NEWTON STUDENT PROBLEMS: SOLUTIONS

PROBLEM 1

Four towns, Alber, Ballin, Coybar, and Dinmar, are to be connected by broadband optical fibre communication cable. Various routes were surveyed to estimate the shortest amounts of cable required between each pair of towns. The results are shown in this table.

town pair	cable length in km
Alber, Ballin	220
Alber, Coybar	250
Alber, Dinmar	210
Ballin, Coybar	190
Ballin, Dinmar	150
Coybar, Dinmar	170

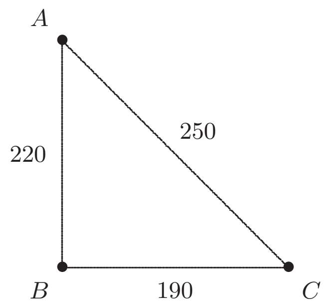
(a) Find the least amount of cable required to connect just Alber, Ballin, and Coybar.

(b) If we connect all four towns with the least amount of cable, explain why the links we use between pairs of towns will not form a circuit.

(c) Find the least amount of cable required to connect all four towns.

SOLUTION 1

(a) Draw a diagram to indicate the distances between Alber (A), Ballin (B), and Coybar (C).

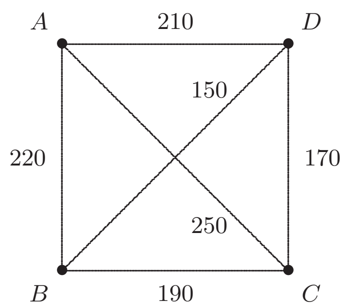


We don't need all three links, so remove the longest which is 250 km. Thus the least amount of cable required to connect just Alber, Ballin, and Coybar is $190 + 220 = 410$ km. **1**

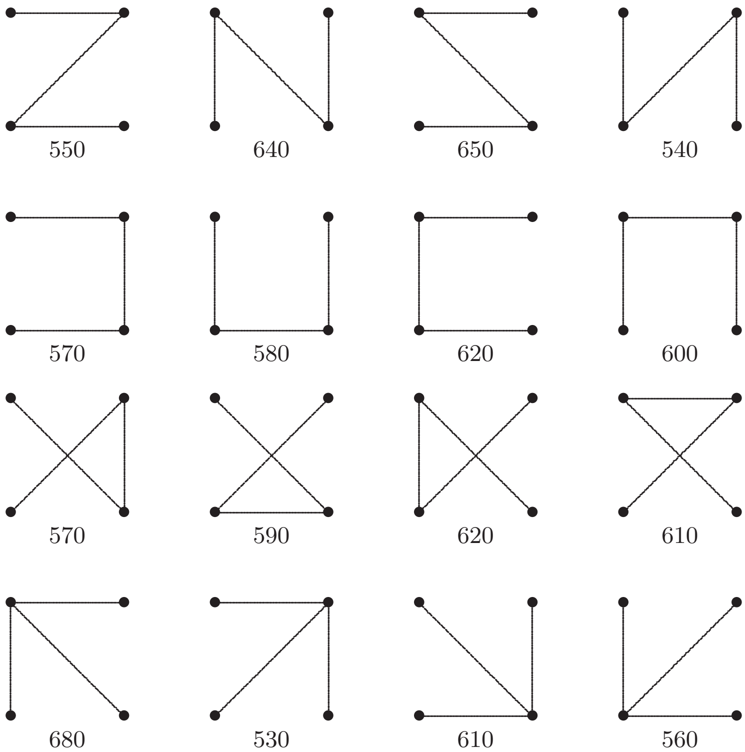
(b) If a network of cables connects all towns and has links that form a circuit, then we can remove one of the links in the circuit and this would leave a network that still connects the towns but uses less cable.

So if we want a network of links that connects all the towns and uses the least amount of cable, then the network must have no circuits. **1**

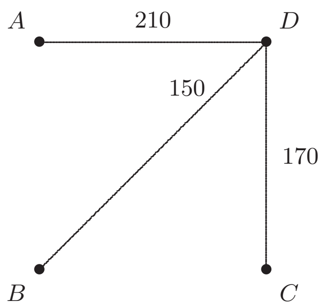
(c) Draw a diagram to indicate the distances between Alber (A), Ballin (B), Coybar (C), and Dinmar (D).



There are 16 ways to choose the links so that all towns are connected and there are no circuits. These are indicated in the following diagram where the town names and link distances have been omitted. The number below each network is the amount of cable in km it uses.



If we remove any link in any of these networks, then not all the towns would be connected. So by comparing the amount of cable used in each network, we see that the least amount of cable required to connect all four towns is $150 + 170 + 210 = 530\,\text{km}$ as in the following diagram.

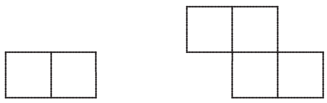


2

PROBLEM 2

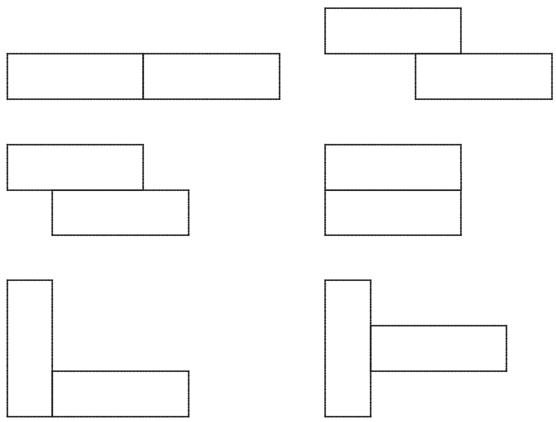
(a) Some hexominoes can be made by joining two straight triominoes together. Draw all of the hexominoes that can be made in this way, showing clearly how the two triominoes are joined.

(b) Some hexominoes can be made by joining a domino and a 'Z' tetromino together. Draw all of the hexominoes that can be made in this way, showing clearly how the two triominoes are joined.



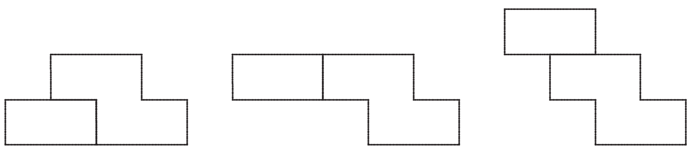
SOLUTION 2

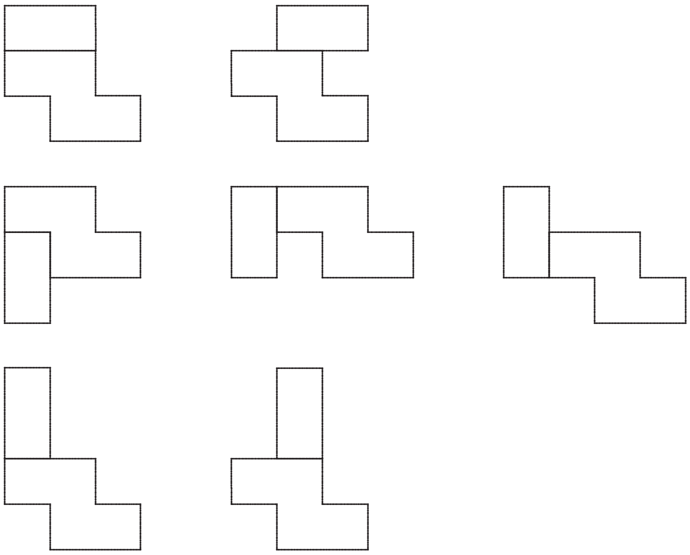
(a) Six hexominoes can be made from two straight triominoes:



2

(b) Ten hexominoes can be made from a domino and a 'Z' tetromino:





2

PROBLEM 3

(a) Write down a rule for multiplying by 31 horizontally.

(b) Show how it works with the example 31×25468.

(c) Show how to check the answer using digit sums.

SOLUTION 3

(a) Go along the number from right to left, adding each digit to three times the digit on its right. **[1]**

(b) 31×25468.

Add 8 to 3×0, giving 8. Write 8.	8
Add 6 to 3×8, giving 30. Write 0. Carry 3.	$^{3}08$
Add 4 to 3×6, then add 3, giving 25. Write 5. Carry 2.	$^{2}508$
Add 5 to 3×4, then add 2, giving 19. Write 9. Carry 1.	$^{1}9508$
Add 2 to 3×5, then add 1, giving 18. Write 8. Carry 1.	$^{1}89508$
Add 0 to 3×2, then add 1, giving 7. Write 7.	789508

Answer: 789508 **[2]**

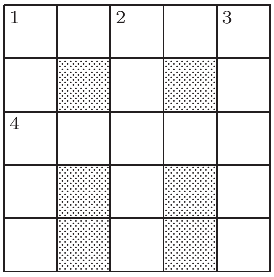
(c) Digit sum of $31 \times 25468 =$ digit sum of $(4 \times 25) =$ digit sum of $100 = 1$.

Digit sum of $789508 =$ digit sum of $37 =$ digit sum of $10 = 1$. **[1]**

PROBLEM 4

The vacant squares in this grid are to be filled with digits so that all the numbers read from left to right and top to bottom are 5-digit cubic numbers.

Find *all* possible solutions if the cubic numbers are different. Show clearly why there are no more solutions.



SOLUTION 4

Make a list of all 5-digit cubic numbers:

10648	19683	32768	50653	74088
12167	21952	35937	54872	79507
13824	24389	39304	59319	85184
15625	27000	42875	64000	91125
17576	29791	46656	68921	97336

1

The numbers at 2 down and 4 across must have the same middle digit, which will go in the centre square of the grid. The next table shows the choices for 2 down and 4 across for each central digit.

0:	27000	64000	74088		
1:	12167	85184	91125		
2:	none				
3:	24389	39304	59319	97336	
4:	none				
5:	17576	79507			
6:	10648	15625	19683	46656	50653
7:	29791	32768			
8:	13824	42875	54872		
9:	21952	35937	68921		

The first digits of 2 down and 4 across are the middle digits of 1 across and 1 down. Since neither 2 nor 4 is the middle digit of any number in the first table, we can eliminate 27000, 24389, 46656, 29791, 42875, and 21952 from the second table. This leaves only one choice if the central digit is 7. So the second table reduces to the following table.

0:	64000	74088		
1:	12167	85184	91125	
2:	none			
3:	39304	59319	97336	
4:	none			
5:	17576	79507		
6:	10648	15625	19683	50653
7:	none			
8:	13824	54872		
9:	35937	68921		

1

If the central digit is 0, then 2 down is 64000 and 4 across is 74088 or 2 down is 74088 and 4 across is 64000. In both cases 1 across and 1 down must have the same first digit and their middle digits must be 6 and 7. There is no such pair of numbers in the first table. So the central digit can not be 0.

By a similar argument, the central digit can not be 1 or 5, but we are left with the following possibilities for 3, 6, 8, 9, together with their reflections about the line through the grid from top left to bottom right.

[1] 2	4	[2] 3	8	[3] 9
1		9		
[4] 9	7	3	3	6
5		0		
2		4		

[1] 1	2	[2] 1	6	[3] 7
7		0		
[4] 5	0	6	5	3
7		4		
6		8		

[1] 1	2	[2] 1	6	[3] 7
7		3		
[4] 5	4	8	7	2
7		2		
6		4		

[1] 5	9	[2] 3	1	[3] 9
0		5		
[4] 6	8	9	2	1
5		3		
3		7		

1

There are no numbers in the first table of the form 9*6**, 2*4**, 7*3**, 6*8**, 7*2**, 6*4**, so we can not complete the first three grids or their reflections.

However the last grid and its reflection give the following two solutions from the first table and these are the only solutions.

[1] 5	9	[2] 3	1	[3] 9
0		5		1
[4] 6	8	9	2	1
5		3		2
3		7		5

[1] 5	0	[2] 6	5	[3] 3
9		8		2
[4] 3	5	9	3	7
1		2		6
9		1		8

1

PROBLEM 5

A certain polyhedron has exactly eight faces. Four of the faces are equilateral triangles and the other faces are regular hexagons.

(a) How many edges does the polyhedron have?

(b) How many vertices does the polyhedron have?

(c) Explain why the polyhedron has at least one vertex at which exactly one triangle and two hexagons meet.

(d) If all vertices are identical, draw a net of the polyhedron and construct the polyhedron out of stiff paper.

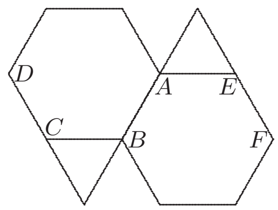
SOLUTION 5

(a) If we count the edges around each triangle and each hexagon in the polyhedron, then we would get a total of $4 \times 3 + 4 \times 6 = 36$. Since each edge lies on exactly two faces, we have counted each edge in the polyhedron exactly twice. So the number of edges in the polyhedron is $36 \div 2 = 18$. **1**

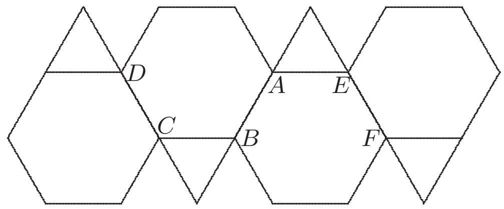
(b) Euler's Rule, $F + V = E + 2$, now gives $8 + V = 18 + 2$. Hence $V = 12$. **1**

(c) At least two hexagons must share an edge otherwise the polyhedron would have at least $4 \times 6 = 24$ edges which is too many. Suppose two hexagons share an edge with vertices A and B. At least three faces must meet at a vertex, so there is another face at A. The internal angle of a regular hexagon is $120°$. Therefore the third face at A can not be a hexagon otherwise the three hexagons at A would form a flat surface. The internal angle of an equilateral triangle is $60°$, so there can not be more than one triangle at A. Hence there are exactly two hexagons and one triangle at vertex A. **1**

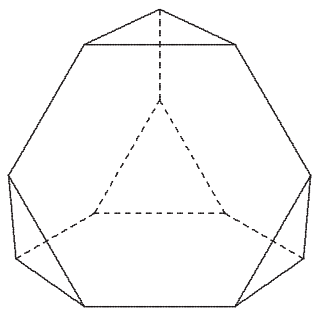
(d) By a similar argument to the one in Part **c**, exactly two hexagons and one triangle meet at vertex B. So we get the following pattern.



Since the vertices are identical, there is another hexagon at each of the vertices C and E and therefore a triangle at each of the vertices D and F. This gives the following net.



Draw this net on stiff paper with tabs on appropriate edges. Cut out the figure and glue each tab to the appropriate face. This will give the following polyhedron. (It is called a truncated tetrahedron.)



1

PROBLEM 6

The counting numbers are arranged in columns as shown.

column:	1	2	3	4	5	6	7	8	9
row 1	1	2	3		13	14	15		–
row 2			4		12		16		–
row 3			5		11		17		–
row 4			6		10		18		–
row 5			7	8	9		19	20	–

(a) Complete the next six columns after column 8.

(b) Find the column in which 50 appears.

(c) Find the row and column in which 2008 appears.

SOLUTION 6

(a)

column:	9	10	11	12	13	14
row 1	25	26	27		37	38
row 2	24		28		36	
row 3	23		29		35	
row 4	22		30		34	
row 5	21		31	32	33	

1

(b) *Alternative 1*

If we continue for a few more columns we will see that 50 is in column 18. **1**

Alternative 2

The pattern in the table repeats after every 12 numbers. For example, the numbers 1, 13, 25, 37, 49 appear in the top of columns 1, 5, 9, 13, 17 respectively. Therefore 50 appears in column 18. **1**

(c) *Alternative 1*

The multiples of 12 occur in row 2 of columns 5, 9, 13, ... List these column numbers as a sequence: $C_1 = 5$, $C_2 = 9$, $C_3 = 13$, ... Thus the number $12 \times n$ occurs in column number C_n. To get the term C_n, we multiply n by 4 and add 1. The multiple of 12 closest to 2008 is 2004 and $2004 = 12 \times 167$. Therefore 2004 occurs in row 2 of column number $C_{167} = 4 \times 167 + 1 = 669$.

Hence 2005 is in row 1 of column 669, 2006 is in row 1 of column 670, 2007 is in row 1 of column 671, and 2008 is in row 2 of column 671. **2**

Similar solutions exist using other sequences of numbers besides the multiples of 12. For example: 2, 14, 26, ...; 3, 15, 27, ...; 8, 20, 32, ..., etc.

Alternative 2

The pattern in the table repeats after every 12 numbers. Since $2008 = 167 \times 12 + 4$, the number 2008 must appear in row 2 after 167 repetitions of the pattern from the number 4. The number 4 is in column 3 and each repetition of the pattern adds another 4 columns. Therefore the number 2008 is in column number $3 + 167 \times 4 = 3 + 668 = 671$. **2**

PROBLEM 7

Fifty-two members of the Highmont Tennis Club bought a ticket each to attend the Australian Open. The ordinary ticket price was \$45 and the concession ticket price was \$36. They spent \$2043 altogether.

(a) Explain why the number of ordinary tickets and the number of concession tickets is odd.

(b) Find the number of each type of ticket that was bought.

SOLUTION 7

(a) **Alternative i**

The total cost of concession tickets is even since the cost of each concession ticket is \$36 which is even. The total cost of all tickets is \$2043, which is odd. Therefore the total cost of the ordinary tickets must be odd, since the sum of two even numbers is even. Hence the number of ordinary tickets must be odd, since the product of an even number with 45 is even. The total number of tickets is 52, which is even. Therefore the number of concession tickets is also odd, since the sum of an even number and an odd number is odd. **2**

Alternative ii

The total number of tickets is 52, an even number. Therefore the number of ordinary and concession tickets are both even or both odd, otherwise one would be even and the other odd so their sum would be odd. If they are both even, then the total cost of the ordinary tickets and the total cost of the concession tickets would both be even. This means the total cost of all tickets would be even, a contradiction. Hence the number of ordinary and concession tickets are both odd. **2**

(b) **Alternative i**

Guess the number of ordinary tickets that were bought and check the total cost of all tickets. If this is too high, then we must replace some of the expensive ordinary tickets with the cheaper concession tickets. If the total cost is too low, then we must have more of the expensive ordinary tickets and less of the cheaper concession tickets. From Part (a), the number of ordinary and concession tickets must be odd.

Number of ordinary tickets	Total cost of ordinary tickets	Number of concession tickets	Total cost of concession tickets	Total cost of all tickets	Check total cost
43	1935	9	324	2259	too high
13	585	39	1404	1989	too low
27	1215	25	900	2115	too high
21	945	31	1116	2061	too high
17	765	35	1260	2025	too low
19	855	33	1188	2043	correct

Thus 19 ordinary and 33 concession tickets were bought. **2**

Alternative ii

If all the tickets were concession tickets, then the total cost would have been $52 \times 36 = \$1872$. The actual total cost was \$2043. The difference is \$171 and this is due to the extra cost of \$9 for each ordinary ticket that was bought. Hence the number of ordinary tickets that were bought was $171 \div 9 = 19$ and the number of concession tickets that were bought was $52 - 19 = 33$. **2**

PROBLEM 8

(a) Insert one digit somewhere in 347835 to make a number that is divisible by 99. Find all such numbers.

(b) Insert one digit between 4 and 7 and another digit between 9 and 8 in the number 473598 to make a new number that is divisible by 24. Find all such numbers.

SOLUTION 8

(a) Since $99 = 9 \times 11$, the new number must be divisible by 9 and 11. The sum of the digits of 347835 is 30. We want the digit sum of the new number to be a multiple of 9. Therefore the digit we insert must be 6. **1**

We tabulate all the possibilities and check if the new number is divisible by 11.

New number	Test	Multiple of 11?
6347835	$(6+4+8+5)-(3+7+3)=10$	no
3647835	$(3+4+8+5)-(6+7+3)=4$	no
3467835	$(3+6+8+5)-(4+7+3)=8$	no
3476835	$(3+7+8+5)-(4+6+3)=10$	no
3478635	$(3+7+6+5)-(4+8+3)=6$	no
3478365	$(3+7+3+5)-(4+8+6)=0$	yes
3478356	$(3+7+3+6)-(4+8+5)=2$	no

Thus the only new number divisible by 99 is 3478365. **1**

(b) Since $24 = 3 \times 8$, the new number must be divisible by 3 and 8.

For the new number to be divisible by 8, the last three digits must form a number that is divisible by 8. The only possible 3-digit numbers are 928 and 968. **1**

For the new number to be divisible by 3, the digit sum must be a multiple of 3. If the new number is 4*735928, then the digit sum is 38 plus the missing digit. Hence the missing digit is 1, 4, or 7. If the new number is 4*735968, then the digit sum is 42 plus the missing digit. Hence the missing digit is 0, 3, 6, or 9.

So the only new numbers divisible by 24 are 41735928, 44735928, 47735928, 40735968, 43735968, 46735968, and 49735968. **1**