Pólya Enrichment Stage

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Chapter 1. Expansion and Factorisation

The following factorisations/expansions are well known:

$$x(x+y) \quad \underset{\text{Factor}}{\overset{\text{Expand}}{\rightleftarrows}} \quad x^2+xy$$

$$(x \pm y)^2 \quad \longleftrightarrow \quad x^2 \pm 2xy + y^2$$

$$(x+y)(x-y) \quad \longleftrightarrow \quad x^2 - y^2$$

$$(a+b)(c+d) \quad \longleftrightarrow \quad ac+ad+bc+bd$$

Extending These Ideas

Example 1.

$$\begin{aligned} &(1+a)(1+b)(1+c)(1+d) \\ &\qquad = \quad 1+(a+b+c+d)+(ab+ac+ad+bc+bd+cd) \\ &\qquad\qquad +(abc+abd+acd+bcd)+abcd \end{aligned}$$

Note the pattern and how the terms are obtained: the single terms $a \times 1 \times 1 \times 1$; the doubles $a \times b \times 1 \times 1$; the triples $a \times b \times c \times 1$ and finally $a \times b \times c \times d$. If you go about the expansions in a systematic way the patterns generally become obvious.

Example 2. $(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$

Example 3. $x - y = (\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$

Example 4.

$$\begin{aligned} x^4+x^2+1 &= x^4+2x^2+1-x^2 \\ &= (x^2+1)^2-x^2 \\ &= (x^2+1+x)(x^2+1-x). \end{aligned}$$

These latter two factoring "tricks" are very useful ideas.

Exercises

1. Write down the expansions of

(a) $(1+a)(1+b)(1+c)$;

(b) $(1+x)^3$;
(c) $(a+b+c)^3$;
(d) $(a+b)^4$.

2. Factor

(a) $a-b$ as the difference of two squares;
(b) $x+2\sqrt{xy}+y$;
(c) a^6-b^6 as the difference of two squares.

3. Factor fully

(a) $(a+b)^2-c^2$;
(b) $a^4+2a^2b^2+b^4$;
(c) $a^4+a^2b^2+b^4$;
(d) x^4+3x^2+4;
(e) $x^4-15x^2y^2+9y^4$;
(f) a^2-2a-b^2+1.

4. Find all the positive integer solutions to the equation

$$x^2+y^2+z^2=10(x+y+z).$$

Prove that there are no other solutions.

5. Express $2(a-b)(a-c)+2(b-c)(b-a)+2(c-a)(c-b)$ as the sum of three squares.

6. If $x+3$ divides $3x^2+x+k$ without remainder, find the value of k.

7. Factor (n^4+4) as the product of two quadratics.
For what positive integer values of n is (n^4+4) a prime number?

8. Factor

(a) $1+y(1+x)^2(1+xy)$;
(b) $1-b-a^2+a^3b+a^2b^3-a^3b^3$.

The Expansion of $(x+y)^n$ Where n is a Positive Integer

$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$
$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$
$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$

The coefficients are the well known Pascal's Triangle numbers, where each number is the sum of the two numbers above it.

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ . & & . & & . & & . & & . & & . & & . \end{array}$$

with

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{r}x^{n-r}y^r + \cdots + \binom{n}{n}y^n$$

where

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r(r-1)(r-2)\ldots 1} \text{ for } r > 0, \text{ and } \binom{n}{0} = 1.$$

Note that there are r factors in both the numerator and denominator. Check the following statements and ensure that you understand them.

- $\binom{11}{4} = \dfrac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2 \times 1} = 330.$
- $\binom{n}{n} = 1.$
- The next row in Pascal's triangle above is

$$\begin{array}{cccccccc} & \binom{6}{0} & \binom{6}{1} & \binom{6}{2} & \binom{6}{3} & \binom{6}{4} & \binom{6}{5} & \binom{6}{6} \\ \text{i.e.} & 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{array}$$

For the expansion of $(x-y)^n$ we set

$$\begin{aligned} (x-y)^n &= (x+(-y))^n \\ &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}(-y)^1 + \binom{n}{2}x^{n-2}(-y)^2 + \cdots + (-y)^n \\ &= x^n - \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 - \cdots + (-1)^n y^n. \end{aligned}$$

Exercises (continued)

9. (a) Find the values of $\binom{21}{3}$ and $\binom{12}{5}$;

(b) Show that

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

where $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$.

10. Show that the sum of the coefficients of the r^{th} term and of the $(r+1)^{th}$ term in the expansion of $(1+x)^n$ is equal to the coefficient of the $(r+1)^{th}$ term in the expansion of $(1+x)^{n+1}$. How is this result connected with Pascal's triangle?

11. Given that $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$ prove that

(a) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$;

(b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n\binom{n}{n} = 0$;

(c) $\binom{n}{0} + 2\binom{n}{1} + 4\binom{n}{2} + 8\binom{n}{3} + \cdots + 2^n\binom{n}{n} = 3^n$;

(d) $\binom{n}{0} - 2\binom{n}{1} + 4\binom{n}{2} - 8\binom{n}{3} + \cdots + (-1)^n 2^n\binom{n}{n} \begin{array}{l} = 1 \text{ if } n \text{ even} \\ = -1 \text{ if } n \text{ odd.} \end{array}$

12. Prove that

$$\binom{n}{r} = \frac{(n-r+1)}{r} \times \binom{n}{r-1}$$

and hence find the value of r that makes $\binom{n}{r}$ the greatest.

13. Check that in any row of Pascal's triangle the sum of the odd-numbered elements is equal to the sum of the even-numbered elements, i.e. $\binom{n}{1} + \binom{n}{3} + \ldots = \binom{n}{0} + \binom{n}{2} + \ldots$. Prove this result.

The Factors of $x^n \pm y^n$, n a Positive Integer

$$\begin{aligned}
x^3 - y^3 &= (x-y)(x^2 + xy + y^2) \\
x^4 - y^4 &= (x-y)(x^3 + x^2y + xy^2 + y^3) \\
x^5 - y^5 &= (x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4) \\
x^n - y^n &= (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1})
\end{aligned}$$

You should multiply out the expressions above to check that they are correct. Note the pattern of the second factor.

But the factorisation of $x^n + y^n$ produces a different result.

$$\begin{aligned} x^3 + y^3 &= (x+y)(x^2 - xy + y^2) \\ x^5 + y^5 &= (x+y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4) \\ x^7 + y^7 &= (x+y)(x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6) \end{aligned}$$

and in general

$$x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots - xy^{n-2} + y^{n-1})$$

for odd positive integral n.
Again, multiply out these expressions to check that they are correct.

Exercises (continued)

14. Explain why

(a) $5^{39} - 2^{39}$ is divisible by 3;

(b) $2^{99} + 3^{99}$ is divisible by 5;

(c) $2^{98} + 3^{98}$ is not divisible by 5;

(d) $2^{99} + 3^{99} + 4^{99} + 5^{99}$ is divisible by 7;

(e) $2^{99} - 4^{99} - 7^{99} + 9^{99}$ is divisible by 10.

15. Find a formula for finding the value of

(a) $1 + x + x^2 + \cdots + x^n$ for all positive integral n;

(b) $1 - x + x^2 - \cdots + x^n$ for all even positive integral n.

16. Factor

(a) $a^6 - b^6$ as the product of four factors;

(b) $a^2(b-c) + b^2(c-a) + c^2(a-b)$.

17. If x and y are positive integers find all solutions to the equation

$$x^2 - 871 = y^6.$$

18. If $\left(a - \dfrac{1}{a}\right)^2 = 3$ and $a - \dfrac{1}{a} > 0$ find the value of

(a) $a^3 - \dfrac{1}{a^3}$;

(b) $a^4 + \dfrac{1}{a^4}$.