau Giay Viet	9	31
Victor Loh WAi Kit	Raffles Institution Sin	9	31
Miles Koumouris	Box Hill High School Vic	10	31
Zefeng Li	Camberwell Grammar School Vic	9	31
Keane Ng	Raffles Institution Sin	8	31
Mikhail Savkin	Gosford High School NSW	8	31
Ethan Tan	Cranbrook School NSW	9	31
Zi Yi Tan	Hwa Chong Institution Sin	9	31
Anthony Tew	Pembroke School Sa	10	31
Shaobo Yang	Anglo-Chinese School (Independent) Sin	9	31
Christopher Ai	Knox Grammar School NSW	7	30
Cheng Arn David Goh	Anglo-Chinese School (Independent) Sin	9	30
Minh Hai Dao	Anglo-Chinese School (Independent) Sin	9	30
Yuhao Li	Hwa Chong Institution Sin	10	30
Philip Liang	James Ruse Agricultural High School NSW	10	30
Forbes Mailler	Canberra Grammar School Act	10	30
Tommy Wei	Scotch College Vic	10	30
Michael Zhao	Box Hill High School Vic	9	30
Xin Zhe	Raffles Institution Sin	10	30
Kieren Connor	Sydney Grammar School NSW	10	29
Matthew Duffy	St Patrick's College Vic	9	29
Enoch Fan	Melbourne Grammar School Vic	10	29
Daniel Gu	Brighton Grammar School Vic	10	29
Hollis Huang	Tintern Grammar Vic	10	29
Jia Nuo, Daniel Chia	Hwa Chong Institution Sin	9	29
Tan Wee Kean	Raffles Institution Sin	8	29
Matheus Calvin Lokadjaja	Anglo-Chinese School (Independent) Sin	9	29
Yu Peng Ng	Hwa Chong Institution Sin	9	29

NAME	SCHOOL	YEAR	SCORE
Anthony Pisani	St Paul's Anglican Grammar School Vic	9	29
William Sutherland	Scotch College Vic	9	29
Andrew Virgona	Smiths Hill High School NSW	10	29
Yaxin Xia	Raffles Girls School Sin	10	29
Ang Ben Xuan	Raffles Institution Sin	10	29
Dimitar Chakarov	Childrens Academy 21St Century Bul	8	28
Liam Coy	Sydney Grammar School NSW	8	28
Grace He	Methodist Ladies' College Vic	8	28
William Hobkirk	Sydney Grammar School NSW	10	28
David Toh Hui Kai	Raffles Institution Sin	8	28
Le Ngoc Vu	Thpt Chuyen Ha Noi - Amsterdam Viet	10	28
Moses Mayer	Surya Institute Idn	10	28
Hugo McCahon-Boersma	Sydney Grammar School NSW	10	28
Chengkai Men	Anglo-Chinese School (Independent) Sin	9	28
Tran Gia Bao	Thcs Nam Tu Liem Viet	9	28
Arran Van Zuylen	Vermont Secondary College Vic	10	28
Zhan Li, Benson Lin	Hwa Chong Institution Sin	8	28
Zihan Zhou	Raffles Girls School Sin	9	28
Amit Ben-Harim	Mckinnon Secondary College Vic	10	27
Andrew Cantrill	Marist College Eastwood NSW	10	27
Chu Khanh An	Thcs Cau Giay Viet	9	27
Seah Fengyu	Raffles Institution Sin	10	27
Jack Gibney	Penleigh And Essendon Grammar School Vic	10	27
Carl Gu	Melbourne High School Vic	10	27
Yasiru Jayasooriya	James Ruse Agricultural High School NSW	8	27
Steven Lim	Hurlstone Agricultural High School NSW	10	27
Nguyen Quang Minh	Thcs Nguyen Truong To Viet	9	27
Pham Quoc Viet	Thpt Chuyen Ha Noi - Amsterdam Viet	10	27
Longxuan Sun	Hwa Chong Institution Sin	9	27
Matthew Scott Tan	Anglo-Chinese School (Independent) Sin	9	27
Ruiqian Tong	Presbyterian Ladies' College Vic	10	27
Chun En Yau	Hwa Chong Institution Sin	8	27
John Min	Baulkham Hills High School NSW	10	26
Katrina Shen	James Ruse Agricultural High School NSW	9	26
Jason Zhang	Killara High School NSW	10	26
Adrian Lo	Newington College NSW	8	26
Rishabh Singh	Parramatta High School NSW	9	26
Ellen Zheng	Smiths Hill High School NSW	10	26
Gorden Zhuang	Sydney Boys High School NSW	10	26
Michael Lin	Trinity Grammar School NSW	8	26

NAME	SCHOOL	YEAR	SCORE
Junhua Chen	Caulfield Grammar School Wheelers Hill Vic		26
Yifan Guo	Glen WAverly Secondary College Vic	10	26
Matthew Lee	Scotch College Vic	9	26
Valeri Vankov	Childrens Academy 21St Century Bul	6	26
Yulong Guo	Hwa Chong Institution Sin	10	26
Quoc Huy Le	Anglo-Chinese School (Independent) Sin	9	26
Tran Duy Phat	Thcs Cau Giay Viet	8	26
Ta Kien Quoc	Thpt Chuyen Ha Noi - Amsterdam Viet	10	26
Tran Duc Anh	Thcs Giang Vo Viet	9	26
Ezra Hui	Baulkham Hills High School NSW	10	25
Charran Kethees	James Ruse Agricultural High School NSW	9	25
Angus Ritossa	St Peter's College Sa	9	25
Ian Chen	Camberwell Grammar School Vic	9	25
Ryan Campbell	Camberwell Grammar School Vic	10	25
Yueqi (Rose) Lin	Shenton College WA	10	25
Nathanial Lukas Christianto	Surya Institute Idn	9	25
Yong Jie Yeoh	Hwa Chong Institution Sin	8	25
Reuben Yoong Ern Wong	Anglo-Chinese School (Independent) Sin	7	25
Matthias Wenqi Liew	Anglo-Chinese School (Independent) Sin	10	25
Pham Duy Tung	Thcs Le Quy Don Viet	9	25
James Phillips	Canberra Grammar School Act	10	24
Ranit Bose	Lyneham High School Act	10	24
Brian Su	James Ruse Agricultural High School NSW	10	24
Mandy Zhu	James Ruse Agricultural High School NSW	9	24
Hanyuan Li	North Sydney Boys High NSW	8	24
Phoebe Zuo	Tara Anglican School For Girls NSW	10	24
Zixuan Chen	Caulfield Grammar School Vic	10	24
Steven Liu	Glen WAverly Secondary College Vic	10	24
Kevin Wu	Scotch College Vic	9	24
Adam Bardrick	Whitefriars College Vic	10	24
Shevanka Dias	All Saints' College WA	9	24
Iliyan Ivanov	Childrens Academy 21St Century Bul	8	24
Yi He, Scott Loo	Hwa Chong Institution Sin	9	24
Kaung Kaung Aung	Hwa Chong Institution Sin	9	24
Quang Huy Bui	Anglo-Chinese School (Independent) Sin	9	24
William Wahyudi	Anglo-Chinese School (Independent) Sin	9	24
Luo Jincheng	Raffles Institution Sin	9	24
Weng Chang Kai Kenneth	Raffles Institution Sin	9	24
Bui Minh Thanh	Thcs Cau Giay Viet	8	24
Tran Van Anh	Thcs Le Quy Don Viet	9	24

NAME	SCHOOL	YEAR	SCORE
Dinh Vu Tung Lam	Thcs Cau Giay Viet	8	24
Ziqi Yuan	Lyneham High School ACT	9	23
William Chen	Brisbane State High School Qld	10	23
Hao Chen	James Ruse Agricultural High School NSW	9	23
Jason Yang	James Ruse Agricultural High School NSW	9	23
Charles Lilley	Sydney Grammar School NSW	10	23
Jee Hwan Kim	Trinity Grammar School NSW	8	23
Yuya Kurokawa	Underdale High School SA	10	23
Harry Zhang	Home Education Network Vic	7	23
John Alealde	Suzanne Cory High School Vic	10	23
Angela Chau	Ruyton Girls' School Vic	10	23
Shivasankaran Jayabalan	Rossmoyne Senior High School WA	10	23
Jacob Smith	All Saints' College WA	10	23
Zhiyao Pan	Anglo-Chinese School (Independent) Sin	9	23
Jonathan Yi Wei Low	Anglo-Chinese School (Independent) Sin	10	23
Anders Mah	Baulkham Hills High School NSW	8	22
Eric Huang	James Ruse Agricultural High School NSW	8	22
Jason Leung	James Ruse Agricultural High School NSW	8	22
Cory Aitchison	Knox Grammar School NSW	10	22
Ethan Ryoo	Knox Grammar School NSW	7	22
Kevin Yan	North Sydney Boys High NSW	10	22
Tay Leung	Plc Sydney NSW	9	22
Junu Choi	The Shore School NSW	9	22
Tianyi Xu	Sydney Boys High School NSW	10	22
Wendi Jin	Methodist Ladies' College Vic	10	22
Evgeniya Artemora	Presbyterian Ladies' College Vic	9	22
Tianqi Geng	Scotch College Vic	9	22
Matthew Hamdorf	Christ Church Grammar School WA	10	22
Yuqing (Sunny) Lu	Perth Modern School WA	10	22
Ivan Ivanov	Childrens Academy 21St Century Bul	8	22
Jiang Yi, Brian Siew	Hwa Chong Institution Sin	8	22
Louth Bin Rawshan	Singapore International Math Contests Centre Sin	8	22
Tran Xuan An	Thcs Le Quy Don Viet	8	22
Ta Son Bach	Thcs Ngo Si Lien Viet	9	22

HONOUR ROLL

Because of changing titles and affiliations, the most senior title achieved and later affiliations are generally used, except for the Interim committee, where they are listed as they were at the time.

Mathematics Challenge for Young Australians

Problems Committee for Challenge

Dr K McAvaney	Victoria, (Director)	11 years; 2006–2016
	Member	1 year; 2005-2006
Mr B Henry	Victoria (Director)	17 years; 1990–2006
	Member	11 years; 2006–2016
Prof P J O'Halloran	University of Canberra, ACT	5 years; 1990–1994
Dr R A Bryce	Australian National University, ACT	23 years; 1990–2012
Adj Prof M Clapper	Australian Mathematics Trust, ACT	4 years; 2013–2016
Ms L Corcoran	Australian Capital Territory	3 years; 1990–1992
Ms B Denney	New South Wales	7 years; 2010–2016
Mr J Dowsey	University of Melbourne, VIC	8 years; 1995–2002
Mr A R Edwards	Department of Education, Qld	27 years; 1990–2016
Dr M Evans	Scotch College, VIC	6 years; 1990–1995
Assoc Prof H Lausch	Monash University, VIC	24 years; 1990–2013
Ms J McIntosh	AMSI, VIC	15 years; 2002–2016
Mrs L Mottershead	New South Wales	25 years; 1992–2016
Miss A Nakos	Temple Christian College, SA	24 years; 1993–2016
Dr M Newman	Australian National University, ACT	27 years; 1990–2016
Ms F Peel	St Peter's College, SA	2 years; 1999, 2000
Dr I Roberts	Northern Territory	4 years; 2013–2016
Ms T Shaw	SCEGGS, NSW	4 years; 2013–2016
Ms K Sims	New South Wales	18 years; 1999–2016
Dr A Storozhev	Attorney General's Department, ACT	23 years; 1994–2016
Prof P Taylor	Australian Mathematics Trust, ACT	20 years; 1995–2014
Mrs A Thomas	New South Wales	18 years; 1990–2007
Dr S Thornton	South Australia	19 years; 1998–2016
Miss G Vardaro	Wesley College, VIC	23 years: 1993–2006, 2008–2016

Visiting members

Prof E Barbeau	University of Toronto, Canada	1991, 2004, 2008
Prof G Berzsenyi	Rose Hulman Institute of Technology, USA	1993, 2002
Dr L Burjan	Department of Education, Slovakia	1993
Dr V Burjan	Institute for Educational Research, Slovakia	1993
Mrs A Ferguson	Canada	1992
Prof B Ferguson	University of Waterloo, Canada	1992, 2005
Dr D Fomin	St Petersburg State University, Russia	1994
Prof F Holland	University College, Ireland	1994
Dr A Liu	University of Alberta, Canada	1995, 2006, 2009
Prof Q Zhonghu	Academy of Science, China	1995
Dr A Gardiner	University of Birmingham, United Kingdom	1996
Prof P H Cheung	Hong Kong	1997
Prof R Dunkley	University of Waterloo, Canada	1997
Dr S Shirali	India	1998
Mr M Starck	New Caledonia	1999
Dr R Geretschlager	Austria	1999, 2013
Dr A Soifer	United States of America	2000
Prof M Falk de Losada	Colombia	2000

Mr H Groves	United Kingdom	2001
Prof J Tabov	Bulgaria	2001, 2010
Prof A Andzans	Latvia	2002
Prof Dr H-D Gronau	University of Rostock, Germany	2003
Prof J Webb	University of Cape Town, South Africa	2003, 2011
Mr A Parris	Lynwood High School, New Zealand	2004
Dr A McBride	University of Strathclyde, United Kingdom	2007
Prof P Vaderlind	Stockholm University, Sweden	2009, 2012
Prof A Jobbings	United Kingdom	2014
Assoc Prof D Wells	United States of America	2015
Dr P Neumann	United Kingdom	2016

Moderators for Challenge

Mr W Akhurst	New South Wales
Ms N Andrews	ACER, Camberwell, VIC
Prof E Barbeau	University of Toronto, Canada
Mr R Blackman	Victoria
Ms J Breidahl	St Paul's Woodleigh, VIC
Ms S Brink	Glen Iris, VIC
Prof J C Burns	Australian Defence Force Academy, ACT
Mr A. Canning	Queensland
Mrs F Cannon	New South Wales
Mr J Carty	ACT Department of Education, ACT
Dr E Casling	Australian Capital Territory
Mr B Darcy	South Australia
Ms B Denney	New South Wales
Mr J Dowsey	Victoria
Mr S Ewington	Sydney Grammar School, NSW
Br K Friel	Trinity Catholic College, NSW
Dr D Fomin	St Petersburg University, Russia
Mrs P Forster	Penrhos College, WA
Mr T Freiberg	Queensland
Mr W Galvin	University of Newcastle, NSW
Mr M Gardner	North Virginia, USA
Mr S Gardiner	University of Sydney, NSW
Ms P Graham	Tasmania
Mr B Harridge	University of Melbourne, VIC
Ms J Hartnett	Queensland
Mr G Harvey	Australian Capital Territory
Ms I Hill	South Australia
Ms N Hill	Victoria
Dr N Hoffman	Edith Cowan University, WA
Prof F Holland	University College, Ireland
Mr D Jones	Coff's Harbour High School, NSW
Ms R Jorgenson	Australian Capital Territory
Dr T Kalinowski	University of Newcastle, NSW
Assoc Prof H Lausch	Victoria
Mr J Lawson	St Pius X School, NSW
Mr R Longmuir	China
Ms K McAsey	Victoria
Dr K McAvaney	Victoria
Ms J McIntosh	AMSI, VIC
Ms N McKinnon	Victoria
Ms T McNamara	Victoria
Mr G Meiklejohn	Queensland School Curriculum Council, QLD

Moderators for Challenge *continued*

Mr M O'Connor	AMSI, VIC
Mr J Oliver	Northern Territory
Mr S Palmer	New South Wales
Dr W Palmer	University of Sydney, NSW
Mr G Pointer	South Australia
Prof H Reiter	University of North Carolina, USA
Mr M Richardson	Yarraville Primary School, VIC
Mr G Samson	Nedlands Primary School, WA
Mr J Sattler	Parramatta High School, NSW
Mr A Saunder	Victoria
Mr W Scott	Seven Hills West Public School, NSW
Mr R Shaw	Hale School, WA
Ms T Shaw	New South Wales
Dr B Sims	University of Newcastle, NSW
Dr H Sims	Victoria
Ms K Sims	New South Wales
Prof J Smit	The Netherlands
Mrs M Spandler	New South Wales
Mr G Spyker	Curtin University, WA
Ms C Stanley	Queensland
Dr E Strzelecki	Monash University, VIC
Mr P Swain	Ivanhoe Girls Grammar School, VIC
Dr P Swedosh	The King David School, VIC
Prof J Tabov	Academy of Sciences, Bulgaria
Mrs A Thomas	New South Wales
Ms K Trudgian	Queensland
Ms J Vincent	Melbourne Girls Grammar School, VIC
Prof J Webb	University of Capetown, South Africa
Dr D Wells	USA

Mathematics Enrichment Development

Enrichment Committee — Development Team (1992–1995)

Mr B Henry	Victoria (Chairman)
Prof P O'Halloran	University of Canberra, ACT (Director)
Mr G Ball	University of Sydney, NSW
Dr M Evans	Scotch College, VIC
Mr K Hamann	South Australia
Assoc Prof H Lausch	Monash University, VIC
Dr A Storozhev	Australian Mathematics Trust, ACT

Polya Development Team (1992–1995)

Mr G Ball	University of Sydney, NSW (Editor)
Mr K Hamann	South Australia (Editor)
Prof J Burns	Australian Defence Force Academy, ACT
Mr J Carty	Merici College, ACT
Dr H Gastineau-Hill	University of Sydney, NSW
Mr B Henry	Victoria
Assoc Prof H Lausch	Monash University, VIC
Prof P O'Halloran	University of Canberra, ACT
Dr A Storozhev	Australian Mathematics Trust, ACT

Euler Development Team (1992–1995)

Dr M Evans	Scotch College, VIC (Editor)
Mr B Henry	Victoria (Editor)
Mr L Doolan	Melbourne Grammar School, VIC

Mr K Hamann	South Australia
Assoc Prof H Lausch	Monash University, VIC
Prof P O'Halloran	University of Canberra, ACT
Mrs A Thomas	Meriden School, NSW

Gauss Development Team (1993–1995)

Dr M Evans	Scotch College, VIC (Editor)
Mr B Henry	Victoria (Editor)
Mr W Atkins	University of Canberra, ACT
Mr G Ball	University of Sydney, NSW
Prof J Burns	Australian Defence Force Academy, ACT
Mr L Doolan	Melbourne Grammar School, VIC
Mr A Edwards	Mildura High School, VIC
Mr N Gale	Hornby High School, New Zealand
Dr N Hoffman	Edith Cowan University, WA
Prof P O'Halloran	University of Canberra, ACT
Dr W Pender	Sydney Grammar School, NSW
Mr R Vardas	Dulwich Hill High School, NSW

Noether Development Team (1994–1995)

Dr M Evans	Scotch College, VIC (Editor)
Dr A Storozhev	Australian Mathematics Trust, ACT (Editor)
Mr B Henry	Victoria
Dr D Fomin	St Petersburg University, Russia
Mr G Harvey	New South Wales

Newton Development Team (2001–2002)

Mr B Henry	Victoria (Editor)
Mr J Dowsey	University of Melbourne, VIC
Mrs L Mottershead	New South Wales
Ms G Vardaro	Annesley College, SA
Ms A Nakos	Temple Christian College, SA
Mrs A Thomas	New South Wales

Dirichlet Development Team (2001–2003)

Mr B Henry	Victoria (Editor)
Mr A Edwards	Ormiston College, QLD
Ms A Nakos	Temple Christian College, SA
Mrs L Mottershead	New South Wales
Mrs K Sims	Chapman Primary School, ACT
Mrs A Thomas	New South Wales

Australian Intermediate Mathematics Olympiad Committee

Dr K McAvaney	Victoria (Chair)	10 years; 2007–2016
Adj Prof M Clapper	Australian Mathematics Trust, ACT	3 years; 2014–2016
Mr J Dowsey	University of Melbourne, VIC	18 years; 1999–2016
Dr M Evans	AMSI, VIC	18 years; 1999–2016
Mr B Henry	Victoria (Chair)	8 years; 1999–2006
	Member	10 years; 2007–2016
Assoc Prof H Lausch	Monash University, VIC	17 years; 1999–2015
Mr R Longmuir	China	2 years; 1999–2000