NOETHER STUDENT SAMPLE PROBLEMS: SOLUTIONS

PROBLEM 1

Automatic Teller Machines in Marchhareland dispense notes of size 30, 40 and 50 rallods. To buy himself a new hat, Mad Hatter has to send a 30 rallods banknote to a factory. What is the largest amount he can withdraw that is certain to contain at least one 30 rallods note?

SOLUTION 1

First we prove that if Mad Hatter withdraws 110 rallods from a machine, he always gets a 30 rallods note.
Since the largest amount that Mad Hatter can get when he is given two notes is 100 rallods, he is given at least three notes when he withdraws 110 rallods.
If none of these notes is of size 30 rallods, the smallest amount is $3 \times 40 = 120$ rallods which is not possible.
Hence when Mad Hatter withdraws 110 rallods from a machine, he always gets a 30 rallods note. $\boxed{1}$

Now we show that if Mad Hatter withdraws an amount which is greater than 110 rallods from a machine, he can be left without a 30 rallods note.
Since
$120 = 40 + 40 + 40$,
$130 = 40 + 40 + 50$,
$140 = 40 + 50 + 50$ and
$150 = 50 + 50 + 50$,
we see that each of the amounts 120, 130, 140 and 150 rallods can be withdrawn without getting a 30 rallods note. $\boxed{1}$

Assume that Mad Hatter withdraws N rallods and $N \geq 120$.
Let M be the remainder obtained when N is divided by 40.

Then M is a multiple of 10 and $0 \leq M < 40$.
Hence $M + 120$ is one of the numbers 120, 130, 140 and 150.
Therefore $N = K + 40 \times L$ where K is one of the numbers 120, 130, 140 or 150 and L is a positive integer or zero which means that the amount of N rallods can be withdrawn without getting a 30 rallods note. $\boxed{1}$

Thus the largest amount Mad Hatter can withdraw which is certain to contain at least one 30 rallods note is 110 rallods. $\boxed{1}$

PROBLEM 2

A positive integer N greater than 100 is written on a blackboard. When its two rightmost digits are erased, the number which is left is $N \div 134$. Find all possible numbers N.

SOLUTION 2

Alternative 1

Let the number which is left on the blackboard when the two digits are erased be a and the number which is formed by the erased digits be b. Then $N = 100a + b$. But we also know that $N \div 134 = a$. [1]

Hence $134a = 100a + b$, which implies $34a = b$. [1]

Since b cannot be greater than 99 and a cannot be zero, we obtain that either $a = 1$ or $a = 2$. [1]

Therefore either $N = 134$ or $N = 268$. [1]

Alternative 2

If $N = 134 \times 1 = 134$, then N is a solution.
If $N = 134 \times 2 = 268$, then again N is a solution. [1]

If $N = 134 \times 3 = 402$, then N is not a solution. [1]

If $N = 134 \times k$ where $k > 3$, then the number, which is left after the two digits are erased, is greater than k since there is carry in the tens column on multiplication. [1]

Thus there are only two solutions: N=134 and N=268. [1]

PROBLEM 3

Simplify:

$$\frac{12a^3 + 4a^2 - 3ab^2 - b^2}{6a^2 - 3ab - 2a + b} \div \frac{9a^2 + 6a + 1}{9a^2 - 1}.$$

SOLUTION 3

We have

$$\begin{aligned} 9a^2 + 6a + 1 &= (3a)^2 + 2 \times 3a \times 1 + 1^2 \\ &= (3a+1)^2\,; \\ 9a^2 - 1 &= (3a)^2 - 1^2 \\ &= (3a-1)(3a+1)\,; \end{aligned}$$

1

and also note

$$\begin{aligned} 6a^2 - 3ab - 2a + b &= (6a^2 - 3ab) - (2a - b) \\ &= 3a(2a-b) - (2a-b) \\ &= (2a-b)(3a-1)\,; \\ 12a^3 + 4a^2 - 3ab^2 - b^2 &= (12a^3 + 4a^2) - (3ab^2 + b^2) \\ &= 4a^2(3a+1) - b^2(3a+1) \\ &= (3a+1)(4a^2 - b^2) \\ &= (3a+1)((2a)^2 - b^2) \\ &= (3a+1)(2a-b)(2a+b)\,. \end{aligned}$$

1

Therefore

$$\begin{aligned} & \frac{12a^3 + 4a^2 - 3ab^2 - b^2}{6a^2 - 3ab - 2a + b} \div \frac{9a^2 + 6a + 1}{9a^2 - 1} \\ = & \frac{12a^3 + 4a^2 - 3ab^2 - b^2}{6a^2 - 3ab - 2a + b} \times \frac{9a^2 - 1}{9a^2 + 6a + 1} \\ = & \frac{(3a+1)(2a-b)(2a+b)}{(2a-b)(3a-1)} \times \frac{(3a-1)(3a+1)}{(3a+1)^2} \\ = & \frac{(3a+1)^2(3a-1)(2a-b)(2a+b)}{(2a-b)(3a-1)(3a+1)^2} \end{aligned}$$

1

i.e.

$$
\begin{aligned}
&= \frac{2a+b}{1} \\
&= 2a+b\,.
\end{aligned}
$$

$\boxed{1}$

PROBLEM 4

Simplify:

$$\frac{2x^2 - 3xy + 4x - 6y}{8x^3 - 27y^3} \times \frac{4x^2 + 6xy + 9y^2}{x^2 + x - 2}\,.$$

SOLUTION 4

$$\begin{aligned} 2x^2 - 3xy + 4x - 6y &= (2x^2 - 3xy) + (4x - 6y) \\ &= x(2x - 3y) + 2(2x - 3y) \\ &= (2x - 3y)(x + 2). \end{aligned}$$

$\boxed{1}$

Also

$$\begin{aligned} 8x^3 - 27y^3 &= (2x)^3 - (3y)^3 \\ &= (2x - 3y)(4x^2 + 6xy + 9y^2) \end{aligned}$$

and

$$x^2 + x - 2 = (x + 2)(x - 1)\,. \qquad \boxed{1}$$

Hence

$$\begin{aligned} &\frac{2x^2 - 3xy + 4x - 6y}{8x^3 - 27y^3} \times \frac{4x^2 + 6xy + 9y^2}{x^2 + x - 2} \\ = \;&\frac{(2x^2 - 3xy + 4x - 6y)(4x^2 + 6xy + 9y^2)}{(8x^3 - 27y^3)(x^2 + x - 2)} \\ = \;&\frac{(2x - 3y)(x + 2)(4x^2 + 6xy + 9y^2)}{(2x - 3y)(4x^2 + 6xy + 9y^2)(x + 2)(x - 1)}, \end{aligned}$$

$\boxed{1}$

i.e.

$$= \frac{1}{x - 1}\,. \qquad \boxed{1}$$

PROBLEM 5

Sum the series:

$$1^2 - 3^2 + 5^2 - 7^2 + 9^2 - 11^2 + \cdots + 10001^2 - 10003^2 .$$

SOLUTION 5

We have

$$\begin{aligned} & 1^2 - 3^2 + 5^2 - 7^2 + \cdots + 10001^2 - 10003^2 \\ = \; & (1^2 - 3^2) + (5^2 - 7^2) + \cdots + (10001^2 - 10003^2) \\ = \; & (1-3)(1+3) + (5-7)(5+7) + \\ & \qquad \cdots + (10001 - 10003)(10001 + 10003), \end{aligned}$$

$\boxed{1}$

i.e.

$$\begin{aligned} = \; & -2 \times 4 - 2 \times 12 - \cdots - 2 \times 20004 \\ = \; & -2(4 + 12 + \cdots + 20004) . \end{aligned}$$

$\boxed{1}$

Since the sequence $4, 12, 20, 28, \ldots$ is an arithmetical progression, we obtain

$$4 + 12 + \cdots + 20004 = \frac{4 + 20004}{2} \times 2501 = 25020004 . \qquad \boxed{1}$$

Therefore,

$$\begin{aligned} & 1^2 - 3^2 + 5^2 - 7^2 + 9^2 - 11^2 + \cdots + 10001^2 - 10003^2 \\ = \; & -2(4 + 12 + \cdots + 20004) \\ = \; & -2 \times 25020004 \\ = \; & -50040008 . \end{aligned}$$

$\boxed{1}$

Note. A computer program or spreadsheet must explain how it has been constructed for full marks (no explanation gets 1 mark - the last).

PROBLEM 6

When a positive integer X is written in base 9, it is a three-digit number. When X is written in base 6, it is a three-digit number that consists of the same digits as the first one but in reverse order. Find the decimal representations of all such numbers X.

SOLUTION 6

Since when X is written in base 9, it is a three-digit number, there exist digits a, b and c such that $X = (abc)_9$.
Since X written in base 6 consists of the same digits but in reverse order, $X = (cba)_6$.
Hence $(abc)_9 = (cba)_6$.
Therefore $81a + 9b + c = 36c + 6b + a$, which means $80a + 3b = 35c$.
[1]

Since both 80 and 35 are divisible by 5, $3b$ is also divisible by 5.
Hence b is either 0 or 5. [1]

If $b = 0$, then $80a = 35c$, which implies that a is divisible by 7.
But a cannot be 0, as $(abc)_9$ is a three-digit number. Therefore $a \geq 7$, which is not possible as a is a digit of base 6 representation.
Hence b cannot be 0.

Let $b = 5$. Then $80a + 15 = 35c$ which means $16a + 3 = 7c$. The only digit $a < 6$ such that $16a + 3$ is divisible by 7 is $a = 2$.
Therefore $c = (32 + 3) \div 7 = 5$.
Hence $X = (255)_9 = 2 \times 81 + 5 \times 9 + 5 = 212$. [1]

Thus the only solution is $X = 212$. [1]

Note. Computer program or spreadsheet solutions must be explained for full marks.

PROBLEM 7

Find all real numbers x such that

$$x^4 \leq 8x^2 - 16.$$

SOLUTION 7

The inequality $x^4 \leq 8x^2 - 16$ can be rewritten as

$$x^4 - 8x^2 + 16 \leq 0.$$

Since $x^4 - 8x^2 + 16 = (x^2)^2 - 2 \times x^2 \times 4 + 4^2 = (x^2 - 4)^2$, we obtain

$$(x^2 - 4)^2 \leq 0. \qquad \boxed{1}$$

Since $(x^2 - 4)^2$ cannot be negative, we come to the conclusion that

$$(x^2 - 4)^2 = 0. \qquad \boxed{1}$$

Hence $x^2 - 4 = 0$ which implies $x^2 = 4$. $\boxed{1}$

Therefore either $x = 2$ or $x = -2$. $\boxed{1}$

PROBLEM 8

On each of the faces of a cube a real number is written. For each of the faces, the number written on it equals the product of the numbers written on all the faces that have a common side with this face. What is the sum of all the numbers that are written on the faces of the cube? Find all possible answers.

SOLUTION 8

First of all, we notice that the numbers written on opposite faces equal the product of the same set of numbers. Hence the numbers written on opposite faces are equal.

Now let a, b and c be numbers written on three faces sharing a common vertex. Then we have the following equations:

$$\begin{aligned} a &= b^2c^2; \\ b &= a^2c^2; \\ c &= a^2b^2. \end{aligned}$$

1

If one of the numbers a, b and c, say c, equals zero, then $a = b^2c^2 = 0$ and $b = a^2c^2 = 0$.
Hence $2a + 2b + 2c = 0$. 1

If none of the numbers a, b and c equals zero, we can use $a = b^2c^2$ and $b = a^2c^2$ to obtain

$$\frac{a}{b} = \frac{b^2c^2}{a^2c^2} = \frac{b^2}{a^2}.$$

Thus $\frac{a}{b} = \frac{b^2}{a^2}$ which yields $a^3 = b^3$.
Hence $a = b$.
Similarly, $a = c$. 1

Therefore the equation $a = b^2c^2$ turns into $a = a^4$, which means $a = 1$ as $a \neq 0$.
Similarly $b = 1$ and $c = 1$.
Hence $2a + 2b + 2c = 6$.

Thus the answer is 0 or 6. 1

Note. Trial and error or similar solutions which do not show that the two answers are the only ones gain only the last two marks.

PROBLEM 9

A bus and a truck left a town at the same time and went along the same road at constant speeds of 60 km/h and 70 km/h respectively. Some time later, a car set off along the same road at a constant speed of 90 km/h. How much time passed between the moments when the truck and the car set off, if the car overtook the bus exactly half an hour before the car caught up with the truck?

SOLUTION 9

Let the time that passed between the moments when the truck and the car set off be T hours and the time that it took the car to catch up with the bus be S hours.

Then the distance that the car had travelled before it caught up with the bus equals

$$90 \text{ km/h} \times S \text{ hours} = 90S \text{ km},$$

and the distance that the bus had covered before it was caught up by the car is

$$60 \text{ km/h} \times (S + T) \text{ hours} = 60(S + T) \text{ km}.$$

Since these two distances are the same, we obtain the following equation:

$$90S = 60(S + T). \qquad \boxed{1}$$

Therefore, $90S = 60S + 60T$, which implies $S = 2T$.

Similarly, since the car overtook the bus exactly half an hour before the car caught up with the truck, we get the following equation:

$$90(S + 0.5) = 70(S + T + 0.5). \qquad \boxed{1}$$

Therefore $90S + 45 = 70S + 70T + 35$, which implies $2S = 7T - 1$.

Since $S = 2T$, we have $2S = 4T$.
On the other hand, $2S = 7T - 1$.
Therefore $4T = 7T - 1$. $\boxed{1}$

Hence $T = \dfrac{1}{3}$ hours $= 20$ min. $\boxed{1}$

PROBLEM 10

Prove that for any positive integer n, the value of the expression

$$5 \times 2^{3n-2} + 3^{3n-1}$$

is divisible by 19.

SOLUTION 10

Proof by induction.

1. If $n = 1$, then $5 \times 2^{3n-2} + 3^{3n-1} = 19$, which is divisible by 19. $\boxed{1}$

2. Assume that for some k the number $5 \times 2^{3k-2} + 3^{3k-1}$ is divisible by 19.

3. Now let us consider the expression $5 \times 2^{3(k+1)-2} + 3^{3(k+1)-1}$.
We have

$$\begin{aligned}
5 \times 2^{3(k+1)-2} + 3^{3(k+1)-1} &= 5 \times 2^{3k+3-2} + 3^{3k+3-1} \\
&= 5 \times 2^3 \times 2^{3k-2} + 3^3 \times 3^{3k-1} \\
&= 40 \times 2^{3k-2} + 27 \times 3^{3k-1} \\
&= 40 \times 2^{3k-2} + 8 \times 3^{3k-1} + 19 \times 3^{3k-1} \\
&= 8(5 \times 2^{3k-2} + 3^{3k-1}) + 19 \times 3^{3k-1}.
\end{aligned}$$

$\boxed{1}$

Thus the number $5 \times 2^{3(k+1)-2} + 3^{3(k+1)-1}$ is the sum of two numbers, namely $8(5 \times 2^{3k-2} + 3^{3k-1})$ and $19 \times 3^{3k-1}$, each of which is divisible by 19 as we have assumed that $5 \times 2^{3k-2} + 3^{3k-1}$ is divisible by 19.

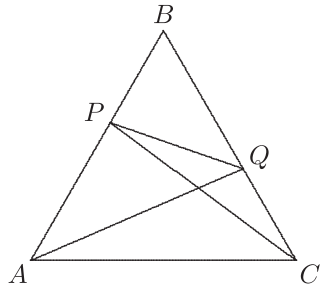
Hence $5 \times 2^{3(k+1)-2} + 3^{3(k+1)-1}$ is divisible by 19. $\boxed{1}$

Therefore, by the Principle of Mathematical Induction, $5 \times 2^{3n-2} + 3^{3n-1}$ is divisible by 19 for every positive integer n. $\boxed{1}$

PROBLEM 11

The side length of an equilateral triangle ABC is 5. P and Q are points on the sides AB and BC respectively such that $BP = 2$ and $BQ = 3$. Find $\angle PAQ + \angle PCQ$.

SOLUTION 11



In $\triangle ABQ$ and $\triangle CAP$, 1

$AB = CA$ and $\angle ABQ = \angle CAP$ as $\triangle ABC$ is equilateral.
Also $BQ = AP$ since $BQ = 3$ and $AP = AB - BP = 5 - 2 = 3$.
Hence $\triangle ABQ$ is congruent to $\triangle CAP$. (SAS) 1

Therefore $\angle BAQ = \angle ACP$. 1

Hence

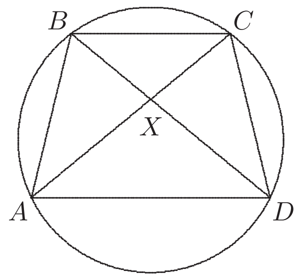
$$\begin{aligned} \angle PAQ + \angle PCQ &= \angle BAQ + \angle PCQ \\ &= \angle ACP + \angle PCQ \\ &= \angle ACQ \\ &= 60^\circ. \end{aligned}$$

Thus $\angle PAQ + \angle PCQ = 60^\circ$. 1

PROBLEM 12

Points A, B, C and D lie on the circumference of a circle. The chords AC and BD meet at the point X. Prove that if $\angle ABC = \angle BCD$, then $AX = XD$.

SOLUTION 12



We have $\angle ABC = \angle BCD$.
Also $\angle ABD = \angle ACD$. (*subtended by the same arc*)
Therefore

$$\angle CBD = \angle ABC - \angle ABD = \angle BCD - \angle ACD = \angle BCA.$$

Thus $\angle CBD = \angle BCA$. [1]

Since $\angle CBD = \angle CAD$ (*subtended by the same arc*)
and $\angle BCA = \angle BDA$, (*subtended by the same arc*)
we have $\angle CAD = \angle BDA$. [1]

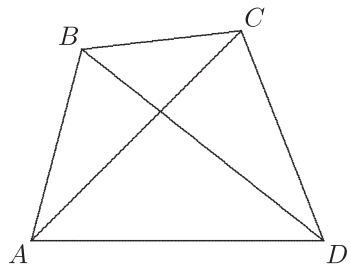
Hence $\angle XAD = \angle XDA$. [1]

Therefore $AX = XD$. [1]

PROBLEM 13

In a quadrilateral $ABCD$, $\angle ABD = 70^\circ$, $\angle CAD = 42^\circ$ and $\angle CDA = 68^\circ$. Find the size of the angle $\angle CBD$.

SOLUTION 13



Since the angle sum of $\triangle ACD$ is 180°, we have

$$\angle ACD = 180^\circ - \angle CAD - \angle CDA = 180^\circ - 42^\circ - 68^\circ = 70^\circ. \quad \boxed{1}$$

Hence $\angle ACD = \angle ABD$ as $\angle ABD = 70^\circ$.
Since $\angle ACD = \angle ABD$, the quadrilateral $ABCD$ is cyclic. $\boxed{1}$

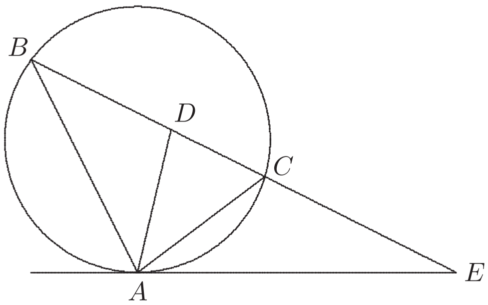
Hence $\angle CBD = \angle CAD$. (*subtended by the same arc*) $\boxed{1}$

Therefore $\angle CBD = 42^\circ$. $\boxed{1}$

PROBLEM 14

Points A, B and C lie on the circumference of a circle. The straight line which is tangent to the circle at A meets the line BC extended at E. A point D is placed on BC so that AD bisects the angle BAC. Prove that $AE = DE$.

Solution



Since $\angle ADE$ is an external angle of the triangle ABD, we have

$$\angle ADE = \angle ABD + \angle BAD. \quad \boxed{1}$$

Since AE is tangent to the circle at A, $\angle ABC = \angle CAE$ by the alternate segment theorem. $\boxed{1}$

Also $\angle BAD = \angle DAC$ as AD bisects $\angle BAC$.
Therefore

$$\begin{aligned} \angle ADE &= \angle ABD + \angle BAD \\ &= \angle ABC + \angle BAD \\ &= \angle CAE + \angle DAC \\ &= \angle DAE. \end{aligned}$$

Thus $\angle ADE = \angle DAE$. $\boxed{1}$

Hence the triangle ADE is isosceles, so that, $AE = DE$. $\boxed{1}$

PROBLEM 15

Helen runs faster than Jane but slower than Lisa. They started at the same time from the same place and ran around a circular route. After a while they stopped simultaneously at the same place from which they started. It turned out that Lisa overtook Jane ten times during this run. How many times did it occur that one of the girls overtook another? Assume that each of the girls ran at a constant speed. (The final stop together is not counted as an overtake.)

SOLUTION 15

It is easily seen that since Lisa overtook Jane ten times, Lisa ran eleven laps more than Jane did. [1]

Similarly, if Helen overtook Jane X times, she ran $X+1$ laps more than Jane, and if Lisa overtook Helen Y times, she ran $Y+1$ laps more than Helen. [1]

Clearly, the number of laps that Helen ran more than Jane and the number of laps that Lisa ran more than Helen add up to the number of laps that Lisa ran more than Jane.
Hence $(X+1)+(Y+1)=11$ which implies $X+Y=9$. [1]

Therefore the total number of the times when one of the girls overtook another equals $X+Y+10=19$.

Thus the answer is 19 times. [1]

PROBLEM 16

Positive integers x and y satisfy the equation

$$3x + 5y = 2xy - 1.$$

What is the value of $x - y$? Find all possible answers.

SOLUTION 16

Since $2xy = xy + xy$, we can rewrite the equation $3x + 5y = 2xy - 1$ as

$$x(y - 3) + y(x - 5) = 1.$$

If $y > 3$ and $x > 5$, then $x(y - 3) + y(x - 5) > 1$, which is not possible.
Hence either $y \leq 3$ or $x \leq 5$. $\boxed{1}$

If $y = 1$, then the equation $3x + 5y = 2xy - 1$ turns into $3x + 5 = 2x - 1$.
Hence $x = -6$, which is not possible as x must be positive.

If $y = 2$, then $3x + 10 = 4x - 1$, which yields $x = 11$.
Hence $x - y = 9$.

If $y = 3$, then $3x + 15 = 6x - 1$, which does not have integer solutions. $\boxed{1}$

If $x = 1$, then $3 + 5y = 2y - 1$, which has no positive solutions.

Similarly x cannot be equal to 2.

If $x = 3$, then $9 + 5y = 6y - 1$, which yields $y = 10$.
Hence $x - y = -7$.

If $x = 4$, then $12 + 5y = 8y - 1$, which has no integer solutions.

Similarly x cannot be equal to 5. $\boxed{1}$

Thus $x - y = 9$ or $x - y = -7$. $\boxed{1}$