POLYA STUDENT SAMPLE PROBLEMS

PROBLEM 1

Factor:

(a) $x^4 + x^2y^4 + 25y^8$;

(b) $a^4 + b^2 + c^2 - 2(a^2b - a^2c + bc) - 2(a^2 - b + c) + 1$.

PROBLEM 2

Prove that

$$55^{62} - 2 \times 13^{62} + 41^{62}$$

is divisible by 182 .

PROBLEM 3

Prove that if p is a prime number, then the equation

$$x^5 - px^4 + (p^2 - p)x^3 + px^2 - (p^3 + p^2)x - p^2 = 0$$

has no integer roots.

PROBLEM 4

For what real numbers k do the equations

$$\begin{aligned} x^{99} + kx^{57} - x^{20} - 1 &= 0, \\ x^{203} - 3x^{201} - (k+2)x^2 - 1 &= 0, \\ x^{201} - 3x^{199} + x^{98} + kx^{56} - x^{19} - k - 2 &= 0 \end{aligned}$$

have a common root?

PROBLEM 5

Is it possible to arrange the integers from 1 to 1000 in a table with 20 rows and 50 columns so that if for each of the rows all the numbers in it are added together, then twenty consecutive integers are obtained?

PROBLEM 6

Simplify:

$$\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\ldots+\frac{1}{2000\times 2003}.$$

PROBLEM 7

Prove that the inequality

$$x(x+1)(x+2)(x+3)\geq -1$$

holds for all real numbers x.

PROBLEM 8

Prove that

$$(ab+bc+ca)^2\geq 3abc(a+b+c)$$

holds for all real numbers a, b and c.

PROBLEM 9

The side length of a square $ABCD$ is 4. P and Q are points on the sides AB and BC respectively such that $BP=1$ and $BQ=3$. Prove that

$$\angle PAQ+\angle PDQ+\angle PCQ=90^\circ.$$

PROBLEM 10

In a quadrilateral $ABCD$, there is a point X on the side BC such that XA and XD are the angle bisectors of $\angle BAD$ and $\angle CDA$ respectively. Prove that if $AB=BX$, then $BC=AB+CD$.

PROBLEM 11

In a quadrilateral $ABCD$, P and R are the midpoints of AB and CD respectively. Also Q and S are points on the sides BC and DA respectively such that $BQ=2QC$ and $DS=2SA$. Prove that the area of the quadrilateral $PQRS$ equals $\dfrac{S}{2}$ where S is the area of the quadrilateral $ABCD$.

PROBLEM 12

In a triangle ABC, $AC = 2AB$ and $\dfrac{BC}{BA} = \dfrac{3}{2}$. H is the foot of the perpendicular from B to AC. Prove that $\dfrac{CH}{AH} = \dfrac{21}{11}$.

PROBLEM 13

A, B, C, D and E are points in order on the circumference of a circle. $\angle ABC = 100°$ and $\angle CDE = 125°$. Prove that $\angle ACE = 45°$.

PROBLEM 14

Two circles C_1 and C_2 meet at the points A and B. A point C is on the circumference of C_1. The chord CB cuts C_2 at the point F and the chord CA produced meets C_2 at D. E is a point on the circumference of C_1 such that EA is parallel to FD. Prove that $EC = CA$.

PROBLEM 15

X and Y are points on the sides BC and AC of a triangle ABC respectively such that $\angle AXC = \angle BYC$ and $BX = XY$. Prove that AX bisects the angle $\angle BAC$.

PROBLEM 16

Two circles meet at the points A and B. A straight line touches one of the circles at X and the other at Y. XA produced meets the chord BY at Q and YA produced meets the chord BX at P. Prove that XY is parallel to PQ.