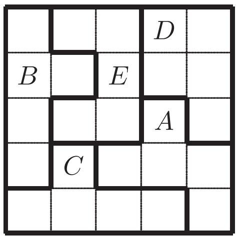
DIRICHLET STUDENT PROBLEMS: SOLUTIONS

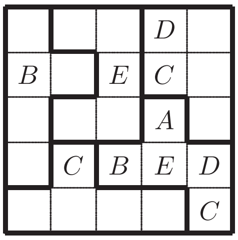
PROBLEM 1

A 5×5 grid is made from five pentominoes and some squares have letters written in them as shown. Fill the remaining squares so that every pentomino and every row and column of the grid contains each of the letters A, B, C, D, E exactly once. Give your reasoning.



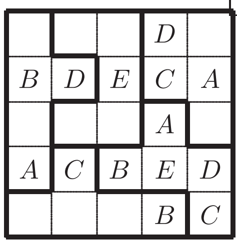
SOLUTION 1

Check each row, column, and pentomino to see where any missing letter could go. In the bottom right pentomino, C can only go in the bottom square. Hence C must go under D in the top right pentomino. So A, B, E must go somewhere in the right column of the top right pentomino. This leaves only D to go above C in the bottom right pentomino. Then E must go next to D leaving only B to go on the left of E in the bottom right pentomino. Thus we get:



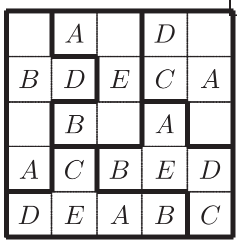
1

Only B is left to go in the remaining square of column 4 and only A is left to go in the remaining square of row 4. This means A can not go next to B in the top left pentomino so must go in the other vacant square of row 2, leaving only D to go next to B in row 2.



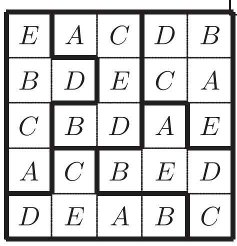
1

In column 2, E must go in the bottom square. Then, in the bottom left pentomino, A must go between E and B leaving only D to go on the left of E. Also in column 2, A must now go in the top square leaving only B to go in the middle square.



1

This forces B to go in the top square of column 5 and D to go in the middle square of column 3. Hence E must go in the middle square of column 5 and C must go in the top square of column 3. Finally, only E is left for the remaining square of row 1 and only C for the remaining square of row 3.



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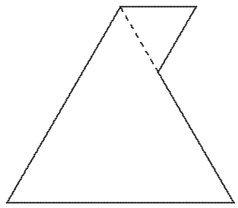
PROBLEM 2

A pentagon is formed from an equilateral triangle by attaching a smaller equilateral triangle to one of its sides as shown.

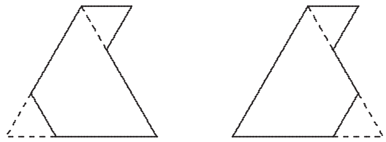
(a) Show in two different ways how to cut a single piece from this pentagon to produce two different polygons that will tessellate separately.

(b) For each polygon from Part (a), draw two different tessellations that show at least 12 copies of the polygon.



SOLUTION 2

(a) From each of the bottom corners of the large triangle, cut off a small equilateral triangle that is congruent to the small triangle in the diagram above. Thus we get two different hexagons.



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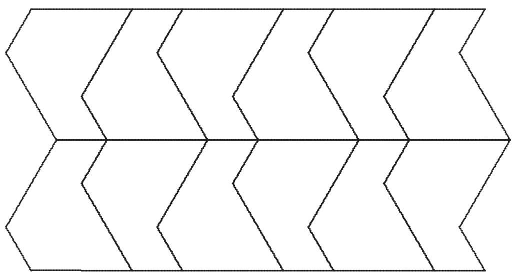
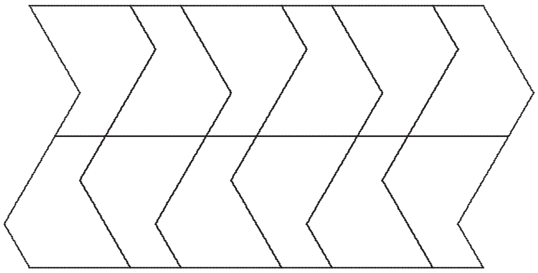
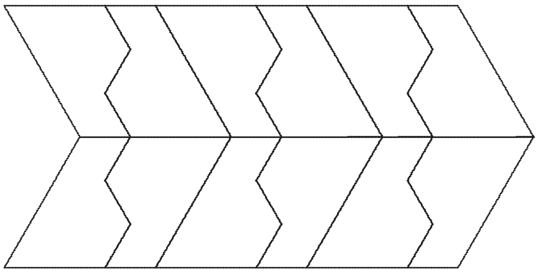
(b) The first hexagon gives the following tessellations. There are others.

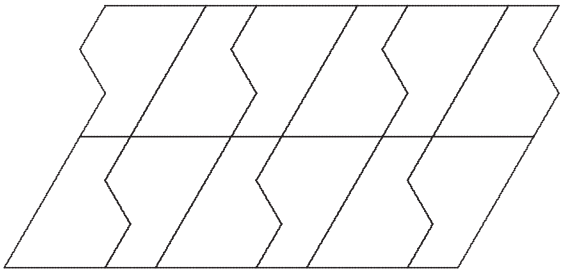




1

The second hexagon gives the following tessellations. There are others.





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PROBLEM 3

(a) Determine the base in which each of the following calculations was done.

$$243 \times 2 = 1041 \qquad 323 \times 3 = 1302$$

(b) Determine the base in which each of the following calculations was done.

$$432 \times 22 = 11724 \qquad 341 \times 23 = 11503$$

SOLUTION 3

(a) The digit 4 appears in the first calculation so the base is at least 5. Perform the multiplication in base 5:

$$\begin{array}{cccc} & 2 & 4 & 3 \\ & & \times & 2 \\ \hline 1 & 0 & {}^{1}4 & {}^{1}1 \end{array}$$

This agrees with the given calculation. No other base will work because base 6 will give 0 as the units digit in the answer, and any higher base will give 6 as the units digit. Hence the calculation was done in base 5. **[1]**

The digit 3 appears in the second calculation so the base is at least 4. Now 3×3 equals 21 in base 4, 14 in base 5, 13 in base 6, 12 in base 7, 11 in base 8, 10 in base 9, and 9 in any higher base. Only base 7 gives the correct units digit of 2. Perform the multiplication in base 7:

$$\begin{array}{cccc} & 3 & 2 & 3 \\ & & \times & 3 \\ \hline 1 & 3 & {}^{1}0 & {}^{1}2 \end{array}$$

This agrees with the given calculation and no other base will work, hence the calculation was done in base 7. **[1]**

(b) The digit 7 appears in the first calculation so the base is at least 8. Perform the multiplication in base 8:

$$\begin{array}{cccccc} & & & 4 & 3 & 2 \\ & & & \times & 2 & 2 \\ \cline{3-6} & & 1 & 0 & 6 & 4 \\ + & 1 & 0 & 6 & 4 & 0 \\ \hline & 1 & 1 & 7 & {}^{1}2 & 4 \end{array}$$

This agrees with the given calculation. No other base will work because base 9 will give 1 as the second digit (from the right) in the answer, base 10 will give 0, and any higher base will give some other symbol as the second digit. Hence the calculation was done in base 8. **1**

The digit 5 appears in the second calculation so the base is at least 6. The second digit from the right in the answer is the units digit in the calculation $3 \times 4 + 2 \times 1$. This digit would be 2 in base 6, 0 in base 7, 6 in base 8, 5 in base 9, 4 in base 10, 3 in base 11, 2 in base 12, 1 in base 13, 0 in base 14, and some other symbol in any higher base. Only base 7 and 14 gives the correct digit of 0. Perform the multiplication in base 7 and 14 (W represents 12 in base 14):

$$
\begin{array}{cccccc}
 & & & 3 & 4 & 1 \\
 & & & \times & 2 & 3 \\
\cline{3-6}
 & & 1 & 3 & {}^{1}5 & 3 \\
+ & 1 & 0 & {}^{1}1 & 2 & 0 \\
\hline
 & 1 & 1 & 5 & {}^{1}0 & 3
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & 3 & 4 & 1 \\
 & & \times & 2 & 3 \\
\cline{3-5}
 & & 9 & \mathrm{W} & 3 \\
+ & 6 & 8 & 2 & 0 \\
\hline
 & 7 & {}^{1}4 & {}^{1}0 & 3
\end{array}
$$

Only the first of these agrees with the given calculation and no other base will work, hence the calculation was done in base 7. **1**

PROBLEM 4

Sue's favourite number is 6, the more the better. She made a big number consisting of lots of consecutive 6s followed by a 7 (her second favourite number) and squared it. What is the maximum number of 6s Sue can have in her number if the sum of the digits in its square is less than 2008?

SOLUTION 4

Sue simplifies the problem by using only a few 6s to start with. She notices on her calculator that $67^2 = 4489$, $667^2 = 444889$, and $6667^2 = 44448889$. There seems to be a pattern: $6\cdots67^2 = 4\cdots48\cdots89$, where the number of 4s is one more than the number of 6s and the number of 8s equals the number of 6s. **1**

To check the pattern, Sue writes $6\cdots67 = 7 + 60 + 600 + \cdots + 60\cdots0$, where the number of 0s in $60\cdots0$ equals the number of 6s in $6\cdots67$. Then

$$\begin{aligned} 6\cdots67^2 &= 6\cdots67 \times 7 + 6\cdots67 \times 60 + \cdots + 6\cdots67 \times 60\cdots0 \\ &= 46\cdots69 + 40\cdots020 + \cdots + 40\cdots020\cdots0. \end{aligned}$$

To see the pattern more clearly, Sue writes the calculation as follows.

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 6 & 6 & \cdot & \cdot & 6 & 6 & 6 & 7 \\
 & & & & & & \times & 6 & 6 & \cdot & \cdot & 6 & 6 & 6 & 7 \\ \cline{7-15}
 & & & & & & 4 & 6 & 6 & \cdot & \cdot & 6 & 6 & 6 & 9 \\
 & & & & & 4 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 & 2 & 0 \\
 & & & & 4 & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 2 & 0 & 0 \\
 & & & & & & & & & & & & & & \vdots \\
+ & 4 & 0 & \cdot & 0 & 0 & 0 & 2 & 0 & \cdot & \cdot & 0 & 0 & 0 & 0 \\ \hline
 & 4 & 4 & \cdot & 4 & 4 & 4 & 8 & 8 & \cdot & \cdot & 8 & 8 & 8 & 9
\end{array}$$

The number of 8s obviously equals the number of 6s in $6\cdots67$ and, since the number of 0s between 4 and 2 in each line stays the same and equals the number of 6s, the number of 4s in the answer is one more than the number of 6s. **2**

If the number of 6s in $6\cdots67$ is n, then the sum of the digits in $6\cdots67^2$ is $4 + 4n + 8n + 9 = 13 + 12n$. By guessing and checking, Sue finds that the maximum value of n for which $13 + 12n$ is less than 2008 is 166. **1**

PROBLEM 5

Each working day Bahir normally rides his bike 3 km from home to the railway station, takes the 36 minute train trip to town, and then rides his bike 2 km downhill from the station to his work place at an average speed of 24 km/h. The train averages 100 km/h and the first bike ride takes 9 minutes.

(a) What distance is the train trip?

(b) What is the average speed for Bahir's trip from home to work?

(c) One day Bahir drove his car to work. The trip was 2 km shorter but the average speed was 36 km/h slower. How much longer did this car trip take?

SOLUTION 5

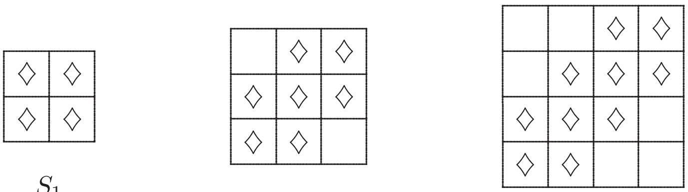
(a) The train trip takes 36 minutes $= \frac{36}{60}\text{h} = \frac{6}{10}$ hours.
Distance = average speed × time taken $= 100 \times \frac{6}{10} = 60\,\text{km}$. **1**

(b) The total distance for the trip $= 3 + 60 + 2 = 65\,\text{km}$.
The time taken for his second bike ride = distance ÷ average speed $= \frac{2}{24} = \frac{1}{12}$ h = 5 minutes. **1**
The total time for the trip $= 9 + 36 + 5 = 50$ minutes $= \frac{50}{60}\,\text{h} = \frac{5}{6}$ hours.
Average speed = total distance ÷ time taken
$= 65 \div \frac{5}{6} = (65 \times 6) \div (\frac{5}{6} \times 6) = (65 \times 6) \div 5 = 78\,\text{km/h}$. **1**

(c) Time taken for the car trip = total distance ÷ average speed
$= \frac{63}{42}$ hours $= \frac{63}{42} \times 60$ minutes = 90 minutes.
So the car trip took an extra $90 - 50 = 40$ minutes. **1**

PROBLEM 6

Deborah drew a sequence of square grid patterns with diamonds in some of the small squares as shown.



S_1 S_2 S_3

(a) Draw S_4.

(b) How many diamonds are there in S_5?

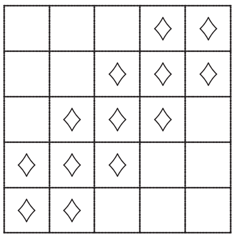
(c) How many diamonds are there in S_n?

(d) How many small squares in S_n do not contain diamonds?

SOLUTION 6

(a)

S_4



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(b) There are 13 diamonds in S_4. There are 3 extra diamonds in S_5, giving a total of 16. 1

(c) Let D_n be the number of diamonds in S_n. We have $D_1 = 4$, $D_2 = 7$, $D_3 = 10$, $D_4 = 13$, and $D_5 = 16$. The number of diamonds increases by 3 from one pattern to the next. By subtracting 1 from each term we get the sequence 3, 6, 9, 12, 15, ... The nth term of this sequence is $3n$. Adding back the 1 gives $D_n = 3n + 1$.

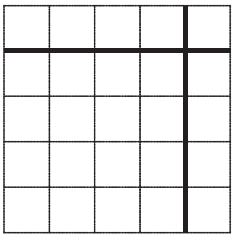
1

(d) **Alternative i**

The number of small squares that do not contain diamonds forms the sequence 0, 2, 6, 12, 20, ... Compare this sequence with the sequence of squares: 1, 4, 9, 16, 25, ... This suggests the formula $n^2 - n$ for the number of squares in S_n that do not contain diamonds. **[1]**

Alternative ii

S_n consists of an $n \times n$ grid of small squares plus a row of n small squares, a column of n small squares, and 1 small square. For example the number of small squares in S_4 is $4^2 + 4 + 4 + 1$:



Thus the total number of small squares in S_n is $n^2 + n + n + 1 = n^2 + 2n + 1$.

So the number of squares that do not contain diamonds is $n^2 + 2n + 1 - (3n + 1) = n^2 - n$. **[1]**

PROBLEM 7

Trevor is playing *Money Trail* in a TV show. There is a 5×5 square grid of glass panels on the studio floor with dollar amounts written on them and lights underneath. He starts on the panel marked 'S' and when the bell rings he steps from one panel to the next and tries to reach the panel marked 'F' before the buzzer goes. At each step Trevor is only allowed to step to the next panel up or to the right. If he does this correctly the light under the panel he moves to flashes on and off. If he breaks this rule a hooter blows, all the lights go out and he has to leave the game. If he reaches 'F' in time he collects all the money he stepped on and the path of panels that would have given the maximum amount lights up.

$10	$30	$60	$10	F
$70	$20	$10	$20	$70
$10	$20	$80	$10	$10
$30	$10	$50	$20	$30
S	$20	$10	$90	$10

(a) What is the maximum amount of money Trevor can win?

(b) Show on the grid the path(s) Trevor must take to win the maximum amount of money.

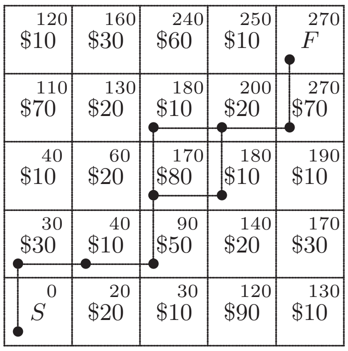
SOLUTION 7

(a) Trevor arrives at a panel P either from the panel below or the panel to the left, if they exist. The maximum amount of money he could collect up to P is the bigger of the money he could collect up to either of those previous panels plus the amount at P. Write these maximum amounts (without dollar signs) on each panel, row by row, starting with those next to S. **1**

120 $10	160 $30	240 $60	250 $10	270 F
110 $70	130 $20	180 $10	200 $20	270 $70
40 $10	60 $20	170 $80	180 $10	190 $10
30 $30	40 $10	90 $50	140 $20	170 $30
0 S	20 $20	30 $10	120 $90	130 $10

Thus the maximum amount Trevor can win is $270. **1**

(b) To find a path Trevor must take to win the maximum amount of money, work backwards. Starting at F move to the panel with the maximum total 270. Subtract the dollar amount on that panel from 270 and move to the panel that has the same total as your answer. Continue until you reach S. We get two paths: **1**



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PROBLEM 8

There is a rule for the decimal expansion of $\frac{1}{m}$, regardless of whether m is prime or not:

the number of digits in the repetend divides the number of integers from 1 to m whose highest common factor with m is 1.

For example $\frac{1}{21} = 0.\overline{047619}$, the number of digits in the repetend is 6, and there are 12 integers from 1 to 21 whose highest common factor with 21 is 1: 1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20.

(a) Check this rule for $\frac{1}{51}$.

(b) Find the family of fractions amongst $\frac{n}{51}$ for $n = 1, 2, 3, ..., 50$, that have the same digits in their repetends as $\frac{1}{51}$ but cyclically rotated.

(c) Check the rule for all other families of fractions amongst $\frac{n}{51}$ for $n = 1, 2, 3, ..., 50$, that have the same digits in their repetends but cyclically rotated.

SOLUTION 8

(a) From a calculator, $\dfrac{1}{51} \approx 0.01960784$, $0.784 \times 51 = 39.984$, $\dfrac{40}{51} \approx 0.78431372$, $0.372 \times 51 = 18.972$, $\dfrac{19}{51} \approx 0.37254902$, $0.902 \times 51 = 46.002$, $\dfrac{46}{51} \approx 0.90196078$.

So $\dfrac{1}{51} = 0.\overline{0196078431372549}$.

The number of digits in the repetend is 16. This divides 32, which is the number of integers from 1 to 51 whose highest common factor with 51 is 1: 1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 19, 20, 22, 23, 25, 26, 28, 29, 31, 32, 35, 37, 38, 40, 41, 43, 44, 46, 47, 49, 50.

1

(b)

Repetend for $\frac{n}{51}$	Approximate value for n	Exact value for n
0196078431372549	$51 \times 0.019 = 0.969$	1
1960784313725490	$51 \times 0.196 = 9.996$	10
9607843137254901	$51 \times 0.960 = 48.960$	49
6078431372549019	$51 \times 0.607 = 30.957$	31
0784313725490196	$51 \times 0.078 = 3.978$	4

Repetend for $\frac{n}{51}$	Approximate value for n	Exact value for n
7843137254901960	$51 \times 0.784 = 39.984$	40
8431372549019607	$51 \times 0.843 = 42.993$	43
4313725490196078	$51 \times 0.431 = 21.981$	22
3137254901960784	$51 \times 0.313 = 15.963$	16
1372549019607843	$51 \times 0.137 = 6.987$	7
3725490196078431	$51 \times 0.372 = 18.972$	19
7254901960784313	$51 \times 0.725 = 36.975$	37
2549019607843137	$51 \times 0.254 = 12.954$	13
5490196078431372	$51 \times 0.549 = 27.999$	28
4901960784313725	$51 \times 0.490 = 24.990$	25
9019607843137254	$51 \times 0.901 = 45.951$	46

1

(c) From similar calculations, we find two more families with 16 digits in their repetends:

$\frac{2}{51} = 0.\overline{0392156862745098}$ and 15 other fractions by rotation,
$\frac{3}{51} = 0.\overline{1176470588235294}$ and 15 other fractions by rotation.

These two families and the family in Part (b) account for $3 \times 16 = 48$ fractions. Since there are 50 fractions altogether, there can not be any more families with 16 digits in their repetends. **1**

There are two families of one fraction each that have 1 digit in their repetends: $\frac{17}{51} = 0.\overline{3}$, $\frac{34}{51} = 0.\overline{6}$. Of course 1 also divides 32.

These five families account for $48 + 2 = 50$ fractions so there are no others. **1**