



MATHEMATICS CHALLENGE FOR YOUNG
AUSTRALIANS
INTERMEDIATE: YEARS 9 and 10
WARM UP PROBLEM 03: SOLUTIONS

Squared Fractions

Alternative 1

1. If $\frac{1}{72}$ can be written with only two terms, then one of the terms must be $\frac{1}{81}$, $\frac{1}{100}$ or $\frac{1}{121}$, since

$$\frac{1}{72} - \frac{1}{144} = \frac{1}{144}$$

and any term used which is smaller than $\frac{1}{144}$ must be paired with one which is bigger than $\frac{1}{144}$.

We note

$$\frac{1}{72} - \frac{1}{81} = \frac{1}{8.9^2},$$

which is not the reciprocal of a square.

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Similarly

$$\frac{1}{72} - \frac{1}{100} = \frac{7}{1800}$$

and

$$\frac{1}{72} - \frac{1}{121} = \frac{49}{(72)(121)},$$

neither of which is the reciprocal of a square.

Thus $\frac{1}{72}$ cannot be written in the desired form.

2. $\frac{1}{72} = \frac{1}{12^2} + \frac{1}{15^2} + \frac{1}{20^2}$.
3. $\frac{1}{8} = \frac{1}{3^2} + \frac{1}{12^2} + \frac{1}{15^2} + \frac{1}{20^2}$.

[This is the best answer, i.e. using fewest terms. Other answers are possible using five or more terms. That it is not possible to write $\frac{1}{8}$ using three terms requires an argument similar to the one in [1.](#), but this is not required for full marks.]

Alternative 2

1. In a problem like this, we have to do some trial and error. However, it is not quite like looking for a needle in a haystack; we can use a couple of ‘magnets’ to help us to find suitable possibilities. The key is to consider the sizes of x and y in a possible representation:

$$\frac{1}{72} = \frac{1}{x^2} + \frac{1}{y^2},$$

with $x > y$.

Clearly, both x^2 and y^2 must be greater than 72.

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Further, since

$$\frac{1}{2} \cdot \frac{1}{72} = \frac{1}{144},$$

if the smaller of x^2 and y^2 is greater than 144, then the larger one must be less than 144.

Hence $72 \leq y^2 \leq 144$, i.e. $9 \leq y \leq 11$.

We now test for $y = 9, 10$ or 11 to see if

$$\frac{1}{72} - \frac{1}{y^2} = \frac{y^2 - 72}{72y^2}$$

can be expressed in the form $\frac{1}{x^2}$.

y	$\frac{y^2 - 72}{72y^2}$	x
9	$\frac{81 - 72}{72 \cdot 81} = \frac{9}{72 \cdot 81} = \frac{1}{8.81}$	No
10	$\frac{100 - 72}{72 \cdot 100} = \frac{28}{72 \cdot 100} = \frac{7}{18.100}$	No
11	$\frac{121 - 72}{72 \cdot 121} = \frac{49}{72 \cdot 121}$	No

In no case can $\frac{y^2 - 72}{72y^2}$ be expressed in the form $\frac{1}{x^2}$.

Thus $\frac{1}{72}$ cannot be expressed as the sum of the reciprocals of two different squares.

2. Now $\frac{1}{72} = \frac{1}{144} + \frac{1}{144} = \frac{1}{12^2} + \frac{1}{12^2}$.

Now if one of the $\frac{1}{12^2}$ s can be expressed as $\frac{1}{x^2} + \frac{1}{y^2}$, with $x > y$, then we have solved the problem.

Let $\frac{1}{144} = \frac{1}{x^2} + \frac{1}{y^2}$, with $x > y$.

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Again by contradiction, $144 < y^2 < 288$, i.e. $13 \leq y \leq 16$ and as before we test for $y = 13, 14, 15, 16$ to see if

$$\frac{1}{144} - \frac{1}{y^2} = \frac{y^2 - 144}{144y^2}$$

can be expressed in the form $\frac{1}{x^2}$.

y	$\frac{y^2-72}{72y^2}$	x
13	$\frac{169-144}{144 \cdot 169} = \frac{25}{144 \cdot 169}$	No
14	$\frac{196-144}{144 \cdot 196} = \frac{52}{144 \cdot 196}$	No
15	$\frac{225-144}{144 \cdot 225} = \frac{1}{16 \cdot 25}$	20
16	$\frac{256-144}{144 \cdot 256} = \frac{112}{144 \cdot 256}$	No

Thus there is a way of expressing $\frac{1}{72}$ as the sum of the reciprocals of three squares, viz

$$\frac{1}{72} = \frac{1}{12^2} + \frac{1}{15^2} + \frac{1}{20^2}.$$

3. Let $\frac{1}{8} = \frac{1}{x^2} + \frac{1}{y^2}$, with $x > y$.

By contradiction, $\frac{1}{16} < \frac{1}{y^2} < \frac{1}{8}$, so $8 < y^2 < 16$ and $y = 3$ only.

Testing for $y = 3$,

$$\frac{1}{8} - \frac{1}{9} = \frac{1}{72},$$

which cannot be expressed in the form $\frac{1}{x^2}$.

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Thus, $\frac{1}{8}$ cannot be expressed as the sum of reciprocals of two different squares.

Let

$$\frac{1}{8} = \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2},$$

with $x > y > z$.

Again using the idea of contradiction, now for three terms, we obtain

$$\frac{1}{24} < \frac{1}{z^2} < \frac{1}{8},$$

i.e. $8 < z^2 < 24$. So $z = 3$ or 4 .

Now we know that when $z = 3$,

$$\frac{1}{8} = \frac{1}{3^2} + \frac{1}{72},$$

and we have shown that $\frac{1}{72}$ cannot be expressed as $\frac{1}{x^2} + \frac{1}{y^2}$, so we test only the value for $z = 4$.

If $z = 4$, then

$$\frac{1}{8} = \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{16},$$

with $x > y$, i.e.

$$\frac{1}{16} = \frac{1}{x^2} + \frac{1}{y^2}.$$

By contradiction, $16 < y^2 < 32$, so $y = 5$.

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However, when $y = 5$ we have

$$\frac{1}{8} = \frac{1}{x^2} + \frac{1}{5^2},$$

i.e.

$$\frac{1}{x^2} = \frac{1}{8} + \frac{1}{25} = \frac{17}{200},$$

which is not of the required form.

Thus $\frac{1}{8}$ cannot be expressed as the sum of the reciprocals of three different squares.

So, as $\frac{1}{8} = \frac{1}{3^2} + \frac{1}{72}$, we can use [2.](#) to obtain

$$\frac{1}{8} = \frac{1}{3^2} + \frac{1}{12^2} + \frac{1}{15^2} + \frac{1}{20^2},$$

thus expressing $\frac{1}{8}$ as the sum of the reciprocals of four different squares.