POLYA STUDENT SAMPLE PROBLEMS: SOLUTIONS

PROBLEM 1

Factor:

1. $x^4 + x^2y^4 + 25y^8\,;$

2. $a^4 + b^2 + c^2 - 2(a^2b - a^2c + bc) - 2(a^2 - b + c) + 1\,.$

SOLUTION 1

(a)

$$\begin{aligned} x^4 + x^2y^4 + 25y^8 &= x^4 + 10x^2y^4 + 25y^8 - 9x^2y^4 \\ &= (x^2)^2 + 2 \times x^2 \times (5y^4) + (5y^4)^2 - 9x^2y^4 \end{aligned}$$

$\boxed{1}$

$$\begin{aligned} &= (x^2 + 5y^4)^2 - (3xy^2)^2 \\ &= (x^2 + 5y^4 - 3xy^2)(x^2 + 5y^4 + 3xy^2). \end{aligned}$$

$\boxed{1}$

(b)

$$\begin{aligned} & a^4 + b^2 + c^2 - 2(a^2b - a^2c + bc) - 2(a^2 - b + c) + 1 \\ = \; & (a^4 + b^2 + c^2 - 2a^2b + 2a^2c - 2bc) - 2(a^2 - b + c) + 1 \end{aligned}$$

$\boxed{1}$

$$\begin{aligned} &= (a^2 - b + c)^2 - 2(a^2 - b + c) + 1 \\ &= (a^2 - b + c - 1)^2. \end{aligned}$$

$\boxed{1}$

PROBLEM 2

Prove that

$$55^{62} - 2 \times 13^{62} + 41^{62}$$

is divisible by 182.

SOLUTION 2

On one hand,

$$\begin{aligned} 55^{62} - 2 \times 13^{62} + 41^{62} &= 55^{62} + 41^{62} - 2 \times 13^{62} \\ &= (55^2)^{31} + (41^2)^{31} - 2 \times 13^{62} \\ &= (55^2 + 41^2)(\ldots) - 2 \times 13^{62} \end{aligned}$$

$\boxed{1}$

$$= 4706(\ldots) - 2 \times 13^{62}\,.$$

Hence $55^{62} - 2 \times 13^{62} + 41^{62}$ is divisible by 13 as 4706 is divisible by 13. $\boxed{1}$

On the other hand,

$$\begin{aligned} 55^{62} - 2 \times 13^{62} + 41^{62} &= 55^{62} - 13^{62} - 13^{62} + 41^{62} \\ &= (55^{62} - 13^{62}) + (41^{62} - 13^{62}) \\ &= (55 - 13)(\ldots) + (41 - 13)(\ldots) \\ &= 42(\ldots) + 28(\ldots)\,. \end{aligned}$$

$\boxed{1}$

Since both 42 and 28 are divisible by 14, the expression $55^{62} - 2 \times 13^{62} + 41^{62}$ is divisible by 14.

Thus $55^{62} - 2 \times 13^{62} + 41^{62}$ is divisible by 13 and 14. Therefore, it is divisible by $13 \times 14 = 182$ as 13 and 14 are relatively prime numbers. $\boxed{1}$

PROBLEM 3

Prove that if p is a prime number, then the equation

$$x^5 - px^4 + (p^2 - p)x^3 + px^2 - (p^3 + p^2)x - p^2 = 0$$

has no integer roots.

SOLUTION 3

Proof by contradiction.

Assume that t is an integer root of the equation in question. Then

$$t^5 - pt^4 + (p^2 - p)t^3 + pt^2 - (p^3 + p^2)t - p^2 = 0\,. \qquad \boxed{1}$$

Hence

$$\begin{aligned} t^5 &= pt^4 - (p^2 - p)t^3 - pt^2 + (p^3 + p^2)t + p^2 \\ &= p(t^4 - (p-1)t^3 - t^2 + (p^2 + p)t + p)\,. \end{aligned}$$

Thus t^5 is divisible by p.
Therefore t is divisible by p as p is a prime. $\boxed{1}$
Since $t^5 - pt^4 + (p^2 - p)t^3 + pt^2 - (p^3 + p^2)t - p^2 = 0$, we obtain

$$p^2 = t^5 - pt^4 + (p^2 - p)t^3 + pt^2 - (p^3 + p^2)t\,.$$

Each of the numbers t^5, pt^4, $(p^2 - p)t^3$, pt^2 and $(p^3 + p^2)t$ is divisible by p^3 as t is divisible by p. $\boxed{1}$

Hence p^2 is divisible by p^3, which is a contradiction.

Thus the equation $t^5 - pt^4 + (p^2 - p)t^3 + pt^2 - (p^3 + p^2)t - p^2 = 0$ has no integer roots. $\boxed{1}$

PROBLEM 4

For what real numbers k do the equations

$$\begin{aligned} x^{99} + kx^{57} - x^{20} - 1 &= 0, \\ x^{203} - 3x^{201} - (k+2)x^2 - 1 &= 0, \\ x^{201} - 3x^{199} + x^{98} + kx^{56} - x^{19} - k - 2 &= 0 \end{aligned}$$

have a common root?

SOLUTION 4

Let t be a common root of the three given equations.
Then t is a common root of the equation

$$x^2(x^{201} - 3x^{199} + x^{98} + kx^{56} - x^{19} - k - 2) - (x^{203} - 3x^{201} - (k+2)x^2 - 1) = 0.$$

Thus t is a common root of the equation $x^{100} + kx^{58} - x^{21} + 1 = 0$.

$\boxed{1}$

Since t is also a root of the equation $x^{99} + kx^{57} - x^{20} - 1 = 0$, we see that t is a root of the equation

$$x^{100} + kx^{58} - x^{21} + 1 - x(x^{99} + kx^{57} - x^{20} - 1) = 0$$

which means that t is a root of the equation $1 + x = 0$.
Hence $t = -1$. $\boxed{1}$
Therefore $(-1)^{99} + k(-1)^{57} - (-1)^{20} - 1 = 0$ which implies $k = -3$.

$\boxed{1}$

But $(-1)^{203} - 3 \times (-1)^{201} - (-3+2) \times (-1)^2 - 1 \neq 0$.
Therefore the given equations have a common root for no real numbers k.

$\boxed{1}$

PROBLEM 5

Is it possible to arrange the integers from 1 to 1000 in a table with 20 rows and 50 columns so that if for each of the rows all the numbers in it are added together, then twenty consecutive integers are obtained?

SOLUTION 5

The answer is no. We shall prove this by contradiction. $\boxed{1}$

Let N be the smallest sum of all numbers in a row. Then the sums of numbers in the other rows are $N+1, N+2, \ldots, N+19$. Therefore the sum of all the numbers in the table equals

$$N+N+1+N+2+\cdots+N+19 = \frac{20(N+N+19)}{2} = 20N+190. \quad \boxed{1}$$

On the other hand, the sum of all numbers in the table equals

$$1+2+\cdots+1000 = \frac{(1000+1)\times 1000}{2} = 1001 \times 500 = 500500\,. \quad \boxed{1}$$

Hence $20N+190 = 500500$ which means $50050-2N = 19$.
Since $50050-2N$ is even, we come to a contradiction as 19 is odd.
$\boxed{1}$

PROBLEM 6

Simplify:

$$\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\ldots+\frac{1}{2000\times 2003}.$$

SOLUTION 6

Since

$$\frac{1}{n(n+3)}=\frac{1}{3}\left(\frac{1}{n}-\frac{1}{n+3}\right), \qquad \boxed{1}$$

we have

$$\begin{aligned}
&\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\ldots+\frac{1}{2000\times 2003}\\
=\;&\frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right)+\frac{1}{3}\left(\frac{1}{5}-\frac{1}{8}\right)+\frac{1}{3}\left(\frac{1}{8}-\frac{1}{11}\right)+\cdots\\
&\qquad+\frac{1}{3}\left(\frac{1}{2000}-\frac{1}{2003}\right)\\
=\;&\frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}+\frac{1}{5}-\frac{1}{8}+\frac{1}{8}-\frac{1}{11}+\cdots+\frac{1}{2000}-\frac{1}{2003}\right)
\end{aligned}$$

$\boxed{1}$

$$=\frac{1}{3}\left(\frac{1}{2}-\frac{1}{2003}\right) \qquad \boxed{1}$$

$$=\frac{667}{4006}.$$

Thus

$$\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\ldots+\frac{1}{2000\times 2003}=\frac{667}{4006}. \qquad \boxed{1}$$

PROBLEM 7

Prove that the inequality

$$x(x+1)(x+2)(x+3) \geq -1$$

holds for all real numbers x.

SOLUTION 7

We have

$$\begin{aligned} x(x+1)(x+2)(x+3) &= x(x+3)(x+1)(x+2) \\ &= (x^2+3x)(x^2+3x+2). \end{aligned}$$

Let $x^2+3x=t$.
Then $x(x+1)(x+2)(x+3) = (x^2+3x)(x^2+3x+2) = t(t+2)$.
Therefore

$$x(x+1)(x+2)(x+3) - (-1) = x(x+1)(x+2)(x+3) + 1 \quad \boxed{1}$$

$$= t(t+2) + 1$$

$$= t^2 + 2t + 1$$

$$= (t+1)^2 \quad \boxed{1}$$

$$\geq 0. \quad \boxed{1}$$

Thus $x(x+1)(x+2)(x+3) \geq -1$ for all real numbers x. $\boxed{1}$

PROBLEM 8

Prove that

$$(ab + bc + ca)^2 \geq 3abc(a + b + c)$$

holds for all real numbers a, b and c.

SOLUTION 8

We have

$$(ab+bc+ca)^2-3abc(a+b+c) \qquad \boxed{1}$$

$$= (a^2b^2 + a^2c^2 + b^2c^2 + 2a^2bc + 2ab^2c + 2abc^2) - (3a^2bc + 3ab^2c + 3abc^2)$$

$$= a^2b^2+b^2c^2+c^2a^2-a^2bc-ab^2c-abc^2 \qquad \boxed{1}$$

$$= \frac{a^2b^2}{2} + \frac{a^2b^2}{2} + \frac{b^2c^2}{2} + \frac{b^2c^2}{2} + \frac{c^2a^2}{2} + \frac{c^2a^2}{2} - a^2bc - ab^2c - abc^2$$

$$= \frac{a^2b^2 - 2a^2bc + a^2c^2}{2} + \frac{a^2b^2 - 2ab^2c + b^2c^2}{2} + \frac{b^2c^2 - 2abc^2 + c^2a^2}{2}$$

$$= \frac{(ab - ac)^2}{2} + \frac{(ab - bc)^2}{2} + \frac{(bc - ca)^2}{2} \qquad \boxed{1}$$

$$\geq 0.$$

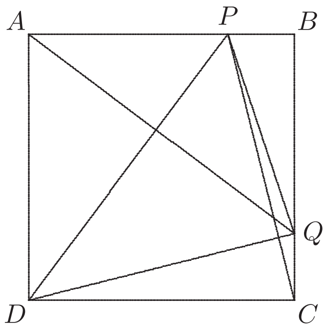
Hence $(ab + bc + ca)^2 \geq 3abc(a + b + c)$. $\boxed{1}$

PROBLEM 9

The side length of a square $ABCD$ is 4. P and Q are points on the sides AB and BC respectively such that $BP = 1$ and $BQ = 3$. Prove that

$$\angle PAQ + \angle PDQ + \angle PCQ = 90°.$$

SOLUTION 9



In $\triangle ABQ$ and $\triangle DAP$, $AB = DA$ and $\angle ABQ = \angle DAP$ as $ABCD$ is a square.
Also $BQ = AP$ since $BQ = 3$ and $AP = AB - BP = 4 - 1 = 3$.
Hence $\triangle ABQ$ is congruent to $\triangle DAP$. (SAS) [1]

Therefore $\angle BAQ = \angle ADP$. [1]

Similarly $\angle BCP = \angle CDQ$ as $\triangle BCP$ and $\triangle CDQ$ are congruent. [1]

Hence

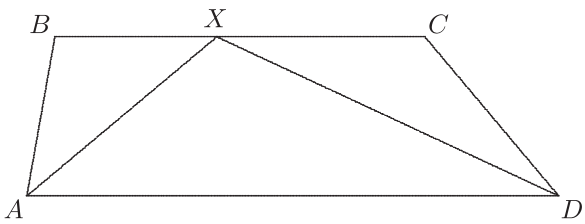
$$\begin{aligned}\angle PAQ + \angle PDQ + \angle PCQ &= \angle BAQ + \angle PDQ + \angle BCP \\ &= \angle ADP + \angle PDQ + \angle CDQ \\ &= \angle ADC \\ &= 90°.\end{aligned}$$

Thus $\angle PAQ + \angle PDQ + \angle PCQ = 90°$. [1]

PROBLEM 10

In a quadrilateral $ABCD$, there is a point X on the side BC such that XA and XD are the angle bisectors of $\angle BAD$ and $\angle CDA$ respectively. Prove that if $AB = BX$, then $BC = AB + CD$.

SOLUTION 10



Since $AB = BX$, $\angle BAX = \angle BXA$.
Since AX bisects $\angle BAD$, $\angle BAX = \angle DAX$.
Therefore $\angle BXA = \angle DAX$.
Hence BX is parallel to AD, which means that BC is parallel to AD. $\boxed{1}$

Since BC is parallel to AD, XC is parallel to AD.
Hence $\angle CXD = \angle ADX$. (*alternate angles*) $\boxed{1}$

Since XD bisects $\angle ADC$, $\angle ADX = \angle CDX$.
Therefore $\angle CXD = \angle CDX$.
Hence $CX = CD$. $\boxed{1}$

Therefore

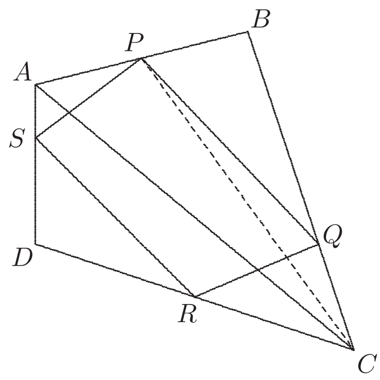
$$BC = BX + XC = AB + CD.$$

Thus $BC = AB + CD$. $\boxed{1}$

PROBLEM 11

In a quadrilateral $ABCD$, P and R are the midpoints of AB and CD respectively. Also Q and S are points on the sides BC and DA respectively such that $BQ = 2QC$ and $DS = 2SA$. Prove that the area of the quadrilateral $PQRS$ equals $\frac{S}{2}$ where S is the area of the quadrilateral $ABCD$.

SOLUTION 11



In $\triangle PBC$ and $\triangle ABC$, $BP = \frac{AB}{2}$ and the triangles have a common altitude from C.
Hence $|PBC| = \frac{|ABC|}{2}$. 1

In $\triangle PBC$ and $\triangle PBQ$, $BQ = \frac{2BC}{3}$ and the triangles have a common altitude from P.
Hence $|PBQ| = \frac{2}{3} \times |PBC| = \frac{2}{3} \times \frac{1}{2} \times |ABC| = \frac{1}{3} \times |ABC|$.
Thus $|PBQ| = \frac{|ABC|}{3}$. 1

Similarly $|SDR| = \frac{|ADC|}{3}$.
Similarly when drawing the diagonal BD, we obtain
$|APS| = \frac{|ABD|}{6}$ and $|CQR| = \frac{|BCD|}{6}$. 1

Therefore

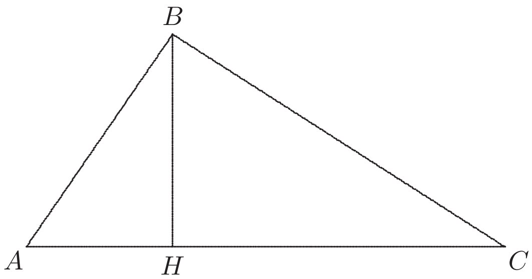
$$\begin{aligned} |PQRS| &= |ABCD| - |PBQ| - |RDS| - |SAP| - |QCR| \\ &= S - \frac{|ABC|}{3} - \frac{|ADC|}{3} - \frac{|DAB|}{6} - \frac{|DCB|}{6} \\ &= S - \frac{|ABC| + |ADC|}{3} - \frac{|DAB| + |DCB|}{6} \\ &= S - \frac{S}{3} - \frac{S}{6} \\ &= \frac{S}{2}. \end{aligned}$$

$\boxed{1}$

PROBLEM 12

In a triangle ABC, $AC = 2AB$ and $\frac{BC}{BA} = \frac{3}{2}$. H is the foot of the perpendicular from B to AC. Prove that $\frac{CH}{AH} = \frac{21}{11}$.

SOLUTION 12



By Pythagoras' theorem applied to $\triangle ABH$, $AB^2 = AH^2 + BH^2$.
Hence $BH^2 = AB^2 - AH^2$.

By Pythagoras' theorem applied to $\triangle CBH$, $CB^2 = CH^2 + BH^2$.
Hence $BH^2 = CB^2 - CH^2$. $\boxed{1}$

Therefore $AB^2 - AH^2 = CB^2 - CH^2$.
Hence

$$CB^2 - AB^2 = CH^2 - AH^2 = (CH - AH)(CH + AH) = AC(CH - AH).$$

Hence

$$CH - AH = \frac{CB^2 - AB^2}{AC} = \frac{\frac{9AB^2}{4} - AB^2}{2AB} = \frac{5AB}{8}.$$

Also $CH + AH = AC = 2AB$. $\boxed{1}$
Hence

$$2CH = (CH - AH) + (CH + AH) = \frac{5AB}{8} + 2AB = \frac{21AB}{8}$$

and

$$2AH = (CH + AH) - (CH - AH) = 2AB - \frac{5AB}{8} = \frac{11AB}{8}. \quad \boxed{1}$$

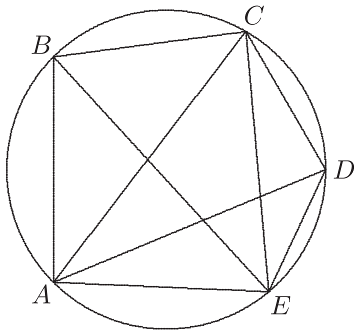
Therefore

$$\frac{CH}{AH} = \frac{2CH}{2AH} = \frac{21AB}{8} \div \frac{11AB}{8} = \frac{21}{11}. \quad \boxed{1}$$

PROBLEM 13

A, B, C, D and E are points in order on the circumference of a circle. $\angle ABC = 100°$ and $\angle CDE = 125°$. Prove that $\angle ACE = 45°$.

SOLUTION 13



Join AC, CE, AD, BE and AE.

$\angle ABC = \angle ABE + \angle CBE$.
$\angle ABE = \angle ACE$. (*subtended by the same arc*)
$\angle CBE = \angle CAE$. (*subtended by the same arc*)
Therefore, $\angle ABC = \angle ACE + \angle CAE$. [1]

Similarly, $\angle CDE = \angle ACE + \angle CEA$.

Therefore, $\angle ABC + \angle CDE = 2\angle ACE + \angle CAE + \angle CEA$.
Hence $\angle ABC + \angle CDE = \angle ACE + 180°$ as $\angle ACE + \angle CAE + \angle CEA = 180°$. [1]

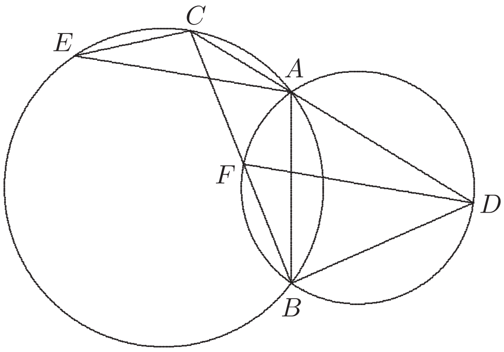
On the other hand, $\angle ABC + \angle CDE = 100° + 125° = 225°$. [1]

Hence $\angle ACE + 180° = 225°$.
Therefore, $\angle ACE = 225° - 180° = 45°$ as required. [1]

PROBLEM 14

Two circles C_1 and C_2 meet at the points A and B. A point C is on the circumference of C_1. The chord CB cuts C_2 at the point F and the chord CA produced meets C_2 at D. E is a point on the circumference of C_1 such that EA is parallel to FD. Prove that $EC = CA$.

SOLUTION 14



Join AB and EC. [1]

$\angle ADF = \angle ABF$ (*subtended by the same arc*)
$\angle ABF = \angle ABC$
$\angle ABC = \angle AEC$ (*subtended by the same arc*)
Hence $\angle ADF = \angle CEA$. [1]

On the other hand, $\angle ADF = \angle CDF = \angle CAE$ as EA is parallel to FD.

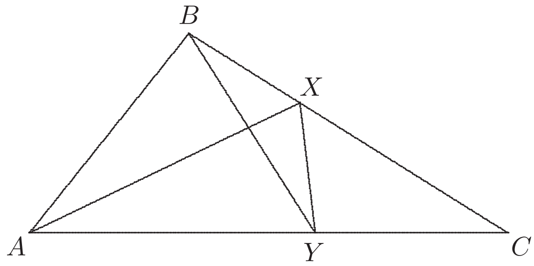
[1]

Therefore $\angle CEA = \angle CAE$, which yields $EC = CA$. [1]

PROBLEM 15

X and Y are points on the sides BC and AC of a triangle ABC respectively such that $\angle AXC = \angle BYC$ and $BX = XY$. Prove that AX bisects the angle $\angle BAC$.

SOLUTION 15



Since $\angle AXC = \angle BYC$, we have

$$\angle XAC = 180° - \angle AXC - \angle XCA = 180° - \angle BYC - \angle YCB = \angle YBC.$$

Thus $\angle XAY = \angle XBY$. 1

Hence $ABXY$ is a cyclic quadrilateral. 1

Hence $\angle BAX = \angle BYX$. (*subtended by the same arc*) 1

Also $\angle BYX = \angle YBX$ as $BX = XY$.
Therefore

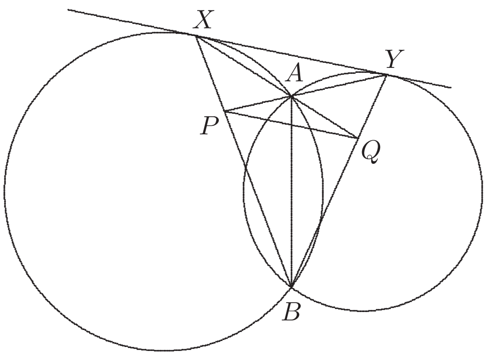
$$\angle BAX = \angle BYX = \angle YBX = \angle XAC.$$

Thus $\angle BAX = \angle YAX$ as required. 1

PROBLEM 16

Two circles meet at the points A and B. A straight line touches one of the circles at X and the other at Y. XA produced meets the chord BY at Q and YA produced meets the chord BX at P. Prove that XY is parallel to PQ.

SOLUTION 16



Join AB.

$\angle AXY = \angle ABX$ by the alternate segment theorem as XY is tangent at X.
$\angle AYX = \angle ABY$ by the alternate segment theorem as XY is tangent at Y. [1]

Therefore

$$\begin{aligned}\angle XAY &= 180^\circ - (\angle AXY + \angle AYX)\\ &= 180^\circ - (\angle ABX + \angle ABY)\\ &= 180^\circ - \angle XBY.\end{aligned}$$

Hence $\angle XAY + \angle XBY = 180^\circ$.
Therefore $\angle PAQ + \angle PBQ = 180^\circ$ as $\angle XAY = \angle PAQ$.
Hence $PAQB$ is a cyclic quadrilateral. [1]

Hence $\angle AQP = \angle ABP$. (*subtended by the same arc*) [1]

Therefore $\angle AQP = \angle ABX = \angle YXA$, which implies that XY is parallel to PQ. [1]