EULER STUDENT SAMPLE PROBLEMS: SOLUTIONS

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PROBLEM 1

Both the leftmost digit and the rightmost digit of a four-digit number N are equal to 1. When these digits are removed, the two-digit number thus obtained is $N \div 21$. Find N.

SOLUTION 1

Alternative 1

Let M denote the two-digit number that is left after the rightmost and leftmost digits of N have been removed.
Then we have

$$N = 1000 + 10 \times M + 1. \qquad \boxed{1}$$

Thus $N = 10M + 1001$.
Since N is twenty one times greater than M, we obtain

$$10M + 1001 = 21M. \qquad \boxed{1}$$

Hence $11M = 1001$.
Therefore $M = 1001 \div 11 = 91$. $\boxed{1}$
Thus $M = 91$ which implies that $N = 1911$. $\boxed{1}$

Alternative 2

Let $N = 1AB1$ where A and B are digits.
Then we have the following multiplication pattern

$$AB \times 21 = 1AB1. \qquad \boxed{1}$$

Since $B \times 1 = B$, the multiplication pattern implies $B = 1$. $\boxed{1}$

Therefore it follows from the multiplication pattern that the last digit of $A + 2$ must be 1 as $B = 1$.
Hence $A = 9$. $\boxed{1}$

Thus N must be 1911 and checking shows that this is a correct answer. $\boxed{1}$

Note. A computer program or spreadsheet must explain how it has been constructed for full marks (no explanation gets 1 mark - the last). Trial and error or similar solutions that do not show that the answer is the only one gain 1 mark.

PROBLEM 2

Find all three-digit even numbers N such that $693 \times N$ is a perfect square, that is, $693 \times N = k^2$ where k is an integer.

SOLUTION 2

First of all, we notice that $693 = 3^2 \times 7 \times 11$.
Hence N is divisible by both 7 and 11 as $693 \times N$ is a perfect square.
Therefore N is divisible by 77. [1]

Since N is even, N is divisible by 2.
Hence N is divisible by $2^2 = 4$ as 693 is odd and $693 \times N$ is a perfect square.
Therefore N is divisible by $77 \times 4 = 308$. [1]

The only three-digit numbers that are divisible by 308 are 308, 616 and 924.
If $N = 308$, then $693 \times N$ is a perfect square as $693 \times N = 2^2 \times 3^2 \times 7^2 \times 11^2$.
If $N = 616$, we have $693 \times N = 2^3 \times 3^2 \times 7^2 \times 11^2$ and $693 \times N$ is not a perfect square. If $N = 924$, we have $693 \times N = 2^2 \times 3^3 \times 7^2 \times 11^2$ and $693 \times N$ is not a perfect square. [1]

Thus the only three-digit even number such that $693 \times N$ is a perfect square is 308. [1]

Note. A computer program or spreadsheet must explain how it has been constructed for full marks (no explanation gets 1 mark - the last). Trial and error or similar solutions that do not show that the answer is the only one gain 1 mark.

PROBLEM 3

The parliament of the land of Achronia consists of two houses. The parliament was elected in 1995 for a period of four years beginning on Monday, January 1st, 1996, when the two houses had their first sessions. According to the rules, the meetings of the first house must occur every ten days for the duration of the term, and the meetings of the second house must occur every twelve days. For example, the second meetings of the first and the second houses were held on the 11th and 13th of January respectively. A new law can be passed only when both houses meet on a Monday. How many opportunities will the parliament members have to pass new laws during this four year term?

SOLUTION 3

We have $10 = 2 \times 5$ and $12 = 2^2 \times 3$.
Hence the least common multiple of 10, 12 and 7 is 420. $\boxed{1}$

Now we notice that January 1st, 1996, falls on a Monday.
Therefore, the next chance for the members of the parliament to pass new laws will be 420 days later. $\boxed{1}$

The whole term consists of $4 \times 365 + 1 = 1461$ days. $\boxed{1}$
Since $1461 = 420 \times 3 + 201$, we come to the conclusion that during the four year term the parliament will have four days when it is able to pass new laws. $\boxed{1}$

Note. A list of all the days when each of the houses meet with appropriate reasoning gets four marks. Correct answer without explanation gets 1 mark.

PROBLEM 4

When a certain number N is divided by d, the remainder is 7. If the original number N is multiplied by 5 and then divided by d, the remainder is 10. Find d.

SOLUTION 4

We know that if N is divided by d, the remainder is 7.
Hence $N = dx + 7$ where x is an integer.
Therefore $5N = 5dx + 35$.
On the other hand, $5N = dy + 10$ where y is an integer.
Hence $5dx + 35 = dy + 10$ which implies $d(y - 5x) = 25$.
Thus 25 is divisible by d. [1]

Since the only divisors of 25 are 1, 5 and 25, d may only be one of these three numbers. [1]
But we know that if a number is divided by another number, the remainder is less than the divisor.
Therefore d can be neither 1 nor 5 as the two remainders obtained are 7 and 10. [1]

Hence d is equal to 25. [1]

PROBLEM 5

Find the sum of all positive integers not greater than 10000 that are divisible by either 3 or 11 but not by both of them.

SOLUTION 5

First of all, we notice that if an integer is divisible by both 3 and 11, it is divisible by 33.

Now let K be the sum of all integers from 1 up to 10000 that are divisible by 3, L be the sum of all integers from 1 up to 10000 that are divisible by 11 and M be the sum of all integers between 1 and 10000 that are divisible by 33. Then the sum we are required to find equals $(K-M)+(L-M) = K+L-2M$.

We have

$$K = 3+6+9+\cdots+9999 = \frac{3333(3+9999)}{2} = 16668333, \qquad \boxed{1}$$

$$L = 11+22+33+\cdots+9999 = \frac{909(11+9999)}{2} = 4549545 \qquad \boxed{1}$$

and

$$M = 33+66+99+\cdots+9999 = \frac{303(9999+33)}{2} = 1519848. \qquad \boxed{1}$$

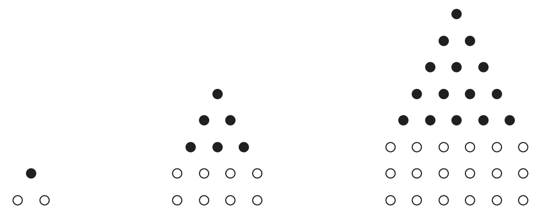
Hence $K+L-2M = 16668333+4549545-2\times 1519848 = 18178182$.

Thus the answer is 18178182. $\boxed{1}$

Note. A computer program or spreadsheet must explain how it has been constructed for full marks. Correct answer without explanation gets 1 mark.

PROBLEM 6

Consider the following pattern:



$Y_1 = 3$ $Y_2 = 14$ $Y_3 = 33$

(a) Find Y_5.

(b) Find a rule for Y_n.

(c) Find Y_{20}.

SOLUTION 6

(a) $Y_5 = 2 \times 25 + 45 = 95$. [1]

(b) $Y_n = 2n^2 + T_{2n-1} = 2n^2 + \frac{2n(2n-1)}{2} = 2n^2 + n(2n-1) = n(4n-1)$. [2]

(c) $Y_{20} = 20(80-1) = 20 \times 79 = 1580$. [1]

PROBLEM 7

(a) Find the remainder when 27^{1001} is divided by 13.

(b) Find the remainder when 38^{101} is divided by 13.

(c) Hence show that $70 \times 27^{1001} + 31 \times 38^{101}$ is divisible by 13.

SOLUTION 7

(a) We have $27 \equiv 1 \pmod{13}$.
Hence $27^{1001} \equiv 1^{1001} \equiv 1 \pmod{13}$.
Thus when 27^{1001} is divided by 13, the remainder is 1. $\boxed{1}$

(b) Since $38 \equiv -1 \pmod{13}$, we obtain

$$38^{101} \equiv (-1)^{101} \equiv -1 \equiv 12 \ (\text{mod } 13).$$

Thus when 38^{101} is divided by 13, the remainder is 12. $\boxed{1}$

(c) We have $70 \equiv 5 \pmod{13}$.
Therefore $70 \times 27^{1001} \equiv 5 \times 1 \equiv 5 \pmod{13}$.

Since $31 \equiv 5 \pmod{13}$, we have $31 \times 38^{101} \equiv 5 \times (-1) \equiv -5 \equiv 8 \pmod{13}$. $\boxed{1}$

Therefore, $70 \times 27^{1001} + 31 \times 38^{101} \equiv 5 + 8 \equiv 13 \equiv 0 \pmod{13}$.
Thus $70 \times 27^{1001} + 31 \times 38^{101}$ is divisible by 13. $\boxed{1}$

PROBLEM 8

The "Good Book" publishing company invited representatives from a certain number of schools to its presentation where it planned to donate 3050 books to the schools so that each school would get the same number of books. But it turned out that eleven of the schools initially invited did not come to the presentation. However, the publishing company managed to divide the books equally among the schools that attended the ceremony. How many books did each of the schools get at the presentation?

SOLUTION 8

Let the number of the schools that were initially invited to the presentation be N.
Then the number of the schools that in fact came to the presentation was $N - 11$.

Therefore both N and $N - 11$ divide 3050. [1]

Since $3050 = 2 \times 5^2 \times 61$, the full list of factors of 3050 is

$$1,\ 2,\ 5,\ 10,\ 25,\ 50,\ 61,\ 122,\ 305,\ 610,\ 1525 \text{ and } 3050.$$ [1]

From this list, we can see that the only pair of factors that differ by 11 is 50 and 61. [1]
Thus the number of the schools that were given books at the presentation is 50.

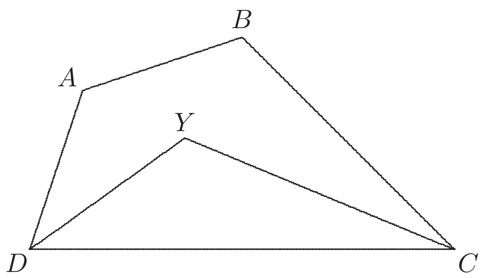
Therefore each of the schools got $3050 \div 50 = 61$ books. [1]

Note. A spreadsheet solution must explain how it has been constructed for full marks. Correct answer without explanation gets 1 mark. Trial and error or similar solutions that do not show that the answer is the only one gain 1 mark.

PROBLEM 9

In a quadrilateral $ABCD$, $\angle A = 120^\circ$ and $\angle B = 140^\circ$. The bisectors of the angles C and D meet at the point Y. Find the size of the angle DYC.

SOLUTION 9



The angle sum of the quadrialteral $ABCD$ is 360°.
Hence $\angle A + \angle B + \angle C + \angle D = 360^\circ$. $\boxed{1}$

Therefore $120^\circ + 140^\circ + \angle C + \angle D = 360^\circ$ as $\angle A = 120^\circ$ and $\angle B = 140^\circ$.
Hence $\angle C + \angle D = 360^\circ - 120^\circ - 140^\circ = 100^\circ$. $\boxed{1}$

In the triangle CYD, $\angle DYC + \angle YDC + \angle YCD = 180^\circ$ as the angle sum of a triangle is 180°.
Hence $\angle DYC = 180^\circ - (\angle YDC + \angle YCD)$. $\boxed{1}$

Since DY bisects the angle ADC, we have $\angle YDC = \dfrac{\angle ADC}{2}$.

Since CY bisects the angle BCD, we have $\angle YCD = \dfrac{\angle BCD}{2}$.

Therefore

$$\begin{aligned}
\angle DYC &= 180^\circ - (\angle YDC + \angle YCD) \\
&= 180^\circ - \left(\frac{\angle ADC}{2} + \frac{\angle BCD}{2}\right) \\
&= 180^\circ - \frac{\angle ADC + \angle BCD}{2} \\
&= 180^\circ - \frac{100^\circ}{2} = 180^\circ - 50^\circ = 130^\circ.
\end{aligned}$$

Thus $\angle DYC = 130^\circ$. $\boxed{1}$

PROBLEM 10

Wetlands High School has done it! It is the lucky school to win a trip to Disneyland for 2 girls, 2 boys and a teacher. The 5 lucky people are to be selected randomly from the school's computer records. If there are 100 boys, 80 girls and 20 teachers on the record, in how many ways can the group of 5 be selected?

SOLUTION 10

There are $\binom{80}{2} = \frac{80 \times 79}{2} = 3160$ ways to select 2 girls for the trip.

$\boxed{1}$

There are $\binom{100}{2} = \frac{100 \times 99}{2} = 4950$ ways to select 2 boys for the trip.

$\boxed{1}$

Finally, there are $\binom{20}{1} = 20$ ways to select a teacher. $\boxed{1}$

Therefore, there are $3160 \times 4950 \times 20 = 312840000$ ways to select a group of 5 for the trip. $\boxed{1}$

Note. A computer program solution must explain how it has been constructed for full marks. Correct answer without explanation gains 1 mark.

PROBLEM 11

Ninety one five-digit numbers are written on a blackboard. Prove that one can find three numbers on the blackboard such that the sums of their digits are equal.

SOLUTION 11

The minimal value that the sum of the digits of a five-digit number can have is 1.
The maximal value that the sum of the digits of a five-digit number can have is $9 \times 5 = 45$. $\boxed{1}$

Let us partition the numbers on the blackboard into 45 groups.
The first group contains the numbers with the sums of their digits equal to 1, the second group the numbers with the sums of their digits equal to 2, the third group the numbers with the sums of their digits equal to 3 and so on.
The last group, 45th, consists of numbers such that the sums of their digits is 45.

Thus we have 91 numbers that are divided into 45 groups. $\boxed{1}$

By the Pigeon-Hole Principle, since $91 > 2 \times 45$, there is a group with at least three numbers in it. $\boxed{1}$

Therefore, there are at least three numbers on the blackboard such that the sums of their digits are equal. $\boxed{1}$

PROBLEM 12

Tom was floating down the river on a raft when, 1 km lower down, Michael took to the water in a rowing boat. Michael rowed downstream at his fastest pace. Then he turned around and rowed back, arriving at his starting point just as Tom drifted by. If Michael's rowing speed in still water is ten times the speed of the current in the river, what distance had Michael covered before he turned his boat around?

SOLUTION 12

Let the speed of the current be u km/h.
Then Michael's speed in still water would be $10u$ km/h.
Hence Michael's speed downstream is $11u$ km/h and his speed upstream is $9u$ km/h. [1]

Let the distance covered by Michael before turning be d km.
Then the time that Michael took to go downstream and back equals

$$\frac{d}{11u} + \frac{d}{9u}. \qquad \boxed{1}$$

Since it took Tom the same time to drift 1 km, we obtain the following equation:

$$\frac{d}{11u} + \frac{d}{9u} = \frac{1}{u}. \qquad \boxed{1}$$

Hence $\frac{20d}{99u} = \frac{1}{u}$ which implies $d = \frac{99}{20} = 4.95$ km.

Thus the answer is 4 kilometres 950 metres. [1]