Learn from yesterday, live for today, hope for tomorrow. The important thing is not to stop questioning.

Albert Einstein
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SUPPORT FOR THE AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE TRAINING PROGRAM

The Australian Mathematical Olympiad Committee Training Program is an activity of the Australian Mathematical Olympiad Committee, a department of the Australian Mathematics Trust.

Trustee

The University of Canberra

Sponsors

The Mathematics Olympiads are supported by the Australian Government Department of Education through the Mathematics and Science Participation Program.

The Australian Mathematical Olympiad Committee (AMOC) also acknowledges the significant financial support it has received from the Australian Government towards the training of our Olympiad candidates and the participation of our team at the International Mathematical Olympiad (IMO).

The views expressed here are those of the authors and do not necessarily represent the views of the government.

Special Thanks

With special thanks to the Australian Mathematical Society, the Australian Association of Mathematics Teachers and all those schools, societies, families and friends who have contributed to the expense of sending the 2014 IMO team to Cape Town, South Africa.
ACKNOWLEDGEMENTS

The Australian Mathematical Olympiad Committee thanks sincerely all sponsors, teachers, mathematicians and others who have contributed in one way or another to the continued success of its activities.

The editors thank sincerely those who have assisted in the compilation of this book, in particular the students who have provided solutions to the 2014 IMO. Thanks also to members of AMOC and Challenge Problems Committees, Adjunct Professor Mike Clapper, staff of the Australian Mathematics Trust and others who are acknowledged elsewhere in the book.
The year 2014 has been an outstanding one for the AMOC training program. The main measure is the performance of the Australian team at the IMO, which this year was held in Cape Town, South Africa. We achieved our best team result (11th out of 101 teams) since 1997 (where we were 9th out of 82 teams), with all of our participants achieving medals. Our Gold medallist, Alex Gunning, achieved a perfect score, the best result ever by an Australian student, finishing equal first in the world with two other students. Together with the Bronze which Alex achieved two years ago and the Gold he won last year, he becomes our most successful Olympian. Three of the team were Year 12 students, so there will be a challenge ahead to build a new team for 2015. In the Mathematics Ashes we lost to a very strong and experienced British team; however, we finished comfortably ahead of them in the IMO competition proper.

Director of Training and IMO Team Leader, Dr Angelo Di Pasquale, and his team of former Olympians continue to innovate and keep the training alive, fresh and, above all, of high quality. We congratulate them again on their success. The Team had a new Deputy Leader this year, Andrew Elvey Price, himself a former IMO Gold medallist.

The Mathematics Challenge for Young Australians (MCYA) also continues to attract strong entries, with the Challenge stage helping students to develop their problem-solving skills. In 2014, we introduced a Middle Primary division to provide a pathway for younger students. The Enrichment stage, containing course work, allows students to broaden their knowledge base in the areas of mathematics associated with the Olympiad programs and more advanced problem-solving techniques. We have continued running workshops for teachers to develop confidence in managing these programs for their more able students—this seems to be paying off with increased numbers in both the Challenge and Enrichment stages.

The final stage of the MCYA program is the Australian Intermediate Mathematics Olympiad (AIMO). This year, we encouraged prize winners from the AMC to enter the AIMO and this resulted in a doubling of numbers. As a result, we unearthed some new talent from schools which had not previously taken part in this competition.

I would particularly wish to thank all the dedicated volunteers without whom this program would not exist. These include the Director of Training and the ex-Olympians who train the students at camps; the AMOC state directors; and the Challenge Director, Dr Kevin McAvaney, and the various members of his Problems Committee, who develop such original problems, solutions and discussions each year.

Our support from the Australian Government for the AMOC program continues, provided through the Department of Education, and this enables us to maintain the quality of our programs. We are most grateful for this support.
The AMOC Senior Problems Committee has been chaired for first time by Dr Norman Do, another former Olympian and Deputy Leader of the IMO team. Norm has done a terrific job in his first year in this position.

The invitational program saw some outstanding results from Australian students, with a number of perfect scores. Details of these achievements are provided in the appropriate section of this book.

For the first time, we are producing Mathematics Contests—The Australian Scene in electronic form only. We hope this will provide greater access to the problems and section reports. Whilst the whole book can be downloaded as a pdf, it is also available in two sections, one containing the MCYA reports and papers and this one which contains the Olympiad training program reports and papers.

Mike Clapper
April 2015
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The Australian Mathematical Olympiad Committee

In 1980, a group of distinguished mathematicians formed the Australian Mathematical Olympiad Committee (AMOC) to coordinate an Australian entry in the International Mathematical Olympiad (IMO).

Since then, AMOC has developed a comprehensive program to enable all students (not only the few who aspire to national selection) to enrich and extend their knowledge of mathematics. The activities in this program are not designed to accelerate students. Rather, the aim is to enable students to broaden their mathematical experience and knowledge.

The largest of these activities is the MCYA Challenge, a problem-solving event held in second term, in which thousands of young Australians explore carefully developed mathematical problems. Students who wish to continue to extend their mathematical experience can then participate in the MCYA Enrichment Stage and pursue further activities leading to the Australian Mathematical Olympiad and international events.

Originally AMOC was a subcommittee of the Australian Academy of Science. In 1992 it collaborated with the Australian Mathematics Foundation (which organises the Australian Mathematics Competition) to form the Australian Mathematics Trust. The Trust, a not-for-profit organisation under the trusteeship of the University of Canberra, is governed by a Board which includes representatives from the Australian Academy of Science, Australian Association of Mathematics Teachers and the Australian Mathematical Society.

The aims of AMOC include:

1. giving leadership in developing sound mathematics programs in Australian schools
2. identifying, challenging and motivating highly gifted young Australian school students in mathematics
3. training and sending Australian teams to future International Mathematical Olympiads.

AMOC schedule from August until July for potential IMO team members

Each year hundreds of gifted young Australian school students are identified using the results from the Australian Mathematics Competition sponsored by the Commonwealth Bank, the Mathematics Challenge for Young Australians program and other smaller mathematics competitions. A network of dedicated mathematicians and teachers has been organised to give these students support during the year either by correspondence sets of problems and their solutions or by special teaching sessions.

It is these students who sit the Australian Intermediate Mathematics Olympiad, or who are invited to sit the AMOC Senior Contest each August. Most states run extension or correspondence programs for talented students who are invited to participate in the relevant programs. The 25 outstanding students in recent AMOC programs and other mathematical competitions are identified and invited to attend the residential AMOC School of Excellence held in December.
In February approximately 100 students are invited to attempt the Australian Mathematical Olympiad. The best 20 or so of these students are then invited to represent Australia in the correspondence Asian Pacific Mathematics Olympiad in March. About 12 students are selected for the AMOC Selection School in April and about 13 younger students are also invited to this residential school. Here, the Australian team of six students plus one reserve for the International Mathematical Olympiad, held in July each year, is selected. A personalised support system for the Australian team operates during May and June.

It should be appreciated that the AMOC program is not meant to develop only future mathematicians. Experience has shown that many talented students of mathematics choose careers in engineering, computing, and the physical and life sciences, while others will study law or go into the business world. It is hoped that the AMOC Mathematics Problem-Solving Program will help the students to think logically, creatively, deeply and with dedication and perseverance; that it will prepare these talented students to be future leaders of Australia.

The International Mathematical Olympiad

The IMO is the pinnacle of excellence and achievement for school students of mathematics throughout the world. The concept of national mathematics competitions started with the Eötvos Competition in Hungary during 1894. This idea was later extended to an international mathematics competition in 1959 when the first IMO was held in Romania. The aims of the IMO include:

(1) discovering, encouraging and challenging mathematically gifted school students
(2) fostering friendly international relations between students and their teachers
(3) sharing information on educational syllabi and practice throughout the world.

It was not until the mid-sixties that countries from the western world competed at the IMO. The United States of America first entered in 1975. Australia has entered teams since 1981.

Students must be under 20 years of age at the time of the IMO and have not enrolled at a tertiary institution. The Olympiad contest consists of two four-and-a-half hour papers, each with three questions.

Australia has achieved varying successes as the following summary of results indicate. HM (Honorable Mention) is awarded for obtaining full marks in at least one question.

The IMO will be held in Chang Mai, Thailand, in 2015.
### Summary of Australia's achievements at previous IMOs

<table>
<thead>
<tr>
<th>YEAR</th>
<th>CITY</th>
<th>GOLD</th>
<th>SILVER</th>
<th>BRONZE</th>
<th>HM</th>
<th>RANK</th>
</tr>
</thead>
<tbody>
<tr>
<td>1981</td>
<td>Washington</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>23 out of 27 teams</td>
</tr>
<tr>
<td>1982</td>
<td>Budapest</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>21 out of 30 teams</td>
</tr>
<tr>
<td>1983</td>
<td>Paris</td>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
<td>19 out of 32 teams</td>
</tr>
<tr>
<td>1984</td>
<td>Prague</td>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
<td>15 out of 34 teams</td>
</tr>
<tr>
<td>1985</td>
<td>Helsinki</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td></td>
<td>11 out of 38 teams</td>
</tr>
<tr>
<td>1986</td>
<td>Warsaw</td>
<td></td>
<td></td>
<td></td>
<td>5</td>
<td>15 out of 37 teams</td>
</tr>
<tr>
<td>1987</td>
<td>Havana</td>
<td></td>
<td>3</td>
<td></td>
<td></td>
<td>15 out of 42 teams</td>
</tr>
<tr>
<td>1988</td>
<td>Canberra</td>
<td>1</td>
<td></td>
<td>1</td>
<td>1</td>
<td>17 out of 49 teams</td>
</tr>
<tr>
<td>1989</td>
<td>Braunschweig</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
<td>22 out of 50 teams</td>
</tr>
<tr>
<td>1990</td>
<td>Beijing</td>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
<td>15 out of 54 teams</td>
</tr>
<tr>
<td>1991</td>
<td>Sigtuna</td>
<td></td>
<td></td>
<td>3</td>
<td>2</td>
<td>20 out of 56 teams</td>
</tr>
<tr>
<td>1992</td>
<td>Moscow</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>19 out of 56 teams</td>
</tr>
<tr>
<td>1993</td>
<td>Istanbul</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td>13 out of 73 teams</td>
</tr>
<tr>
<td>1994</td>
<td>Hong Kong</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td></td>
<td>12 out of 69 teams</td>
</tr>
<tr>
<td>1995</td>
<td>Toronto</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td></td>
<td>21 out of 73 teams</td>
</tr>
<tr>
<td>1996</td>
<td>Mumbai</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td>23 out of 75 teams</td>
</tr>
<tr>
<td>1997</td>
<td>Mar del Plata</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td></td>
<td>9 out of 82 teams</td>
</tr>
<tr>
<td>1998</td>
<td>Taipei</td>
<td>4</td>
<td>2</td>
<td></td>
<td></td>
<td>13 out of 76 teams</td>
</tr>
<tr>
<td>1999</td>
<td>Bucharest</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>15 out of 81 teams</td>
</tr>
<tr>
<td>2000</td>
<td>Taegon</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td></td>
<td>16 out of 82 teams</td>
</tr>
<tr>
<td>2001</td>
<td>Washington D.C.</td>
<td>1</td>
<td></td>
<td>4</td>
<td></td>
<td>25 out of 83 teams</td>
</tr>
<tr>
<td>2002</td>
<td>Glasgow</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>26 out of 84 teams</td>
</tr>
<tr>
<td>2003</td>
<td>Tokyo</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
<td>26 out of 82 teams</td>
</tr>
<tr>
<td>2004</td>
<td>Athens</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>27 out of 85 teams</td>
</tr>
<tr>
<td>2005</td>
<td>Merida</td>
<td></td>
<td></td>
<td></td>
<td>6</td>
<td>25 out of 91 teams</td>
</tr>
<tr>
<td>2006</td>
<td>Ljubljana</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td>26 out of 90 teams</td>
</tr>
<tr>
<td>2007</td>
<td>Hanoi</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td></td>
<td>22 out of 93 teams</td>
</tr>
<tr>
<td>2008</td>
<td>Madrid</td>
<td>5</td>
<td>1</td>
<td></td>
<td></td>
<td>19 out of 97 teams</td>
</tr>
<tr>
<td>2009</td>
<td>Bremen</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>23 out of 104 teams</td>
</tr>
<tr>
<td>2010</td>
<td>Astana</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>15 out of 96 teams</td>
</tr>
<tr>
<td>2011</td>
<td>Amsterdam</td>
<td>3</td>
<td>3</td>
<td></td>
<td></td>
<td>25 out of 101 teams</td>
</tr>
<tr>
<td>2012</td>
<td>Mar del Plata</td>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
<td>27 out of 100 teams</td>
</tr>
<tr>
<td>2013</td>
<td>Santa Marta</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td>15 out of 97 teams</td>
</tr>
<tr>
<td>2014</td>
<td>Cape Town</td>
<td></td>
<td></td>
<td></td>
<td>3</td>
<td>25 out of 101 teams</td>
</tr>
</tbody>
</table>

1 Perfect Score by Alexander Gunning

10 Mathematics Contests The Australian Scene 2014
Membership of AMOC Committees

Australian Mathematical Olympiad Committee 2014

CHAIR
Prof C Praeger, University of Western Australia

DEPUTY CHAIR
Assoc Prof D Hunt, University of New South Wales

EXECUTIVE DIRECTOR
Adj Prof Mike Clapper, Australian Mathematics Trust, ACT

TREASURER
Dr P Swedosh, The King David School, VIC

CHAIR, SENIOR PROBLEMS COMMITTEE
Dr N Do, Monash University, VIC

CHAIR, CHALLENGE
Dr K McAvaney, Deakin University, VIC

DIRECTOR OF TRAINING AND IMO TEAM LEADER
Dr A Di Pasquale, University of Melbourne, VIC

IMO DEPUTY TEAM LEADER
Mr A Elvey Price, University of Melbourne, VIC

STATE DIRECTORS
Dr K Dharmadasa, University of Tasmania
Dr G Gamble, University of Western Australia
Dr Ian Roberts, Northern Territory
Dr W Palmer, University of Sydney, NSW
Mr D Martin, South Australia
Dr V Scharaschkin, University of Queensland
Dr P Swedosh, The King David School, VIC
Dr Chris Wetherell, Radford College, ACT

REPRESENTATIVES
Ms A Nakos, Challenge Committee
Prof M Newman, Challenge Committee
Mr H Reeves, Challenge Committee
AMOC TIMETABLE FOR SELECTION OF THE TEAM TO THE 2015 IMO

August 2014—July 2015

Hundreds of students are involved in the AMOC programs which begin on a state basis. The students are given problem-solving experience and notes on various IMO topics not normally taught in schools.

The students proceed through various programs with the top 25 students, including potential team members and other identified students, participating in a ten-day residential school in December.

The selection program culminates with the April Selection School during which the team is selected.

Team members then receive individual coaching by mentors prior to assembling for last minute training before the IMO.

<table>
<thead>
<tr>
<th>MONTH</th>
<th>ACTIVITY</th>
</tr>
</thead>
<tbody>
<tr>
<td>AUGUST</td>
<td>Outstanding students are identified from AMC results, MCYA, other competitions and recommendations; and eligible students from previous training programs</td>
</tr>
<tr>
<td></td>
<td>AMOC state organisers invite students to participate in AMOC programs</td>
</tr>
<tr>
<td></td>
<td>Various state-based programs</td>
</tr>
<tr>
<td></td>
<td>AMOC Senior Contest</td>
</tr>
<tr>
<td>SEPTEMBER</td>
<td>Australian Intermediate Mathematics Olympiad</td>
</tr>
<tr>
<td>DECEMBER</td>
<td>AMOC School of Excellence</td>
</tr>
<tr>
<td>JANUARY</td>
<td>Summer Correspondence Program for those who attended the School of Excellence</td>
</tr>
<tr>
<td>FEBRUARY</td>
<td>Australian Mathematical Olympiad</td>
</tr>
<tr>
<td>MARCH</td>
<td>Asian Pacific Mathematics Olympiad</td>
</tr>
<tr>
<td>APRIL</td>
<td>AMOC Selection School</td>
</tr>
<tr>
<td>MAY–JUNE</td>
<td>Personal Tutor Scheme for IMO team members</td>
</tr>
<tr>
<td>JULY</td>
<td>Short mathematics school for IMO team members</td>
</tr>
<tr>
<td></td>
<td>2015 IMO in Chiang Mai, Thailand.</td>
</tr>
</tbody>
</table>
This committee has been in existence for many years and carries out a number of roles. A central role is the collection and moderation of problems for senior and exceptionally gifted intermediate and junior secondary school students. Each year the Problems Committee provides examination papers for the AMOC Senior Contest and the Australian Mathematical Olympiad. In addition, problems are submitted for consideration to the Problem Selection Committees of the annual Asian Pacific Mathematics Olympiad and the International Mathematical Olympiad.

AMOC Senior Problems Committee October 2013–September 2014

Dr A Di Pasquale, University of Melbourne, VIC
Dr N Do, Monash University, VIC
Dr M Evans, Australian Mathematical Sciences Institute, VIC
Dr J Guo, University of Sydney, NSW
Assoc Prof D Hunt, University of NSW
Dr J Kupka, Monash University, VIC
Assoc Prof H Lausch, Monash University, VIC (Chair)
Dr K McAvaney, Deakin University, VIC
Dr D Mathews, Monash University, VIC
Dr A Offer, Queensland
Dr C Rao, NEC Australia, VIC
Dr B B Saad, Monash University, VIC
Assoc Prof J Simpson, Curtin University of Technology, WA
Emeritus Professor P J Taylor, Australian Capital Territory
Dr I Wanless, Monash University, VIC

* In 2013 AssocProf H Lausch retired as Chair and the position was taken up by Dr N Do.

1. 2014 Australian Mathematical Olympiad

The Australian Mathematical Olympiad (AMO) consists of two papers of four questions each and was sat on 11 and 12 February. There were 99 participants including 12 from New Zealand, one more participant than 2013. One student, Mel Shu, achieved a perfect score and nine other students were awarded Gold certificates, 18 students were awarded Silver certificates and 21 students were awarded Bronze certificates.

2. 2014 Asian Pacific Mathematics Olympiad

On Tuesday 11 March students from 36 nations around the Asia-Pacific region were invited to write the Asian Pacific Mathematics Olympiad (APMO). Of the top ten Australian students who participated, there were 1 Gold, 2 Silver, 4 Bronze and 3 HM certificates awarded. Australia finished in 9th place overall, a significant improvement from 15th last year.

The paper was more difficult this year, with the mean score of all contestants being just 10 marks. In particular, the last two problems, one of which was submitted by Australia, proved difficult for all contestants.
3. 2014 **International Mathematical Olympiad, Cape Town.**

The IMO consists of two papers of three questions worth seven points each. They were attempted by teams of six students from 101 countries on 8 and 9 July in Cape Town, South Africa. Australia was placed 11th of 101 countries.

The Australian team had its most successful results since it began participating with Alexander Gunning achieving a perfect score. He was ranked equal first in the world with only two other contestants. It was an outstanding effort winning him a Gold medal. The medals for Australia were one Gold, three Silver and two Bronze.

4. 2014 **AMOC Senior Contest**

Held on Tuesday 12 August, the Senior Contest was sat by 81 students (compared to 74 in 2013). There were three students who obtained perfect scores who were the Prize winners, eight High Distinctions and ten Distinctions.
1. Each point in the plane is labelled with a real number. For each cyclic quadrilateral $ABCD$ in which the line segments $AC$ and $BD$ intersect, the sum of the labels at $A$ and $C$ equals the sum of the labels at $B$ and $D$.

Prove that all points in the plane are labelled with the same number.

2. For which integers $n \geq 2$ is it possible to separate the numbers $1, 2, \ldots, n$ into two sets such that the sum of the numbers in one of the sets is equal to the product of the numbers in the other set?

3. Consider functions $f$ defined for all real numbers and taking real numbers as values such that

$$f(x + 14) - 14 \leq f(x) \leq f(x + 20) - 20,$$

for all real numbers $x$.

Determine all possible values of $f(8765) - f(4321)$.

4. Let $ABC$ be a triangle such that $\angle ACB = 90^\circ$. The point $D$ lies inside triangle $ABC$ and on the circle with centre $B$ that passes through $C$. The point $E$ lies on the side $AB$ such that $\angle DAE = \angle BDE$. The circle with centre $A$ that passes through $C$ meets the line through $D$ and $E$ at the point $F$, where $E$ lies between $D$ and $F$.

Prove that $\angle AFE = \angle EBF$.

5. Ada tells Byron that she has drawn a rectangular grid of squares and placed either the number 0 or the number 1 in each square. Next to each row, she writes the sum of the numbers in that row. Below each column, she writes the sum of the numbers in that column. After Ada erases all of the numbers in the squares, Byron realises that he can deduce each erased number from the row sums and the column sums.

Prove that there must have been a row containing only the number 0 or a column containing only the number 1.
1. Each point in the plane is labelled with a real number. For each cyclic quadrilateral \(ABCD\) in which the line segments \(AC\) and \(BD\) intersect, the sum of the labels at \(A\) and \(C\) equals the sum of the labels at \(B\) and \(D\).

Prove that all points in the plane are labelled with the same number.

**Solution 1** (Angelo Di Pasquale)

For any point \(P\) in the plane, let \(f(P)\) denote its label. Consider two points \(A\) and \(B\), and construct any cyclic pentagon \(ABPQR\) whose vertices lie in that order.

Then we have \(f(A) + f(Q) = f(P) + f(R) = f(B) + f(Q)\).

This implies that the arbitrarily chosen points \(A\) and \(B\) satisfy \(f(A) = f(B)\).

So it is necessarily true that all points in the plane are labelled with the same number.

**Solution 2** (Chaitanya Rao)

For any point \(P\) in the plane, let \(f(P)\) denote its label. Take an arbitrary cyclic quadrilateral \(ABCD\), where \(AC\) and \(BD\) intersect, and consider points on the circumcircle. Note that every point \(P\) on the circular arc \(DB\) containing \(A\) satisfies

\[
f(P) = f(B) + f(D) - f(C) = f(A) + f(C) - f(C) = f(A).
\]

Similarly, every point \(Q\) on the circular arc \(AC\) containing \(B\) satisfies

\[
f(Q) = f(A) + f(C) - f(D) = f(B) + f(D) - f(D) = f(B).
\]

We can choose \(P = Q\) on the arc \(AB\) not containing \(C\) or \(D\) in order to deduce that \(f(A) = f(B)\).

Since the points \(A\) and \(B\) can be chosen arbitrarily, it follows that all points in the plane are labelled with the same number.
2. For which integers \( n \geq 2 \) is it possible to separate the numbers \( 1, 2, \ldots, n \) into two sets such that the sum of the numbers in one of the sets is equal to the product of the numbers in the other set?

**Solution 1** (Angelo Di Pasquale)

Suppose that \( x, y, z \) are in one of the groups and that the rest of the numbers are in the other group. This information leads to the equation

\[
xyz = 1 + 2 + \cdots + n - x - y - z = \frac{n(n + 1)}{2} - x - y - z.
\]

If we substitute \( z = 1 \) in the equation above, we can rearrange to obtain

\[
(x + 1)(y + 1) = \frac{n(n + 1)}{2}.
\]

If \( n \) is even, we can take \( x = \frac{n}{2} - 1 \) and \( y = n \).

If \( n \) is odd, we can take \( x = \frac{n-1}{2} \) and \( y = n - 1 \).

These constructions can be carried out as long as \( x, y, z \) are all different, which holds for \( n \geq 5 \). It is easy to check that the task is impossible for \( n = 2 \) and \( n = 4 \). On the other hand, we have \( 1 + 2 = 3 \) for \( n = 3 \). Therefore, the task is possible only for \( n = 3 \) and all integers \( n \geq 5 \).

**Solution 2** (Daniel Mathews and Kevin McAvaney)

By examining small values of \( n \), one can directly observe the following patterns and verify them.

- If \( n = 2k \) for an integer \( k \geq 3 \), we have
  \[
  [1 + 2 + \cdots + (2k)] - 1 - (k - 1) - (2k) = 1 \times (k - 1) \times (2k).
  \]
  This fact follows from the identity \( 1 + 2 + \cdots + (2k) = k(2k + 1) \).

- When \( n = 2k + 1 \) for an integer \( k \geq 2 \), we have
  \[
  [1 + 2 + \cdots + (2k + 1)] - 1 - k - (2k) = 1 \times k \times (2k).
  \]
  This fact follows from the identity \( 1 + 2 + \cdots + (2k + 1) = (k + 1)(2k + 1) \).

The remaining cases \( n = 1, 2, 3, 4 \) can be handled individually, as in the previous solution. Therefore, the task is possible only for \( n = 3 \) and all integers \( n \geq 5 \).
3. Consider functions $f$ defined for all real numbers and taking real numbers as values such that

$$f(x + 14) - 14 \leq f(x) \leq f(x + 20) - 20, \quad \text{for all real numbers } x.$$  

Determine all possible values of $f(8765) - f(4321)$.

**Solution 1**

Replace $x$ by $x - 14$ in the left inequality and $x$ by $x - 20$ in the right inequality to obtain

$$f(x - 20) + 20 \leq f(x) \leq f(x - 14) + 14.$$  

It follows by induction that the following inequalities hold for every positive integer $n$.

$$f(x - 20n) + 20n \leq f(x) \leq f(x - 14n) + 14n$$

$$f(x + 14n) - 14n \leq f(x) \leq f(x + 20n) - 20n$$

Therefore, we have the following chains of inequalities.

$$f(x + 2) = f(x + 20 \times 5 - 14 \times 7) \quad f(x + 2) = f(x + 14 \times 3 - 20 \times 2)$$

$$\geq f(x + 20 \times 5) - 98 \quad \leq f(x + 14 \times 3) - 40$$

$$\geq f(x) + 100 - 98 \quad \leq f(x) + 42 - 40$$

$$\geq f(x) + 2 \quad \leq f(x) + 2$$

So we have deduced that $f(x + 2) = f(x) + 2$ and it follows by induction that

$$f(x + 2n) = f(x) + 2n$$

for all real numbers $x$ and all positive integers $n$.

Substituting $x = 4321$ and $n = 2222$ into this equation yields

$$f(8765) - f(4321) = 4444.$$  

Since $f(x) = x$ satisfies the conditions of the problem, the only possible value of the expression $f(8765) - f(4321)$ is 4444.
Solution 2 (Angelo Di Pasquale)

Taking advantage of the fact that $140 = 7 \times 20 = 10 \times 14$, we have the following chains of inequalities.

\[
\begin{align*}
f(x) &\leq f(x + 20) - 20 \\
f(x) &\geq f(x + 14) - 14
\end{align*}
\]

So by the squeeze principle, equality holds throughout and we have $f(x) = f(x + 20) - 20$ and $f(x) = f(x + 14) - 14$ for all real numbers $x$.

Then since $8765 - 4405 = 4360$ is a multiple of 20 and $4405 - 4321 = 84$ is a multiple of 14, it follows that

\[
\begin{align*}
f(8765) - f(4405) &= 8765 - 4405 \\
f(4405) - f(4321) &= 4405 - 4321
\end{align*}
\]

Adding these equations yields $f(8765) - f(4321) = 8765 - 4321 = 4444$.

Since $f(x) = x$ satisfies the conditions of the problem, the only possible value of the expression $f(8765) - f(4321)$ is 4444.

Solution 3 (Joe Kupka)

We have the following chain of inequalities for all real numbers $x$.

\[
\begin{align*}
\cdots &\leq f(x + 28) - 28 \\
&\leq f(x + 14) - 14 \\
&\leq f(x) \\
&\leq f(x + 20) - 20 \\
&\leq f(x + 40) - 40 \\
&\leq \cdots
\end{align*}
\]

Using $f(x + 42) - 42 \leq f(x + 40) - 40$ yields $f(x + 2) \leq f(x) + 2$.

It follows that $f(x + 4) \leq f(x) + 4$ for all real numbers $x$.

Similarly, using $f(x + 56) - 56 \leq f(x + 60) - 60$ yields $f(x + 4) \geq f(x) + 4$ for all real numbers $x$.

It follows now that $f(x + 4) = f(x) + 4$ and by induction, $f(x + 4n) = f(x) + 4n$ for all real numbers $x$ and positive integers $n$.

Setting $x = 4321$ and $n = 1111$ gives $f(8765) - f(4321) = 4444$.

Since $f(x) = x$ satisfies the conditions of the problem, the only possible value of the expression $f(8765) - f(4321)$ is 4444.
4. Let $ABC$ be a triangle such that $\angle ACB = 90^\circ$. The point $D$ lies inside triangle $ABC$ and on the circle with centre $B$ that passes through $C$. The point $E$ lies on the side $AB$ such that $\angle DAE = \angle BDE$. The circle with centre $A$ that passes through $C$ meets the line through $D$ and $E$ at the point $F$, where $E$ lies between $D$ and $F$. Prove that $\angle AFE = \angle EBF$.

Solution 1

Since $\angle BAD = \angle BDE$ by assumption and $\angle DBA = \angle EBD$, we know that triangles $BAD$ and $BDE$ are similar.

Hence, we have the equal ratios

$$\frac{EB}{DB} = \frac{DB}{AB} \Rightarrow EB = \frac{DB^2}{AB}.$$
Therefore, we can deduce the following sequence of equalities.

\[ AE = AB - EB \]
\[ = AB - \frac{BD^2}{AB} \]
\[ = \frac{AB^2 - BD^2}{AB} \]
\[ = \frac{AB^2 - BC^2}{AB} \quad (D \text{ lies on the circle with centre } B \text{ through } C) \]
\[ = \frac{AC^2}{AB} \quad (\text{Pythagoras’ theorem in triangle } ABC) \]
\[ = \frac{AF^2}{AB} \quad (F \text{ lies on the circle with centre } A \text{ through } C) \]

This implies that \( \frac{AF}{AB} = \frac{AE}{AF} \), which combines with the fact that \( \angle BAF = \angle FAE \) to show that triangles \( AFB \) and \( AEF \) are similar.

Therefore, we conclude that

\[ \angle AFE = \angle AFB = \angle EBF. \]

**Solution 2** (Angelo Di Pasquale)

Since we know that \( \angle DAE = \angle BDE \), it follows from the alternate segment theorem that \( BD \) is tangent to the circumcircle of triangle \( ADE \) at \( D \).

So the power of a point theorem implies that

\[ BD^2 = AB \cdot BE \quad \Rightarrow \quad BC^2 = AB \cdot (AB - AE) = AB^2 - AB \cdot AE. \]

The second equation follows from the first since \( BC = BD \) and \( BE = AB - AE \).

By Pythagoras’ theorem, we know that \( AC^2 + BC^2 = AB^2 \). Combining this with the previous equation and the fact that \( AC = AF \), we deduce that

\[ AB \cdot AE = AC^2 \quad \Rightarrow \quad AB \cdot AE = AF^2. \]

By the power of a point theorem, this implies that \( AF \) is tangent to the circumcircle of triangle \( BEF \) at \( F \).

Now invoke the alternate segment theorem to conclude that \( \angle AFE = \angle EBF \).
5. Ada tells Byron that she has drawn a rectangular grid of squares and placed either the number 0 or the number 1 in each square. Next to each row, she writes the sum of the numbers in that row. Below each column, she writes the sum of the numbers in that column. After Ada erases all of the numbers in the squares, Byron realises that he can deduce each erased number from the row sums and the column sums.

Prove that there must have been a row containing only the number 0 or a column containing only the number 1.

**Solution 1**

Call a rectangular array of numbers *amazing* if each number in the array is equal to 0 or 1 and no other array has the same row sums and column sums. We are required to prove that in an amazing array, there must be a row containing only the number 0 or a column containing only the number 1.

Call the four entries in the intersection of two rows and two columns a *rectangle*. We say that a rectangle is *forbidden* if

- its top-left and bottom-right entries are 0, while its top-right and bottom-left entries are 1; or
- its top-left and bottom-right entries are 1, while its top-right and bottom-left entries are 0.

It should be clear that an amazing array cannot have forbidden rectangles, since switching 0 for 1 and vice versa in a forbidden rectangle will produce another array with the same row sums and column sums.

Suppose that there exists an amazing array in which there is no row containing only the number 0 nor a column containing only the number 1. Let row $a$ have the maximum number of entries equal to 0. Then row $a$ contains the number 1, in column $m$, say. Furthermore, this column contains the number 0, in row $b$, say. In order to avoid a forbidden rectangle, row $b$ must have a 0 in every column in which row $a$ has a 0.
Since row $b$ also has a 0 in column $m$, while row $a$ has a 1 in column $m$, this contradicts the fact that row $a$ has the maximum number of entries equal to 0.

Therefore, every amazing array has a row containing only the number 0 or a column containing only the number 1.

**Solution 2** (Ivan Guo and Alan Offer)

We will use the notion of an amazing array, defined in Solution 1.

Suppose that every row contains the number 1 and every column contains the number 0 in an amazing array. So for an entry equal to 0 in the array, we can find an entry equal to 1 in its row. Then for that entry equal to 1, we can find an entry equal to 0 in its column, and so on.

Eventually, we must return to an entry already considered. At this point, we have identified a sequence of distinct squares in the array

$$a_1, b_1, a_2, b_2, \ldots , a_n, b_n,$$

such that $a_i$ contains a 0 and $b_i$ contains a 1 for all $i$. Furthermore, $a_i$ and $b_i$ are in the same row, while $b_i$ and $a_{i+1}$ are in the same column for all $i$, where we take $a_{n+1} = a_1$.

After switching the entry in each square $a_i$ from 0 to 1 and the entry in each square $b_i$ from 1 to 0, we obtain another array with identical row and column sums. This contradicts the fact that the array is amazing.

**Solution 3** (Daniel Mathews)

We will use the notion of an amazing array, defined in Solution 1.

**Lemma.** If $A$ is an amazing array, and $B$ is obtained from $A$ by removing a row or a column, then $B$ is also amazing.

**Proof.** If $B$ is not amazing, then there are two distinct arrays $B, B'$ with the same row and column sums. We can then add in the removed row or column to obtain two distinct arrays $A, A'$ with the same row and column sums, contradicting the fact that $A$ is amazing. 

\[\square\]
We now use this lemma to prove the desired result.

Suppose that $A$ is an amazing array in which every row has a 1 and every column has a 0, in order to derive a contradiction. We claim that there either exists a row $r$ of $A$ such that each 0 in row $r$ lies in a column containing another 0, or there exists a column $c$ of $A$ such that each 1 in column $c$ lies in a row containing another 1. Remove this row or column from $A$ to obtain the array $A'$, which is amazing by the lemma above. By construction, every row still has a 1 and every column still has a 0.

To prove the claim, suppose to the contrary that every row of $A$ has a unique 0 in its column, and every column of $A$ has a unique 1 in its row. If $A$ has $m$ rows and $n$ columns, then we have $m \leq n$, since each row has a 0 unique in its column. Similarly, we have $m \geq n$, since each column has a 1 unique in its row. Hence, we have $m = n$, with precisely one 0 in each column and one 1 in each row. But this means that there are $n$ occurrences of 0 and $n$ occurrences of 1 in the entire array, leading to $2n = n^2$ and $n = 2$.

The only $2 \times 2$ arrays with precisely one 0 in each column and one 1 in each row are

$$
\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}
\quad \text{and} \quad
\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}.
$$

So $A$ must be one of these possibilities. But these two arrays have the same row and column sums, contradicting the fact that $A$ is amazing. This proves the claim, and so there exists a row $r$ or column $c$ of the desired type.

The claim allows us to successively remove rows or columns from $A$ while preserving its amazingness and so that, at each stage, every row contains a 1 and every column contains a 0. Eventually, we must arrive at a $1 \times n$ or $n \times 1$ or $2 \times 2$ array.

If we arrive at a $1 \times n$ array, then as every column contains a 0, so the entire array is 0, contradicting that every row contains a 1. Similarly, if we arrive at an $n \times 1$ array, then as every row contains a 1, so the entire array is 1, contradicting that every column contains a 0. If we arrive at a $2 \times 2$ array, the only arrays with a 1 in every row and a 0 in every column are the two arrays shown above, neither of which is amazing.
In any case, we obtain a contradiction. Hence, an amazing array must have a row containing only the number 0 or a column containing only the number 1.

**Solution 4 (Daniel Mathews)**

We will use the notion of an amazing array, defined in Solution 1. We also use the lemma from Solution 3.

Suppose for the sake of contradiction that $A$ is an amazing array in which every row contains an entry equal to 1 and every column contains an entry equal to 0. Using the lemma, we may delete any duplicate rows or columns, to obtain an amazing array $B$ in which all rows are distinct, all columns are distinct, every row contains a 1, and every column contains a 0.

If $B$ has two rows with equal sums, then it is not amazing. As the rows are distinct, we can swap them to obtain a distinct array with the same row and column sums. Similarly, if $B$ has two columns with equal sums, then it is not amazing. Thus, all row sums of $B$ are distinct, and all column sums of $B$ are distinct.

Let $B$ have $m$ rows and $n$ columns. As each row contains a 1, there are $m$ distinct row sums, each of which is an integer from 1 to $n$ inclusive — hence, $m \geq n$. As each column contains a 0, there are $n$ distinct column sums, each of which is an integer from 0 to $m - 1$ inclusive — hence, $n \leq m$. It follows that $m = n$, the row sums are precisely 1, 2, \ldots, $n$, and the column sums are precisely 0, 1, \ldots, $n - 1$.

Thus the sum of all the elements in the array is both $1 + 2 + \cdots + n$ and $0 + 1 + \cdots + n - 1$, a contradiction. Hence, an amazing array $A$ must have a row containing only the number 0 or a column containing only the number 1.
# AMOC SENIOR CONTEST STATISTICS

## DISTRIBUTION OF AWARDS/SCHOOL YEAR

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### AMOC SENIOR CONTEST RESULTS

**TOP ENTRIES:**

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The 2013 AMOC School of Excellence was held 2-11 December at Newman College, University of Melbourne. The main qualifying exams for this are the AIMO and the AMOC Senior Contest, from which 25 Australian students are selected for invitation to the school. A further student from New Zealand also attended.

The students are divided into a senior group and a junior group. There were 13 junior students, 7 of whom were attending for the first time. There were 13 students making up the senior group.

The program covered the four major areas of number theory, geometry, combinatorics and algebra. Each day the students would start at 8am with lectures or an exam and go until 12noon or 1pm. After a one hour lunch break they would have a lecture at 2pm. At 4pm they would usually have free time, followed by dinner at 6pm. Finally, each evening would round out with a problem session, topic review, or exam review from 7pm until 9pm.

Many thanks to Andrew Elvey Price, Ivan Guo, Konrad Pilch and Sampson Wong, who served as live-in staff. Also my thanks go to Adrian Agisilaou, Ross Atkins, Aaron Chong, Norman Do, Charles Li, Daniel Mathews and Sally Tsang, who assisted in lecturing and marking.

*Angelo Di Pasquale*
*Director of Training, AMOC*
## PARTICIPANTS AT THE 2014 AMOC SCHOOL OF EXCELLENCE

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<td>Harry Dinh</td>
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<td>Karen Gong</td>
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<td>Richard Gong</td>
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<td>Alan Guo</td>
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<td>Leo Li</td>
<td>Christ Church Grammar School WA</td>
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<td>Allen Lu</td>
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* Equivalent to year 10 in Australia.
1. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 0$ and, for $n \geq 2$,

$$a_n = \max_{i=1, \ldots, n-1} \left\{ i + a_i + a_{n-i} \right\}.$$ (For example, $a_2 = 1$ and $a_3 = 3$.)

Determine $a_{200}$.

2. Let $ABC$ be a triangle with $\angle BAC < 90^\circ$. Let $k$ be the circle through $A$ that is tangent to $BC$ at $C$. Let $M$ be the midpoint of $BC$, and let $AM$ intersect $k$ a second time at $D$. Finally, let $BD$ (extended) intersect $k$ a second time at $E$.

Prove that $\angle BAC = \angle CAE$.

3. Consider labelling the twenty vertices of a regular dodecahedron with twenty different integers. Each edge of the dodecahedron can then be labelled with the number $|a - b|$, where $a$ and $b$ are the labels of its endpoints. Let $e$ be the largest edge label.

What is the smallest possible value of $e$ over all such vertex labellings?

(A regular dodecahedron is a polyhedron with twelve identical regular pentagonal faces.)

4. Let $\mathbb{N}^+$ denote the set of positive integers, and let $\mathbb{R}$ denote the set of real numbers.

Find all functions $f : \mathbb{N}^+ \rightarrow \mathbb{R}$ that satisfy the following three conditions:

(i) $f(1) = 1$,

(ii) $f(n) = 0$ if $n$ contains the digit 2 in its decimal representation,

(iii) $f(mn) = f(m)f(n)$ for all positive integers $m, n$. 
5. Determine all non-integer real numbers $x$ such that

$$x + \frac{2014}{x} = \lfloor x \rfloor + \frac{2014}{\lfloor x \rfloor}.$$  

(Note that $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to the real number $x$. For example, $\lfloor 20.14 \rfloor = 20$ and $\lfloor -20.14 \rfloor = -21$.)

6. Let $S$ be the set of all numbers

$$a_0 + 10 a_1 + 10^2 a_2 + \cdots + 10^n a_n \quad (n = 0, 1, 2, \ldots)$$

where

(i) $a_i$ is an integer satisfying $0 \leq a_i \leq 9$ for $i = 0, 1, \ldots, n$ and $a_n \neq 0$, and

(ii) $a_i < \frac{a_{i-1} + a_{i+1}}{2}$ for $i = 1, 2, \ldots, n - 1$.

Determine the largest number in the set $S$.

7. Let $ABC$ be a triangle. Let $P$ and $Q$ be points on the sides $AB$ and $AC$, respectively, such that $BC$ and $PQ$ are parallel. Let $D$ be a point inside triangle $APQ$. Let $E$ and $F$ be the intersections of $PQ$ with $BD$ and $CD$, respectively. Finally, let $O_E$ and $O_F$ be the circumcentres of triangle $DEQ$ and triangle $DFP$, respectively.

Prove that $O_E O_F$ is perpendicular to $AD$.

8. An $n \times n$ square is tiled with $1 \times 1$ tiles, some of which are coloured. Sally is allowed to colour in any uncoloured tile that shares edges with at least three coloured tiles. She discovers that by repeating this process all tiles will eventually be coloured.

Show that initially there must have been more than $\frac{n^2}{3}$ coloured tiles.
1. **Solution 1** (Leo Li, year 10, Christ Church Grammar School, WA)

*Answer:* 19900.

We prove \( a_n = 0 + 1 + \cdots + (n - 1) \) by strong induction.

The base case \( n = 1 \) is given.

For the inductive part, assume \( a_n = 0 + 1 + \cdots + (n - 1) \) for \( n \leq k \).

We know
\[
a_{k+1} = \max_{i=1,\ldots,k} \{ i + a_i + a_{k+1-i} \}. \tag{1}
\]

We claim that the max in (1) occurs when \( i = k \). For this it is sufficient to prove that whenever \( i < k \), we have
\[
k + a_k + a_1 > i + a_i + a_{k+1-i}.
\]

But since \( a_1 = 0 \) and \( k > i \), it suffices to prove that
\[
a_k - a_i \geq a_{k+1-i}. \tag{2}
\]

Using the inductive assumption, we have
\[
a_k - a_i = (0 + 1 + \cdots + (k - 1)) - (0 + 1 + \cdots + (i - 1))
\]
\[
= i + (i + 1) + \cdots + (k - 1) \tag{3}
\]

and
\[
a_{k+1-i} = 0 + 1 + \cdots + (k + 1 - i - 1)
\]
\[
= 1 + 2 \cdots + (k - i). \tag{4}
\]

Observe that the right hand sides of (3) and (4) both consist of the sum of \( (n - k) \) consecutive integers and that the first term in (3) is at least as big as the first term in (4). This establishes the truth of (2), and hence also our claim.

Therefore, using our claim, we may substitute \( i = k \) in (1) to find
\[
a_{k+1} = k + 0 + 1 + \cdots + (k - 1) + 0 = 0 + 1 + \cdots + k.
\]

This completes the induction.

Finally, the well-known formula for summing the first so many consecutive positive integers may be used to deduce
\[
a_{200} = 1 + 2 + \cdots + 200 = \frac{199 \times 200}{2} = 19900,
\]
as required. \(\square\)
Solution 2 (Kevin Xian, year 10, James Ruse Agricultural High School, NSW)

We prove $a_n = \frac{n(n-1)}{2}$ by strong induction.

The base case $a_1 = 0$ is already given.

For the inductive part, assume $a_n = \frac{n(n-1)}{2}$ for $n \leq k$. For $n = k + 1$ we have

$$a_{k+1} = \max_{i=1,\ldots,k} \{i + a_i + a_{k+1-i}\}$$

$$= \max_{i=1,\ldots,k} \left\{i + \frac{i(i-1)}{2} + \frac{(k+1-i)(k-i)}{2}\right\}$$

$$= \max_{i=1,\ldots,k} \left\{i^2 - ki + \frac{k^2 + k}{2}\right\}$$

$$= \frac{k^2 + k}{2} + \max_{i=1,\ldots,k} \{i(i-k)\}.$$

However, $i(i-k) < 0$ for $1 \leq i \leq k - 1$, while $i(i-k) = 0$ for $i = k$. Therefore,

$$\max_{i=1,\ldots,k} \{i(i-k)\} = 0,$$

and so,

$$a_{k+1} = \frac{k^2 + k}{2}$$

$$= \frac{(k+1)((k+1)-1)}{2}.$$

This completes the induction. Hence in particular,

$$a_{200} = \frac{200 \times 199}{2}$$

$$= 19900,$$

as desired. \(\square\)
Solution 3 (Jerry Mao, year 8, Caulfield Grammar School, VIC)
We prove \( a_n = \frac{n(n-1)}{2} \) by strong induction.

The base case \( n = 1 \) is given.

For the inductive part, assume \( a_n = \frac{n(n-1)}{2} \) for \( n \leq k \). We know

\[
a_{k+1} = \max_{i=1,\ldots,k} \{i + a_i + a_{k+1-i}\}.
\]

We claim that the max in (1) occurs when \( i = k \).

Assume, for the sake of contradiction, that the max occurs for some integer \( i = r \) satisfying \( 1 \leq r \leq k - 1 \).

Case 1. \( r < \frac{k}{2} \).

Consider \( s = k + 1 - i \). Note that \( \frac{k}{2} < s \leq k \). Furthermore,

\[
\begin{align*}
    r + a_r + a_{k+1-r} &< s + a_{k+1-r} + a_r \\
    &= s + a_s + a_{k+1-s},
\end{align*}
\]
in contradiction to the assumption that the max in (1) occurs at \( i = r \).

Case 2. \( \frac{k}{2} \leq r \leq k - 1 \).

We claim that \( r + a_r + a_{k+1-r} < r + 1 + a_{r+1} + a_{k-(r+1)} \).

Using the inductive assumption we know

\[
\begin{align*}
    r + a_r + a_{k+1-r} &= r + \frac{r(r-1)}{2} + \frac{(k+1-r)(k-r)}{2} \\
                         &= r^2 - rk + \frac{k^2}{2} + \frac{k}{2},
\end{align*}
\]

and

\[
\begin{align*}
    r + 1 + a_{r+1} + a_{k-(r+1)} &= r + 1 + \frac{(r+1)r}{2} + \frac{(k-r)(k-r-1)}{2} \\
                          &= r^2 - rk + \frac{k^2}{2} + \frac{k}{2} + 2r + 1 - k.
\end{align*}
\]

So our claim is true because \( k \leq 2r \). Since the claim is true we have contradicted the assumption that the max in (1) occurs at \( i = r \).

Since cases 1 and 2 both end up in contradictions, the max must occur at \( i = k \). Substituting \( i = k \) in (1), using the inductive assumption and simplifying yields \( a_{k+1} = \frac{(k+1)k}{2} \). This completes the induction.

Thus we may conclude \( a_{200} = \frac{200 \times 199}{2} = 19900 \), as required. \( \square \)
Since $MC$ is tangent to circle $k$ at $C$, then by the power of a point theorem we have

$$MC^2 = MD \cdot MA.$$ 

Since $MB = MC$ it follows that

$$MB^2 = MD \cdot MA.$$ 

Hence considering the power of $M$ with respect to circle $ADB$, it follows that $MB$ is tangent to circle $ADB$ at $M$.

In the angle chase that follows, $AST$ is an abbreviation for the alternate segment theorem.

$$\angle BAC = \angle BAM + \angle MAC$$
$$\quad = \angle MBD + \angle DAC \quad (AST \ circle \ ADB)$$
$$\quad = \angle CBD + \angle BCD \quad (AST \ circle \ k)$$
$$\quad = \angle CDE \quad \text{(exterior angle } \triangle BCD)$$
$$\quad = \angle CAE, \quad \text{(AECD cyclic)}$$

which is the desired result. $\square$
Solution 2 (Hannah Sheng, year 10, Rossmoyne Senior High School, WA)

Refer to the diagram used in solution 1.

Since $BC$ is tangent to circle $k$ at $C$, we may apply the alternate segment theorem to deduce

$$\angle MAC = \angle DAC = \angle MCD. \quad (1)$$

Furthermore, since $\angle CMA = \angle CMD$, it follows by (AA) that

$$\triangle MCD \sim \triangle MAC.$$

Therefore,

$$\frac{MC}{MA} = \frac{MD}{MC}.$$

Since $MC = MB$, we have

$$\frac{MB}{MA} = \frac{MD}{MB}.$$

But now $\angle BMA = \angle BMD$, and so by (PAP) we have

$$\triangle MBD \sim \triangle MAB.$$

Hence

$$\angle BAM = \angle MBA = \angle CBA. \quad (2)$$

Finally, adding (1) and (2) together yields

$$\angle BAC = \angle MCD + \angle MBD$$

$$= \angle CDE \quad \text{(exterior angle $\triangle BCD$)}$$

$$= \angle CAE, \quad \text{($AECD$ cyclic)}$$

as required. \qed

Comment Solutions 1 and 2 are essentially the same solution. This is because the similar triangles used in solution 2 are exactly the same similar triangles that are normally used to prove the power of a point theorem that was used in solution 1.
Solution 3 (Yang Song, year 11, James Ruse Agricultural High School, NSW)

Refer to the diagram used in solution 1.

As in solution 2, we deduce

$$\triangle MBD \sim \triangle MAB.$$ 

It follows that

$$\angle ABC = \angle ABM$$

$$\quad = \angle BDM \quad (\triangle MBD \sim \triangle MAB)$$

$$\quad = \angle ADE$$

$$\quad = \angle ACE. \quad (AECD \text{ cyclic})$$

Since also $\angle ACE = \angle AEC$ from the alternate segment theorem applied to circle $k$, it follows by (AA) that

$$\triangle ABC \sim \triangle ACE.$$ 

From this we may immediately conclude that $\angle BAC = \angle CAE.$
Solution 4 (Matthew Sun, year 12, Penleigh and Essendon Grammar School, VIC)

Let $N$ be the midpoint of $CE$.

Since $M$ is the midpoint of $BC$ and $N$ is the midpoint of $EC$, it follows that $MN \parallel BE$. Using this along with the fact that $AECD$ is cyclic, we find

$$\angle CNM = \angle CEB = \angle CED = \angle CAD = \angle CAM,$$

from which it follows that $CMAN$ is cyclic.

Hence, $\angle CMA = \angle ENA$. By the alternate segment theorem we have $\angle ACM = \angle AEC = \angle AEN$. So by (AA) we have $\triangle ACM \sim \triangle AEN$.

One implication of this is

$$\angle MAC = \angle NAE. \tag{3}$$

Another implication is

$$\frac{AM}{AN} = \frac{MC}{NE} = \frac{MB}{NC},$$

since $MC = MB$ and $NE = NC$. And we have $\angle AMB = \angle ANC$ due to $CMAN$ being cyclic. Thus by (PAP) we have $\triangle AMB \sim \triangle ANC$ and so

$$\angle BAM = \angle CAN. \tag{4}$$

Adding together (3) and (4) yields the required result. \qed
**Solution 5** (Richard Gong, year 9, Sydney Grammar School, NSW)

Let $D'$ be the point so that $BDCD'$ is a parallelogram. Since the
diagonals of a parallelogram bisect each other and $M$ is the midpoint
of $BC$, it follows that $M$ is the midpoint of $DD'$. Therefore, $D'$
is collinear with $D$ and $M$.

![Diagram](image)

It follows that

$$\angle BD'A = \angle D'DC \quad (BD' \parallel DC)$$

$$= \angle AEC \quad (AECD \text{ cyclic})$$

$$= \angle BCA, \quad \text{(alternate segment theorem)}$$

and so $ABD'C$ is cyclic.

Therefore, $\angle ABC = \angle AD'C = \angle DD'C$. Part of the above angle
chase yielded $\angle D'DC = \angle BCA$. Hence $\triangle BAC \sim \triangle D'CD$ by (AA).

Therefore,

$$\angle BAC = \angle D'CD$$

$$= \angle CDE \quad (D'C \parallel BD)$$

$$= \angle CAE, \quad (AECD \text{ cyclic})$$

as required. □
Solution 6 (Allen Lu, year 11, Sydney Grammar School, NSW)

Let lines \(CD\) and \(AB\) intersect at point \(X\), and let lines \(BE\) and \(AC\) intersect at point \(Y\).

Since \(AM, BY\) and \(CX\) we may apply Ceva’s theorem to find

\[
\frac{BM}{MC} \cdot \frac{CY}{YA} \cdot \frac{AX}{XB} = 1
\]

\[
\Rightarrow \quad \frac{AX}{XB} = \frac{AY}{YC} \quad \text{(since } BM = MC).\]

It follows that \(XY \parallel BC\).

Therefore,

\[
\angle DXY = \angle DCB \quad \text{\((XY \parallel BC)\)}
\]

\[
= \angle DAC, \quad \text{(alternate segment theorem)}
\]

from which it follows that \(AXDY\) is cyclic.

Therefore,

\[
\angle BAC = \angle EDC \quad \text{\((AXYD\) cyclic)}
\]

\[
= \angle CAE, \quad \text{\((AECD\) cyclic)}
\]

as desired. \(\square\)

**Comment** Solutions 5 and 6 are quite similar underneath the surface. See if you can find the connection!
3. **Solution** (Jerry Mao, year 8, Caulfield Grammar School, VIC)

*Answer: $e = 6$."

Since the edge labels only depend on the difference of the vertex labels we can assume without loss of generality that the vertex with minimal label is labelled with the number 1. The first diagram below shows a graph of the dodecahedron along with a numbering for which $e = 6$.

In the second diagram below, the vertices marked with $A$, $B$ and $C$, require at least 1, 2 and 3 edges, respectively, to reach them from the vertex labelled 1.

If we assume $e \leq 5$, then each $A$-vertex has label at most 6, each $B$-vertex has label at most 11 and each $C$-vertex has label at most 16. Thus all of the 16 vertices including the one labelled with 1 and the 15 marked with $A$, $B$ or $C$, must be labelled with different positive integers less than or equal to 16. Therefore, they have exactly the labels 1, 2, \ldots, 16 in some order.

Consequently, the four unmarked vertices have labels at least equal to 17. But all six $C$-vertices are adjacent to one of these unmarked vertices. Since $e \leq 5$, the labels of the $C$-vertices must all be at least $17 - e \geq 12$. Thus the six $C$-vertices have labels lying in the range 12 to 16. This is clearly impossible by the pigeonhole principle because all the labels are different. Hence $e \geq 6$. 

\[\square\]
4. Solution 1 (Michael Cherryh, year 11, Gungahlin College, ACT)

Answer: \( f(n) = 0 \) for all integers \( n \geq 2 \).

Suppose \( n = 2k \) is even. Then \( f(n) = f(2)f(k) = 0 \).

Our strategy will be to prove that for each odd integer \( n > 1 \), there exists a positive integer \( k \) such that \( n^k \) begins with a 2. Then since \( f(n)^k = f(n^k) = 0 \) it will follow that \( f(n) = 0 \).

By the pigeonhole principle there exist two integers \( a > b \) such that \( n^a \) and \( n^b \) have the same first two digits, which we shall denote by \( x \) and \( y \). Let us write \( n^a \) and \( n^b \) in scientific notation. That is,

\[
    n^a = x.ya_2a_3\ldots \times 10^k \quad \text{and} \quad n^b = x.yb_2b_3\ldots \times 10^\ell,
\]

for some non-negative integers \( k \geq \ell \). Then

\[
    n^{a-b} = \frac{x.ya_2a_3\ldots}{x.yb_2b_3\ldots} \times 10^{k-\ell} = r \times 10^{k-\ell}.
\]

Note that \( r \neq 1 \) because \( n \) is odd and \( a > b \).

If we can find a positive integer \( k \) such that \( r^k \) starts with a 2 when written in scientific notation, then we will be done because \( n^{k(a-b)} \) will also start with a 2.

Our idea is as follows. If \( r > 1 \), then the sequence \( r, r^2, r^3, \ldots \) grows arbitrarily large. But since \( r \) is close to 1 we cannot jump from being less than 2 to at least 3. Thus there is a power of \( r \) that lies between 2 and 3. Similarly, if \( r < 1 \), then the sequence \( r, r^2, r^3, \ldots \) converges to 0. But since \( r \) is close to 1 we cannot jump from being at least 0.3 to less than 0.2 and so there is a power of \( r \) lying between 0.2 and 0.3.

If \( r > 1 \), then

\[
    1 < r < \frac{x.y + 0.1}{x.y} = 1 + \frac{0.1}{x.y} \leq 1 + 0.1 = 1.1
\]

Consider the least positive integer \( k \) such that \( r^k \geq 2 \). Then we have \( r^{k-1} < 2 \). Thus \( r^k < 2 \times 1.1 = 2.2 \), and so \( r^k \) starts with a 2.

If \( r < 1 \), then

\[
    1 > r > \frac{x.y}{x.y + 0.1} = 1 - \frac{0.1}{x.y + 0.1} \geq 1 - \frac{0.1}{1.1} > 0.9.
\]

Consider the least positive integer \( k \) such that \( r^k < 0.3 \). Then we have \( r^{k-1} \geq 0.3 \). Thus \( r^k > 0.3 \times 0.9 = 0.27 \), and so \( r^k \) starts with a 2.

In both cases we have shown that the required \( k \) exists. \( \square \)
Solution 2 (Mel Shu, year 12, Melbourne Grammar School, VIC)

We shall prove that \( f(n) = 0 \) for all integers \( n \geq 2 \). Since \( f \) is completely multiplicative it suffices to show that \( f(p) = 0 \) for all primes \( p \). Since \( f(p^k) = f(p)^k \) for any positive integer \( k \) it is enough to prove that any prime has a power that contains the digit 2. In fact we shall prove that any prime has a power whose first digit is 2.

Let \( p \) be any prime. We seek integers \( i > 0 \) and \( j \geq 0 \) such that

\[
2 \cdot 10^j \leq p^i < 3 \cdot 10^j
\]

\[
\Leftrightarrow j + \log 2 \leq i \log p < j + \log 3,
\]

where the logarithm is to base 10. It would be a good idea to estimate the sizes of \( \log 2 \) and \( \log 3 \). Indeed since \( 2^9 < 10^3 \) and \( 3^9 > 10^4 \) we have \( \log 2 < \frac{3}{9} \) and \( \log 3 > \frac{4}{9} \). Hence it suffices to find \( i \) and \( j \) satisfying

\[
\frac{3}{9} < i\alpha - j < \frac{4}{9}
\]

\[
\Leftrightarrow \frac{3}{9} < \{i\alpha\} < \frac{4}{9}, \tag{\ast}
\]

where \( \alpha = \log p \), and \( \{i\alpha\} \) denotes the fractional part of \( i\alpha \).

We claim that \( \alpha \), and consequently also \( \{i\alpha\} \), are irrational. Indeed, if \( \alpha = \frac{a}{b} \) for \( a, b \in \mathbb{N}^+ \), then \( p^b = 10^a \). But then \( p^b \) would be divisible by both 2 and 5. This is impossible and so \( \alpha \notin \mathbb{Q} \) as claimed.

Consider the ten irrational numbers \( \{\alpha\}, \{2\alpha\}, \ldots, \{10\alpha\} \) and the nine open intervals \( (0, \frac{1}{9}), (\frac{1}{9}, \frac{2}{9}), \ldots, (\frac{8}{9}, 1) \). By the pigeonhole principle at least one of these intervals contains at least two of the ten values. Suppose that \( \{k\alpha\} \) and \( \{\ell\alpha\} \), where \( 1 \leq \ell < k \leq 10 \), both lie within one of these nine intervals. Let \( \beta = \{k\alpha\} - \{\ell\alpha\} \). Note that \( |\beta| < \frac{1}{9} \) and that \( \beta \neq 0 \), because \( \{(k - \ell)\alpha\} \) is irrational.

Case 1: \( \beta > 0 \).

By adding \( \beta \) to itself enough times, we see that there is a positive integer \( m \) such that \( \frac{3}{9} < m\beta < \frac{4}{9} \). This is because we cannot jump the interval \( (\frac{3}{9}, \frac{4}{9}) \) just by adding \( \beta \).

Case 2: \( \beta < 0 \).

By adding \( \beta \) to itself enough times, we see that there is a positive integer \( m \) such that \( \frac{3}{9} - 1 < m\beta < \frac{4}{9} - 1 \).

In both cases 1 and 2, we can satisfy \( (\ast) \) by taking \( i = (k - \ell)m \). This concludes the proof. \( \square \)
Solution 3 (Angelo Di Pasquale, AMOC Senior Problems Committee)

As in solution 2, it is sufficient to prove that any prime \( p \) has a power that contains the digit 2.

We verify directly that \( 2 = 2^1 \) and \( 5^2 = 25 \).

Consider any other prime \( p \). It has last digit 1, 3, 7 or 9. Since \( 1^4 \equiv 3^4 \equiv 7^4 \equiv 9^4 \equiv 1 \) (mod 10), we have \( p^4 = 10x + 1 \) for some positive integer \( x \). It follows that \( p^8 = 100x^2 + 20x + 1 \). Hence the last two digits of \( p^8 \) are 01, 21, 41, 61 or 81.

One can easily check that \( 41 \rightarrow 81 \rightarrow 61 \rightarrow 21 \) (mod 100) upon repeated squaring. This shows that if \( p^8 \) does not end in 01, then \( p \) has a power that contains the digit 2 in its second last position.

**Lemma.** Let \( m > 1 \) be any integer that ends in 01. Let \( k \geq 3 \) be the integer such that the last \( k \) digits of \( m \) are \( x0\ldots01 \) where \( x \neq 0 \) and there are \( k-2 \) zeros. If \( x \neq 5 \), then there exists a power of \( m \) that contains a 2.

**Proof.** The last \( k \) digits of \( m^2 \) are \( y0\ldots01 \) where \( y \equiv 2x \) (mod 10). Since \( x \neq 0 \) or 5, it follows that \( y \) is a nonzero even digit. One may check that the last \( k \) digits go as

\[ 40\ldots01 \rightarrow 80\ldots01 \rightarrow 60\ldots01 \rightarrow 20\ldots01, \]

upon repeated squaring. This establishes that some power of \( m \) will contain the digit 2 in the \( k \)th last position. \( \square \)

The lemma solves the problem unless the last \( k \) digits of \( p^8 \) are 50\ldots01. In such a case let \( w \) be the next digit to the left of the digit 5. Thus the last \( k+1 \) digits of \( p^8 \) are \( w50\ldots01 \).

If \( w = 2 \), we are finished.

If \( w \neq 2 \), we square again and note that the last \( k+1 \) digits of \( p^{16} \) are \( z0\ldots01 \) where \( z \equiv 2w + 1 \) (mod 10). Note that \( z \neq 0 \). If \( z \neq 5 \), we may use the lemma to solve the problem. If \( z = 5 \), this corresponds to \( w = 2 \) or 7. Since \( w \neq 2 \) we have \( w = 7 \).

We are now left with the case of when the last \( k+1 \) digits of \( p^8 \) are 750\ldots01. Since \( k \geq 3 \) there is at least one zero among the last \( k+1 \) digits. Cubing 750\ldots01, we find that the last \( k+1 \) digits of \( p^{24} \) are 250\ldots01. This has a 2 in the \((k+1)\)th place from the right and therefore solves the problem. \( \square \)
Solution 4 (Alexander Gunning, year 11, Glen Waverley Secondary College, VIC)

Clearly \( f(2) = 0 \). Also \( f(5)^2 = f(25) = 0 \), which implies \( f(5) = 0 \). Thus if \( n \) is any positive integer that is divisible by 2 or 5, then \( f(n) = 0 \). From here on we assume that \( n \) is not divisible by 2 or 5.

By Fermat’s little theorem we have \( 5 \mid n^4 - 1 \). Let \( 5^m \parallel n^4 - 1 \).

Lemma 1. If \( 5^m \parallel n^4 - 1 \), then also \( 5^m \parallel n^{4a} - 1 \) for \( a \in \mathbb{N}^+ \) and \( 5 \nmid a \).

Proof. Write \( n^4 = 5^m k + 1 \) where \( 5 \nmid k \). Then

\[
n^{4a} - 1 = (5^m k + 1)^a - 1 = \sum_{i=1}^{a} \binom{a}{i} (5^m k)^i \equiv 5^m k a \pmod{5^{m+1}}.
\]

Since \( 5^m k a \) is divisible by \( 5^m \) but not \( 5^{m+1} \), the lemma is proven. \( \square \)

Lemma 2. For any positive integer \( b \) we have \( n^{2^m b} \equiv 1 \pmod{2^{m+1}} \).

Proof. An application of Euler’s theorem yields \( n^{2^m} \equiv 1 \pmod{2^{m+1}} \). The lemma follows once we raise both sides to the power of \( b \). \( \square \)

Let \( d = \max\{2^m, 4\} = \text{lcm}\{2^m, 4\} \). Then the two lemmas tell us that \( n^d \equiv 1 \pmod{2^{m+1}} \) and \( 5^m \parallel n^d - 1 \). Thus we may write

\[
n^d = 5^m 2^{m+1} c + 1 = 2c \cdot 10^m + 1,
\]

where \( 5 \nmid c \). Then for any integer \( e \geq 2 \) we have

\[
n^{de} = (2c \cdot 10^m + 1)^e
\]

\[
= 1 + \binom{e}{1} 2c \cdot 10^m + \binom{e}{2} (2c \cdot 10^m)^2 + \cdots
\]

\[
\equiv 2ce \cdot 10^m + 1 \pmod{10^{m+1}}.
\]

Since \( \gcd(c, 5) = 1 \), we may choose \( e \) so that \( ce \equiv 1 \pmod{5} \). This implies that \( 2ce \equiv 2 \pmod{10} \) and so we have

\[
n^{de} \equiv 2 \cdot 10^m + 1 \pmod{10^{m+1}}.
\]

This contains the digit 2 in the \((m+1)\)th place from the right. Thus \( f(n)^{de} = f(n^{de}) = 0 \) and so \( f(n) = 0 \). \( \square \)

---

1For a prime \( p \) and an integer \( k \), the notation \( p^m \parallel k \) means that \( p^m \mid k \) but \( p^{m+1} \nmid k \).
5. **Solution** (Kevin Xian, year 10, James Ruse Agricultural High School, NSW)

*Answer:* $x = -\frac{2014}{45}$.

The given equation may be rewritten as

$$x - \lfloor x \rfloor = 2014 \left( \frac{1}{\lfloor x \rfloor} - \frac{1}{x} \right) = \frac{2014(x - \lfloor x \rfloor)}{x \lfloor x \rfloor}.$$ 

Since $x$ is not an integer it follows that $x \neq \lfloor x \rfloor$. Hence we may divide both sides by $x - \lfloor x \rfloor$ and rearrange to find

$$x \lfloor x \rfloor = 2014.$$  \hfill (1)

*Case 1.* $\lfloor x \rfloor \geq 45$.

Then $x > 45$, and so $x \lfloor x \rfloor > 45^2 = 2025 > 2014$.

*Case 2.* $-44 \leq \lfloor x \rfloor \leq 44$.

Then $-44 < x < 45$, and so $x \lfloor x \rfloor < 44 \times 45 = 1980 < 2014$.

*Case 3.* $\lfloor x \rfloor \leq -46$.

Then $x < -45$, and so $x \lfloor x \rfloor > 45 \times 46 = 2070 > 2014$.

*Case 4.* $\lfloor x \rfloor = -45$.

Then from (1) we derive $x = -\frac{2014}{45} = -44\frac{34}{45}$.

Checking this in the original equation we have

$$\text{LHS} = x + \frac{2014}{x} = -\frac{2014}{45} + \frac{2014}{-\frac{2014}{45}} = -\frac{2014}{45} - 45$$

and

$$\text{RHS} = \lfloor x \rfloor + \frac{2014}{\lfloor x \rfloor} = -45 + \frac{2014}{-45} = \text{LHS},$$

as required. \hfill \Box
6. **Solution 1** (Seyoon Ragavan, year 10, Knox Grammar School, NSW)

*Answer:* 96433469.

It is straightforward to verify that 96433469 is in $S$. Assume that $S$ contains a number $N > 96433469$. If $N$ has nine or more digits, let the first nine such digits in order from the left be $a, b, c, d, e, f, g, h, i$. Condition (ii) implies that

\[ 2b < a + c \]
\[ \Rightarrow a > 2b - c \]
\[ \Rightarrow a \geq 2b - c + 1, \] (1)

since $a, b, c$ are all integers. Similarly, we deduce the following.

\[ b \geq 2c - d + 1 \] (2)
\[ c \geq 2d - e + 1 \] (3)
\[ d \geq 2e - f + 1 \] (4)
\[ e \geq 2f - g + 1 \] (5)
\[ f \geq 2g - h + 1 \] (6)
\[ g \geq 2h - i + 1 \] (7)

Since $a \leq 9$, we may use successive substitution to find the following.

\[ a \geq 2b - c + 1 \] \hspace{1cm} (using (1))
\[ \Rightarrow 8 \geq 2b - c \] \hspace{1cm} (1')
\[ \geq 2(2c - d + 1) - c + 1 \] \hspace{1cm} (using (2))
\[ \Rightarrow 6 \geq 3c - 2d \] \hspace{1cm} (2')
\[ \geq 3(2d - e + 1) - 2d \] \hspace{1cm} (using (3))
\[ \Rightarrow 3 \geq 4d - 3e \] \hspace{1cm} (3')
\[ \geq 4(2e - f + 1) - 3e \] \hspace{1cm} (using (4))
\[ \Rightarrow -1 \geq 5e - 4f \] \hspace{1cm} (4')
\[ \geq 5(2f - g + 1) - 4f \] \hspace{1cm} (using (5))
\[ \Rightarrow -6 \geq 6f - 5g \] \hspace{1cm} (5')
\[ \geq 6(2g - h + 1) - 5g \] \hspace{1cm} (using (6))
\[ \Rightarrow -12 \geq 7g - 6h \] \hspace{1cm} (6')
\[ \geq 7(2h - i + 1) - 6h \] \hspace{1cm} (using (7))
\[ \Rightarrow -19 \geq 8h - 7i \] \hspace{1cm} (7')

Concentrating on (4') we have $-1 \geq 5e - 4f \geq -4f$. Hence $f \geq 1$. 

*Part 2: Invitational Competitions* 47
Substituting successively into (5’), (6’) and (7’) yields

\[-6 \geq 6f - 5g \geq 6 - 5g \quad \Rightarrow \quad g \geq 3\]
\[-12 \leq 7g - 6h \geq 21 - 6h \quad \Rightarrow \quad h \geq 6\]
\[-19 \leq 8h - 7i \geq 48 - 7i \quad \Rightarrow \quad i \geq 10.\]

However, this is in contradiction with \(i\) being a digit. Therefore, \(N\) cannot contain more than eight digits.

We are left to deal with the case where \(N\) is an eight-digit number. Let the digits of \(N\) in order from the left be \(a, b, c, d, e, f, g, h\). We deduce inequalities (1)–(6) and (1’)–(6’) on the previous page in the same way as we did earlier.

Since \(h\) is a digit we know that \(h \leq 9\). If we substitute successively into (6’), (5’), (4’), (3’), (2’) and (1’), we find the following.

\[-12 \geq 7g - 6h \quad \Rightarrow \quad 7g \leq 6h - 12 \leq 6 \times 9 - 12 = 42 \quad \Rightarrow \quad g \leq 6\]
\[-6 \geq 6f - 5g \quad \Rightarrow \quad 6f \leq 5g - 6 \leq 5 \times 6 - 6 = 24 \quad \Rightarrow \quad f \leq 4\]
\[-1 \geq 5e - 4f \quad \Rightarrow \quad 5e \leq 4f - 1 \leq 4 \times 4 - 1 = 15 \quad \Rightarrow \quad e \leq 3\]
\[3 \geq 4d - 3e \quad \Rightarrow \quad 4d \leq 3e + 3 \leq 3 \times 3 + 3 = 12 \quad \Rightarrow \quad d \leq 3\]
\[6 \geq 3c - 2d \quad \Rightarrow \quad 3c \leq 2d + 6 \leq 2 \times 3 + 6 = 12 \quad \Rightarrow \quad c \leq 4\]
\[8 \geq 2b - c \quad \Rightarrow \quad 2b \leq c + 8 \leq 4 + 8 = 12 \quad \Rightarrow \quad b \leq 6\]

We also know that \(a \leq 9\) because \(a\) is a digit.

Therefore, each digit of \(N\) is less than or equal to the corresponding digit of 96433469. It follows that \(N \leq 96433469\). This contradicts that \(N > 96433469\). Hence 96433460 is the largest number in \(S\). \(\square\)
Solution 2 (Found independently by Norman Do and Ivan Guo, AMOC Senior Problems Committee)

Consider the differences $b_i = a_{i+1} - a_i$ for $i = 0, 1, 2, \ldots$. Condition (ii) is equivalent to $b_0, b_1, b_2, \ldots$ being a strictly increasing sequence.

Lemma. At most three $b_i$ are strictly positive and at most three $b_i$ are strictly negative.

Proof. Suppose there are four $b_i$ that are strictly positive. If $b_s$ is the smallest such $b_i$, then we have $b_s \geq 1, b_{s+1} \geq 2, b_{s+2} \geq 3$ and $b_{s+3} \geq 4$. Therefore,

$$a_{s+4} - a_s = b_s + b_{s+1} + b_{s+2} + b_{s+3} \geq 1 + 2 + 3 + 4 = 10.$$  

However, this is a contradiction because $a_{r+4}$ and $a_r$ are single digits and hence differ by at most 9. A similar argument shows that no four $b_i$ are strictly negative. □

It follows from the lemma that $n \leq 7$. If $n = 7$, then we must have $b_0 < b_1 < b_2 < 0, \ b_3 = 0$ and $0 < b_4 < b_5 < b_6$.

Since the $b_i$ are distinct integers this implies that $b_0 \leq -3, b_1 \leq -2$ and $b_2 \leq -1$. Hence we have the following.

$$a_0 \leq 9$$
$$a_1 = a_0 + b_0 \leq 9 - 3 = 6$$
$$a_2 = a_1 + b_1 \leq 6 - 2 = 4$$
$$a_3 = a_2 + b_2 \leq 4 - 1 = 3$$

Similarly, coming from the other end we have $b_6 \geq 3, b_5 \geq 2$ and $b_4 \geq 1$. Hence we have following.

$$a_7 \leq 9$$
$$a_6 = a_7 - b_6 \leq 9 - 3 = 6$$
$$a_5 = a_6 + b_5 \leq 6 - 2 = 4$$
$$a_4 = a_5 + b_4 \leq 4 - 1 = 3$$

It follows that no number in $S$ exceeds 96433469. Since it is readily verified that 96433469 is in $S$, it is the largest number in $S$. □
7. The common chord of two intersecting circles is always perpendicular to the line joining their centres. All the solutions we present reduce the matter to proving that $A$ lies on the common chord of circles $DEQ$ and $DFP$. That is, $A$ is on the radical axis of that pair of circles.

**Solution 1** (Mel Shu, year 12, Melbourne Grammar School, VIC)

Let the line through $A$ and $D$ intersect $PQ$ at $K$ and $BC$ at $L$.

![Diagram showing the lines and points](image)

The parallel lines imply $\triangle DKE \sim \triangle DLB$ and $\triangle DKF \sim \triangle DLC$. Therefore,

$$\frac{KE}{LB} = \frac{DK}{DL} = \frac{KF}{LC}$$

$$\Rightarrow \quad \frac{KE}{KF} = \frac{LB}{LC}. \quad (1)$$

We also have $\triangle AKP \sim \triangle ALB$ and $\triangle AKQ \sim \triangle ALC$. Therefore,

$$\frac{KP}{LB} = \frac{AK}{AL} = \frac{KQ}{LC}$$

$$\Rightarrow \quad \frac{KP}{KQ} = \frac{LB}{LC}. \quad (2)$$

Comparing (1) and (2) we find

$$\frac{KE}{KF} = \frac{KP}{KQ}$$

$$\Rightarrow \quad KE \cdot KF = KQ \cdot KP.$$

Thus $K$ has equal power with respect to circles $DEQ$ and $DFP$ and so the line $ADK$ is the radical axis of the two circles. □
Solution 2 (Alexander Gunning, year 11, Glen Waverley Secondary College, VIC)

Refer to the diagram in solution 1.

The parallel lines imply $\triangle APQ \sim \triangle ABC$. Thus

\[
\frac{AP}{AB} = \frac{AQ}{AC} \Rightarrow 1 - \frac{AP}{AB} = 1 - \frac{AQ}{AC} \Rightarrow \frac{BP}{AB} = \frac{CQ}{AC}.
\]

(3)

Applying Menelaus’ theorem to triangle $APK$ with transversal $DEB$ and then again to triangle $AQK$ with transversal $DFC$ we have

\[
\frac{KD}{DA} \cdot \frac{AB}{BP} \cdot \frac{PE}{EK} = -1 = \frac{KD}{DA} \cdot \frac{AC}{CQ} \cdot \frac{QF}{FK}.
\]

Using (3) we can cancel most of this down to derive

\[
\frac{PE}{EK} = \frac{QF}{FK} \Rightarrow 1 + \frac{PE}{EK} = 1 + \frac{QF}{FK} \Rightarrow \frac{PK}{EK} = \frac{QK}{FK} \Rightarrow EK \cdot QK = FK \cdot PK.
\]

Thus $K$ has equal power with respect to circles $DEQ$ and $DFP$ and so the line $ADK$ is the radical axis of the two circles. \(\Box\)
Solution 3 (Seyoon Ragavan, year 10, Knox Grammar School, NSW)

Let circles $DFP$ and $DEQ$ intersect for a second time at point $D'$. Let circle $DFP$ intersect line $AB$ for the second time at point $X$ and let circle $DEQ$ intersect line $AC$ for the second time at point $Y$.

Then

$$\angle DXA = DFP \quad (DXPF \text{ cyclic})$$
$$= \angle DCB \quad (PQ \parallel BC).$$

Hence $DXBC$ is cyclic and so $X$ lies on circle $DBC$. Similarly, $Y$ lies on circle $DBC$. Thus $DXBCY$ is a cyclic pentagon.

In particular, $XBCY$ is cyclic. From this we have

$$\angle AYX = \angle ABC \quad (XBCY \text{ cyclic})$$
$$= \angle APQ \quad (PQ \parallel BC).$$

Therefore, $XPQY$ is cyclic.

Applying the radical axis theorem to circles $DFPX$, $DEQY$ and $XPQY$ we have that $PQ$, $QY$ and $DD'$ are concurrent. Since $PX$ and $QY$ intersect at $A$, we conclude that $A$ lies on the line $DD'$, as required. \qed
8. **Solution 1** (Jeremy Yip, year 11, Trinity Grammar School, VIC)

Assume at the beginning that \( k \) tiles are coloured and \( n^2 - k \) tiles are uncoloured. Then the perimeter \( P \) of the coloured tiles is at most \( 4k \).
(An edge counts towards the perimeter if it is adjacent to a coloured and an uncoloured tile.)

Every tile Sally colours in reduces the perimeter by 2 or 4 according to whether the newly coloured tile is adjacent to three or four coloured tiles. Therefore, when all the tiles have been coloured, \( P \) has been reduced by at least \( 2(n^2 - k) \). Thus the final perimeter \( P_{\text{end}} \) satisfies

\[
P_{\text{end}} \leq 4k - 2(n^2 - k) = 6k - 2n^2.
\]

However, if \( k \leq \frac{n^2}{3} \), then \( P_{\text{end}} \leq 0 \). This is a contradiction because \( P_{\text{end}} = 4n \).

**Comment** A careful reading of this solution reveals the stronger result \( k \geq \frac{n^2 + 2n}{3} \).
Solution 2 (George Han, year 12, Westlake Boys’ High School, NZ)

Let there be initially $k$ coloured tiles and $n^2 - k$ uncoloured tiles. We start giving money to uncoloured tiles as follows.

(i) For each of the $k$ coloured tiles we give $1$ to each of its uncoloured neighbours.

(ii) If an uncoloured tile amasses $3$, we colour it in and give $1$ to each of its uncoloured neighbours.

If all the tiles are eventually coloured, then all of the $n^2 - k$ tiles, which were originally uncoloured, now each have at least $3$ in them. Thus $D \geq 3(n^2 - k)$ where $D$ is the total amount of dollars at the end.

All dollars in the array come from (i) and (ii). The amount of dollars coming from (i) is at most $4k$. The amount of dollars coming from (ii) is at most $n^2 - k$. Thus $D \leq 4k + n^2 - k$.

Combining the two inequalities for $D$ we deduce

$$4k + n^2 - k \geq 3(n^2 - k)$$

$$\Rightarrow \quad k \geq \frac{n^2}{3}.$$

However, since a corner tile has only two neighbours, at least one of the inequalities for $D$ is strict. Thus the final inequality is strict. □
Solution 3 (Alexander Babidge, year 12, Sydney Grammar School, NSW)

It is convenient for us to use some biology language in this solution. Coloured tiles correspond to organisms, which we shall call *squarelings*. Each unit square of the $n \times n$ array may be occupied by at most one squareling. Furthermore, each squareling has one unit of genes. If $k$ squarelings ($k = 3$ or $4$) are adjacent to a vacant square, they produce a *child* in the vacant square. The $k$ squarelings are then said to be *parents* of the child. The child is also a squareling with one unit of genes made up of $\frac{1}{k}$ of a unit of genes from each of its parents.

Each square is adjacent to at most four other squares. Hence at the beginning, before any children are produced, each squareling, which we shall call a *founder*, has the potential to be a parent to at most four children. Since each parent contributes at most one-third of its genes to any child, the total direct gene contribution from any such founder is at most $\frac{4}{3}$.

Consider any squareling that is not a founder. At least three of its neighbouring squares are occupied by its parents. Hence such a squareling has the potential to be the parent of at most one child. Thus the total direct gene contribution from this squareling is at most $\frac{1}{3}$.

Now children can also become parents to other children, but they only pass on genes from their parents. Thus the total gene count from any given founder is at most $1$ from itself, $\frac{4}{3}$ from its children, $\frac{1}{3} \cdot \frac{4}{3}$ from its children’s children, and so on. If the number of generations is $g$, then by summing the geometric series, the total gene count from any given founder is at most

$$1 + \frac{4}{3} + \frac{4}{9} + \cdots + \frac{4}{3^{g-1}} = 1 + \frac{4}{3} \left( \frac{1 - \frac{1}{3^{g-1}}}{1 - \frac{1}{3}} \right)$$

$$< 1 + \frac{4}{3} \left( \frac{1}{1 - \frac{1}{3}} \right)$$

$$= 3.$$

If the total number of founders is at most $\frac{n^2}{3}$, then the total gene contribution from these founders is less than $n^2$, which means that not every square of the array has a squareling in it. This contradiction concludes the proof. □
## AUSTRALIAN MATHEMATICAL OLYMPIAD STATISTICS

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AUSTRALIAN MATHEMATICAL OLYMPIAD RESULTS

* indicates a perfect score
** indicates New Zealand school year.

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Time allowed: 4 hours
No calculators are to be used
Each problem is worth 7 points

Problem 1. For a positive integer \( m \) denote by \( S(m) \) and \( P(m) \) the sum and product, respectively, of the digits of \( m \). Show that for each positive integer \( n \), there exist positive integers \( a_1, a_2, \ldots, a_n \) satisfying the following conditions:

\[
S(a_1) < S(a_2) < \cdots < S(a_n) \quad \text{and} \quad S(a_i) = P(a_{i+1}) \quad (i = 1, 2, \ldots, n).
\]
(We let \( a_{n+1} = a_1 \).)

Problem 2. Let \( S = \{1, 2, \ldots, 2014\} \). For each non-empty subset \( T \subseteq S \), one of its members is chosen as its representative. Find the number of ways to assign representatives to all non-empty subsets of \( S \) so that if a subset \( D \subseteq S \) is a disjoint union of non-empty subsets \( A, B, C \subseteq S \), then the representative of \( D \) is also the representative of at least one of \( A, B, C \).

Problem 3. Find all positive integers \( n \) such that for any integer \( k \) there exists an integer \( a \) for which \( a^3 + a - k \) is divisible by \( n \).

Problem 4. Let \( n \) and \( b \) be positive integers. We say \( n \) is \( b \)-discerning if there exists a set consisting of \( n \) different positive integers less than \( b \) that has no two different subsets \( U \) and \( V \) such that the sum of all elements in \( U \) equals the sum of all elements in \( V \).

(a) Prove that 8 is 100-discerning.
(b) Prove that 9 is not 100-discerning.

Problem 5. Circles \( \omega \) and \( \Omega \) meet at points \( A \) and \( B \). Let \( M \) be the midpoint of the arc \( AB \) of circle \( \omega \) (\( M \) lies inside \( \Omega \)). A chord \( MP \) of circle \( \omega \) intersects \( \Omega \) at \( Q \) (\( Q \) lies inside \( \omega \)). Let \( \ell_P \) be the tangent line to \( \omega \) at \( P \), and let \( \ell_Q \) be the tangent line to \( \Omega \) at \( Q \). Prove that the circumcircle of the triangle formed by the lines \( \ell_P, \ell_Q \) and \( AB \) is tangent to \( \Omega \).
1. **Solution** (Mel Shu, year 12, Melbourne Grammar School, VIC)

We will construct a solution where each $a_i$ consists only of the digits 1 and 2. Since $S(a_1) < S(a_2) < \cdots < S(a_n)$ and $S(a_i) = P(a_{i+1})$ for $i = 1, 2, \ldots, n$, it is sufficient to ensure that

$$P(a_i) = 2^{r+i-1} \text{ for } i = 2, 3, \ldots, n + 1,$$

where $r$ is a positive integer to be decided upon later.

To do this, let

$$a_2 = \underbrace{2\ldots2}_{r+1}1\ldots1_{b_2},$$

$$a_3 = \underbrace{2\ldots2}_{r+2}1\ldots1_{b_3},$$

$$\vdots$$

$$a_n = \underbrace{2\ldots2}_{r+n-1}1\ldots1_{b_n},$$

$$a_1 = \underbrace{2\ldots2}_{r+n}1\ldots1_{b_1},$$

where $b_1, b_2, \ldots, b_n$ are yet to be determined.

In order to satisfy $S(a_i) = P(a_{i+1})$ for $i = 1, 2, \ldots, n$, we require

$$b_1 = 2^{r+1} - 2(r + n)$$

$$b_2 = 2^{r+2} - 2(r + 1)$$

$$\vdots$$

$$b_{n-1} = 2^{r+n-1} - 2(r + n - 2)$$

$$b_n = 2^{r+n} - 2(r + n - 1).$$

Observe that $b_1$ is the smallest of the $b_i$. However, since $n$ is fixed, by choosing $r$ sufficiently large, we can ensure that $b_1$, and consequently all of the $b_i$, are non-negative integers. These values for $b_i$ yield a corresponding valid set of values for $a_i$. \qed
2. **Solution** (Mel Shu, year 12, Melbourne Grammar School, VIC)

Answer: 108 \times 2014!.

For any set \( X \), let \( g(X) \) denote the representative of \( X \). For any positive integer \( k \) let \( S_k = \{1, 2, \ldots, k\} \), and let \( f(S_k) \) denote the number of ways of assigning representatives to all the non-empty subsets of \( S_k \). We will prove by induction that \( f(S_k) = 108k! \) for each integer \( k \geq 4 \), which is sufficient to complete the problem.

For the base case, \( k = 4 \), we must have

\[
g(\{1\}) = 1, \quad g(\{2\}) = 2, \quad g(\{3\}) = 3, \quad g(\{4\}) = 4.
\]

We also have \( g(S_4) = 1, 2, 3 \) or 4. Without loss of generality

\[
g(S_4) = 1.
\]

Note that this will give us a quarter of all possible assignments.

Then since \( S_4 = \{1, 2\} \cup \{3\} \cup \{4\} \) and \( g(\{3\}) \neq 1 \) and \( g(\{4\}) \neq 1 \), we must have

\[
g(\{1, 2\}) = 1.
\]

Similarly,

\[
g(\{1, 3\}) = 1, \quad g(\{1, 4\}) = 1.
\]

Consider the four 3-element subsets \( \{1, 2, 3\} \), \( \{1, 2, 4\} \), \( \{1, 3, 4\} \) and \( \{2, 3, 4\} \). None of these can be part of a disjoint union with two other non-empty subsets to create another subset of \( S_4 \). Furthermore, the only way to write \( \{1, 2, 3\} \) as a disjoint union of three non-empty subsets is \( \{1, 2, 3\} = \{1\} \cup \{2\} \cup \{3\} \). Thus any of the three elements of \( \{1, 2, 3\} \) can be a representative of \( \{1, 2, 3\} \). A similar argument applies to the other 3-element subsets. Thus there are \( 3^4 \) possible assignments here.

Consider the three 2-element subsets \( \{2, 3\} \), \( \{2, 4\} \) and \( \{3, 4\} \). None of these is a disjoint union of three non-empty subsets. Furthermore, the only way \( \{2, 3\} \) can be part of a disjoint union is in the case \( S_4 = \{1\} \cup \{2, 3\} \cup \{4\} \). But \( g(S_4) = g(\{1\}) = 1 \) already. Thus either of the two elements of \( \{2, 3\} \) can be its representative. A similar argument applies to \( \{2, 4\} \) and \( \{3, 4\} \). Thus there are \( 2^3 \) possible assignments here.

Putting all our information together, there are 4 ways of choosing the representative of \( S_4 \), \( 3^4 \) ways of choosing the representatives of the four 3-element subsets, and \( 2^3 \) ways of choosing the representatives of the three 2-element subsets not containing \( g(S_4) \). Since all these choices
are independent, it follows that the total number of assignments is $4 \times 3^4 \times 2^3 = 108 \times 4!$, as desired.

For the inductive step, let us assume that $f(S_{n-1}) = 108(n - 1)!$ for some integer $n \geq 5$. Suppose that $g(S_n) = a$. Then for any subset $D$ of $S_n$ containing $a$ with $|D| \leq n - 2$ we can write $S_n$ as a disjoint union of three non-empty sets

$$S_n = D \cup X \cup Y.$$ 

Since $a \not\in X$ and $a \not\in Y$, it follows that $g(X) \neq a$ and $g(Y) \neq a$. Hence $g(D) = a$. Thus $g(D) = a$ for all subsets $D$ of $S_n$ except perhaps for some of size $n - 1$. In particular $g(D) = a$ for all 2-element subsets $D$ containing $a$.

Suppose, for the sake of contradiction that there exists a set $D$ of size $n - 1$ such that $g(D) = b$, where $b \neq a$. Then since $n - 1 \geq 4$, we may write $D$ as a disjoint union of non-empty sets

$$D = \{a, b\} \cup X \cup Y.$$ 

Since $g(D) = b$, then because $b \not\in X$ and $b \not\in Y$, we must also have $g(\{a, b\}) = b$. But since $\{a, b\}$ is a 2-element subset of $S_n$ containing $a$ we have from earlier that $g(\{a, b\}) = a$, a contradiction.

Thus $g(D) = a$ for all subsets $D$ of $S_n$ containing $a$. The remaining subsets are precisely those not containing $a$. These are exactly the subsets of $S_{n-1}$. Applying the inductive assumption, there are $f(S_{n-1}) = 108(n - 1)!$ ways of making assignments for these subsets, and all such assignments produce valid global assignments for the subsets of $S_n$.

Since there are $n$ possible choices for $a$ we have

$$f(S_n) = nf(S_{n-1})$$
$$= n(108(n - 1)!)$$
$$= 108n!,$$

thus completing the induction and the problem. $\square$
3. **Solution 1** (Based on the presentation by Vaishnavi Calisa, year 12, North Sydney Girls High School, NSW)

Answer: $n = 3^r$ for every non-negative integer $r$.

We seek all values of $n$ such that $a^3 + a$ covers all residues modulo $n$ as $a$ ranges over the integers. However, since $a \equiv b \pmod{n}$ implies $a^3 + a \equiv b^3 + b \pmod{n}$, we seek all $n$ such that $a^3 + a$ covers all residues modulo $n$ where $a$ ranges over the remainders modulo $n$. Since there are $n$ different remainders modulo $n$, this is equivalent to finding all $n$ such that $a^3 + a \not\equiv b^3 + b \pmod{n}$ whenever $a \not\equiv b \pmod{n}$. That is, we seek all values of $n$ such that

$$a^3 + a \equiv b^3 + b \pmod{n} \implies a \equiv b \pmod{n}.$$

We check directly that $n = 1$ is a valid solution.

If $n = 3^r$ for some positive integer $r$ and $a^3 + a \equiv b^3 + b \pmod{n}$, then

$$3^r \mid (a - b)(a^2 + ab + b^2 + 1).$$

Suppose that $3^r \nmid a - b$, then it follows that

$$a^2 + ab + b^2 + 1 \equiv 0 \pmod{3} \implies (a - b)^2 \equiv -1 \pmod{3}.$$

However, this is impossible because $-1$ is not a quadratic residue modulo $3$. Therefore, $a \equiv b \pmod{n}$, verifying that $n = 3^r$ is indeed a solution for any positive integer $r$.

We shall now prove that there are no other solutions. Suppose that there is some other solution $n$ with $n \neq 3^r$. Then $n$ has a prime factor $p \neq 3$. If $a^3 + a$ does not cover all residues modulo $p$, then $a^3 + a$ certainly does not cover all residues modulo any multiple of $p$. Therefore, it suffices to show that there exist integers $a$ and $b$ such that $a \not\equiv b \pmod{p}$ and

$$a^3 + a \equiv b^3 + b \pmod{p} \iff (a - b)(a^2 + ab + b^2 + 1) \equiv 0 \pmod{p} \iff a^2 + ab + b^2 + 1 \equiv 0 \pmod{p},$$

where the last line follows because $p \nmid a - b$.

For $p = 2$ we may take $a = 0$ and $b = 1$.

For $p \neq 2$ we may use the quadratic formula in (1) to find

$$a \equiv \frac{-b \pm \sqrt{-3b^2 - 4}}{2}.$$
We claim that there exists a value for $b$ modulo $p$ for which $-3b^2 - 4$ is a quadratic residue. That is, there is an integer solution to the equation

$$m^2 \equiv -3b^2 - 4 \pmod{p}.$$  

Note that $b_1^2 \equiv b_2^2 \pmod{p}$ if and only if $b_1 \equiv \pm b_2 \pmod{p}$. Thus the $\frac{p+1}{2}$ numbers $0^2, 1^2, \ldots, (\frac{p-1}{2})^2$ are all different modulo $p$. Hence $b^2$, and consequently also $-3b^2 - 4$, take on exactly $\frac{p+1}{2}$ different values modulo $p$. Since there are only $\frac{p-1}{2}$ non quadratic residues, it follows that $-3b^2 - 4$ is a quadratic residue for some $b$. For such a value of $b$ we use (2) to find a corresponding value for $a$.

If $a \not\equiv b \pmod{p}$, then we are done.

If $a \equiv b \pmod{p}$, then choose the other root of the quadratic so that $a \not\equiv b \pmod{p}$. This is impossible if only if

$$-3b^2 - 4 \equiv 0 \pmod{p}.$$  

But in such a case using $a \equiv b \pmod{p}$ in (1) also yields

$$3b^2 + 1 \equiv 0 \pmod{p}.$$  

Adding the last two congruences implies $p = 3$, a contradiction. □
**Solution 2** (Seyoon Ragavan, year 10, Knox Grammar School, NSW)

We prove by induction on $r$ that $n = 3^r$ is a solution for any non-negative integer $r$.

The case $r = 0$ is trivial while the case $r = 1$ is easily verified by taking $a = 0, 1, 2$.

Assume for some $r \geq 1$ that $a^3 + a$ attains every value modulo $3^r$. Consider the congruence

$$a^3 + a \equiv k \pmod{3^{r+1}},$$

for some integer $k$. By the inductive assumption there is an integer $a$ such that

$$a^3 + a \equiv k \pmod{3^r}.$$ 

Therefore, $a^3 + a = k + 3^rj$ for some integer $j$. But then we have

$$(a - 3^rj)^3 + (a - 3^rj) \equiv a^3 + a - 3^rj \pmod{3^{r+1}} \equiv k \pmod{3^{r+1}}.$$ 

This completes the inductive step and the proof that $n = 3^r$ is always a solution.

We shall now prove that there are no further solutions. As in solution 1, we reduce the problem to proving that for every prime $p > 3$, there are integers $a$ and $b$ such that $a \not\equiv b \pmod{p}$ and

$$a \equiv \frac{-b \pm \sqrt{-3b^2 - 4}}{2} \pmod{p}. \quad (2)$$

Let $\left(\frac{u}{p}\right)$, as usual, denote the Legendre symbol.\(^1\)

**Case 1:** $\left(\frac{-1}{p}\right) = 1$.

Choose $b$ to satisfy $b^2 \equiv -1 \pmod{p}$. If we also take $a \equiv 0 \pmod{p}$, we have $a \not\equiv b \pmod{p}$ and

$$a^2 + ab + b^2 + 1 \equiv 0 \pmod{p},$$

as desired.

**Case 2:** $\left(\frac{-1}{p}\right) = -1$.

Note that the equation $-3b^2 - 4 \equiv 0 \pmod{p}$ has a solution if and only if $-\frac{4}{3}$ is a quadratic residue modulo $p$. We compute that

$$\left(\frac{-\frac{4}{3}}{p}\right) = \left(\frac{-1}{p}\right) \left(\frac{2}{3}\right)^2 \left(\frac{3}{p}\right) = -\left(\frac{3}{p}\right).$$

\(^1\)If you do not know what the Legendre symbol is, please refer to the supplementary information following the last solution to this question for more information.
Case 2a: \( \left( \frac{3}{p} \right) = -1 \).
From the preceding remarks we know there exists an integer \( b \) such that \(-3b^2 - 4 \equiv 0 \pmod{p} \). For such a value of \( b \) we use (2) to find \( a \equiv -\frac{b}{2} \pmod{p} \).

If \( a \not\equiv b \pmod{p} \), then we are done.

If \( a \equiv b \pmod{p} \), then \(-\frac{b}{2} \equiv b \pmod{p} \) from which it easily follows that \( b \equiv 0 \pmod{p} \). However, since \(-3b^2 - 4 \equiv 0 \pmod{p} \), it would then follow that \( p \mid 4 \), a contradiction.

Case 2b: \( \left( \frac{3}{p} \right) = 1 \).
First we shall show for this case that there exist integers \( b \) and \( m \) satisfying \( m^2 \equiv -3b^2 - 4 \pmod{p} \). That is, \( \frac{m^2 + 4}{-3} \equiv b^2 \pmod{p} \).

If this were not true, then for all integers \( m \) we would have

\[
\left( \frac{m^2 + 4}{-3} \right) = -1
\]

\[
\Rightarrow \left( \frac{m^2 + 4}{p} \right) \left( \frac{-1}{p} \right) \left( \frac{3}{p} \right) = -1
\]

\[
\Rightarrow \left( \frac{m^2 + 4}{p} \right) = 1.
\]

In other words, if \( q \) is a quadratic residue, then so is \( q + 4 \). Then by induction \( q + 4x \) is a quadratic residue for any non-negative integer \( x \). However, since \( \gcd(4, p) = 1 \), we know that \( q + 4x \) covers all residues modulo \( p \). Thus all residues modulo \( p \) are quadratic residues, which is impossible.

Thus we can find integers \( b \) and \( m \) satisfying \( m^2 \equiv -3b^2 - 4 \pmod{p} \).
Then using (2) we have \( a \equiv -\frac{b + m}{2} \pmod{p} \). Since \( \left( \frac{3}{p} \right) = 1 \), from the introductory remarks to case 2 we know that \( m \not\equiv 0 \pmod{p} \). Hence we can choose the sign of \( \pm \) so that \( a \not\equiv b \pmod{p} \), as desired. \( \square \)
Solution 3 (Alex Gunning, year 11, Glen Waverley Secondary College, VIC)

We prove that $n = 3^r$ for any non-negative integer $r$ as in solution 2.

To show that there are no further solutions, then as in solution 1, we reduce the matter to proving that for all primes $p \geq 5$, there exist integers $a$ and $b$ such that $a \not\equiv b \pmod{p}$ and

$$a^2 + ab + b^2 + 1 \equiv 0 \pmod{p}.$$

Consider the change of variables $(c, d) \equiv \left(\frac{a+b}{2}, \frac{a-b}{2}\right) \pmod{p}$. Note that $(a, b) \equiv (c + d, c - d) \pmod{p}$ and that

$$a^2 + ab + b^2 + 1 \equiv 3c^2 + d^2 + 1 \pmod{p}.$$

With this change of variables it suffices to show that there exist integers $c$ and $d$ such that $d \not\equiv 0 \pmod{p}$ and

$$3c^2 + d^2 \equiv -1 \pmod{p}. \tag{3}$$

If $-1$ is a quadratic residue, then we may simply take $c \equiv 0 \pmod{p}$ and $d$ satisfying $d^2 \equiv -1 \pmod{p}$.

If $-3$ is a quadratic residue, let $e \equiv \sqrt{-3}c \pmod{p}$. (This change of variable is invertible.) Equation (3) now becomes

$$d^2 - e^2 \equiv -1 \pmod{p} \quad \tag{4}$$

$$\Leftrightarrow (d - e)(d + e) \equiv -1 \pmod{p}.$$

Trying $d + e \equiv -2 \pmod{p}$ and $d - e \equiv \frac{1}{2} \pmod{p}$, then solving for $d$ and $e$ yields the valid values $(d, e) \equiv \left(-\frac{3}{4}, -\frac{5}{4}\right)$ satisfying (4).

If neither $-1$ nor $-3$ are quadratic residues modulo $p$, then since the product of two quadratic nonresidues is a quadratic residue, we deduce that $3$ is a quadratic residue. Let $e \equiv \sqrt{3}c$. Equation (3) becomes

$$d^2 + e^2 \equiv -1 \pmod{p}. \tag{5}$$

Let $m$ be the smallest positive integer such that $m + 1$ is a quadratic nonresidue. Then $m$ is a quadratic residue. Since the quotient of two quadratic nonresidues is a quadratic residue, there is an integer $f$ such that

$$f^2 \equiv \frac{-1}{m + 1} \pmod{p}.$$

Then taking $(d, e) \equiv (f, \sqrt{mf})$ yields valid values satisfying (5). \qed
Solution 4 (Andrew Elvey Price, Deputy Leader of the 2014 Australian IMO team)

We prove that \( n = 3^r \) is a solution for every non-negative integer \( r \) as in solution 1.

In order to prove there are no other solutions we reduce the matter to primes \( p \geq 5 \) as in solution 1.

**Lemma.** If \( n \) is a positive integer, then

\[
\sum_{i=1}^{p-1} i^n \equiv \begin{cases} 
0 & \text{if } p - 1 \nmid n \\
-1 & \text{if } p - 1 \mid n.
\end{cases} \quad \pmod{p}
\]

**Proof.** Let \( g \) be a generator modulo \( p \).

If \( p - 1 \nmid n \), then \( g^n - 1 \not\equiv 0 \pmod{p} \). It follows that

\[
\sum_{i=1}^{p-1} i^n \equiv \sum_{i=0}^{p-2} g^i \equiv g^{(p-1)n} - 1 \pmod{p}
\]

\[
\equiv g^{(p-1)n} - 1 \pmod{p}
\]

\[
\equiv 0 \pmod{p}.
\]

If \( p - 1 \mid n \), then

\[
\sum_{i=1}^{p-1} i^n \equiv \sum_{i=1}^{p-1} 1 \pmod{p}.
\]

\[
\equiv -1 \pmod{p}. \quad \square
\]

Returning to the problem, suppose for the sake of contradiction that \( a^3 + a \) covers all residues modulo \( p \). Then \( a^3 + a \) covers the nonzero residues modulo \( p \) as \( a \) ranges over the nonzero residues modulo \( p \).

Note that \( p \) is either of the form \( p = 3k + 1 \) or \( p = 3k - 1 \) for some positive integer \( k \). In either case, let us consider the quantity

\[
S = \sum_{a=1}^{p-1} (a^3 + a)^k.
\]

Since \( a^3 + a \) runs over all nonzero residues modulo \( p \) and \( 0 < k < p - 1 \), the lemma tells us that

\[
S \equiv 0 \pmod{p}. \quad (*)
\]
Using the binomial theorem to expand \((a^3 + a)^k\), we have

\[
S = \sum_{a=1}^{p-1} \sum_{i=0}^{k} \binom{k}{i} a^{3i} a^{k-i} = \sum_{i=0}^{k} \sum_{a=1}^{p-1} \binom{k}{i} a^{2i+k} = \sum_{i=0}^{k} \left( \binom{k}{i} \sum_{a=1}^{p-1} a^{2i+k} \right)
\]

**Case 1: \(p = 3k + 1\).**

For \(i < k\) we have \(0 < 2i + k < p - 1\) and so \(p - 1 \nmid 2i + k\). Therefore, using the lemma we have

\[
\sum_{a=1}^{p-1} a^{2i+k} \equiv 0 \pmod{p}.
\]

However, for \(i = k\), using the lemma we have

\[
\binom{k}{k} \sum_{a=1}^{p-1} a^{2i+k} = \sum_{a=1}^{p-1} a^{p-1} \equiv -1 \pmod{p}.
\]

Thus \(S \equiv -1 \pmod{p}\), which contradicts (*)..

**Case 2: \(p = 3k - 1\).**

For \(i \leq k - 2\) we have \(0 < 2i + k < p - 1\) and for \(i = k\) we have \(p - 1 < 2i + k < 2(p - 1)\). In either case we have \(p - 1 \nmid 2i + k\). Therefore, using the lemma we again have

\[
\sum_{a=1}^{p-1} a^{2i+k} \equiv 0 \pmod{p}.
\]

However, for \(i = k - 1\), using the lemma we have

\[
\binom{k}{k-1} \sum_{a=1}^{p-1} a^{2i+k} = k \sum_{a=1}^{p-1} a^{p-1} \equiv -k \pmod{p}.
\]

Thus \(S \equiv -k \neq 0 \pmod{p}\), which contradicts (*). \(\square\)
Supplementary Information About the Legendre Symbol

Recall that for any prime \( p \), an integer \( a \) is called a \textit{quadratic residue} modulo \( p \) if there is an integer \( x \) satisfying
\[
x^2 \equiv a \pmod{p},
\]
and a \textit{quadratic nonresidue} otherwise.

For a prime \( p \) and any integer \( a \) the Legendre symbol \( \left( \frac{a}{p} \right) \) is defined as follows.

\[
\left( \frac{a}{p} \right) = \begin{cases} 
0 & \text{if } p \mid a \\
1 & \text{if } p \nmid a \text{ and } a \text{ is a quadratic residue modulo } p \\
-1 & \text{if } p \nmid a \text{ and } a \text{ is a quadratic nonresidue modulo } p
\end{cases}
\]

We list some properties of the Legendre symbol below. In what follows \( a \) and \( b \) are any integers and \( p \) and \( q \) are any odd primes with \( p \neq q \).

\[
\left( \frac{a}{p} \right) = \left( \frac{b}{p} \right) \quad \text{if } a \equiv b \pmod{p} \quad (1)
\]

\[
\left( \frac{a}{p} \right) \left( \frac{b}{p} \right) = \left( \frac{ab}{p} \right) \quad (2)
\]

\[
\left( \frac{-1}{p} \right) = \begin{cases} 
1 & \text{if } p \equiv 1 \pmod{4} \\
-1 & \text{if } p \equiv -1 \pmod{4}
\end{cases} \quad (3)
\]

\[
\left( \frac{2}{p} \right) = \begin{cases} 
1 & \text{if } p \equiv \pm 1 \pmod{8} \\
-1 & \text{if } p \equiv \pm 3 \pmod{8}
\end{cases} \quad (4)
\]

\[
\left( \frac{p}{q} \right) \left( \frac{q}{p} \right) = (-1)^{(p-1)(q-1)/4} \quad (5)
\]

Property (5) is known as the law of \textit{quadratic reciprocity}. Note also that property (5) tells us that \( \left( \frac{2}{q} \right) = \left( \frac{2}{p} \right) \) unless both \( p \) and \( q \) are of the form \( 4k + 3 \) in which case \( \left( \frac{2}{q} \right) = - \left( \frac{2}{p} \right) \).

Here is an example of how one might compute the value of a Legendre symbol. To find out if 21 is a quadratic residue modulo 37, we compute
\[
\left( \frac{21}{37} \right) = \left( \frac{3}{37} \right) \left( \frac{7}{37} \right) = \left( \frac{37}{3} \right) \left( \frac{37}{7} \right) = \left( \frac{1}{3} \right) \left( \frac{2}{7} \right) = 1.
\]

Therefore, 21 is a quadratic residue modulo 37.
4. (a) **Solution** (Yong See Foo, year 10 Nossal High School, VIC)

Consider the set $S = \{3, 6, 12, 24, 48, 96, 97, 98\}$. The subsets of $S$ can be partitioned into the following categories.

- **Group 1.** Subsets that do not contain 97 or 98.
- **Group 2.** Subsets that contain 97 but not 98.
- **Group 3.** Subsets that contain 98 but not 97.
- **Group 4.** Subsets that contain both 97 and 98.

Group 1 generates subsets that have sums 0, 3, 6, 9, ..., 189. That is, all the multiples of 3 from 0 to 189 exactly once.

Group 2 generates subsets that have sums 97, 100, 103, ..., 286. These are all congruent to 1 modulo 3.

Group 3 generates subsets that have sums 98, 101, 104, ..., 287. These are all congruent to 2 modulo 3.

Group 4 generates subsets that have sums 195, 198, 201, ..., 384. These are all congruent to 0 modulo 3 and greater than 189.

Note that the four groups cover all possible sums of subsets of $S$ with no sum appearing twice. Since $S$ has 8 elements, we have shown that 8 is 100-discerning.

(b) **Solution** (Problem Selection Committee)

Suppose, for the sake of contradiction, that 9 is 100-discerning. Then there is a set $S = \{s_1, \ldots, s_9\}$ with $0 < s_1 < \cdots < s_9 < 100$ that has no two different subsets with equal sums of elements.

Let $X$ be the collection of all subsets of $S$ having at least 3 and at most 6 elements. Note that $X$ consists of exactly

$$\binom{9}{3} + \binom{9}{4} + \binom{9}{5} + \binom{9}{6} = 84 + 126 + 126 + 84 = 420$$

subsets of $S$. The greatest possible sum of elements of a member of $X$ is $s_4 + s_5 + s_6 + s_7 + s_8 + s_9$, while the smallest possible sum is $s_1 + s_2 + s_3$. Since all sums of elements of members of $X$ are different, we must have

$$s_4 + s_5 + s_6 + s_7 + s_8 + s_9 - s_1 - s_2 - s_3 \geq 419. \quad (1)$$

Let $Y$ be the collection of all subsets of $S$ having exactly 2 or 3 or 4 elements greater than $s_3$. Observe that $\{s_4, s_5, s_6, s_7, s_8, s_9\}$ has $\binom{6}{2}$ 2-element subsets, $\binom{6}{3}$ 3-element subsets and $\binom{6}{4}$ 4-element subsets, while $\{s_1, s_2, s_3\}$ has exactly 8 subsets. Hence the number of members of $Y$ is equal to

$$8 \left( \binom{6}{2} + \binom{6}{3} + \binom{6}{4} \right) = 8(15 + 20 + 15) = 400.$$
Notice that the greatest possible sum of elements of a member of $Y$ is $s_1 + s_2 + s_3 + s_6 + s_7 + s_8 + s_9$, while the smallest possible sum is $s_4 + s_5$. Since all the sums of elements of members of $Y$ are assumed different, we must have

$$s_1 + s_2 + s_3 + s_6 + s_7 + s_8 + s_9 - s_4 - s_5 \geq 399. \quad (2)$$

Adding equations (1) and (2) we find

$$2(s_6 + s_7 + s_8 + s_9) \geq 818.$$

However, this is impossible because $s_6, s_7, s_8, s_9 < 100$. □

Comment: This problem completely resolves the following question.

What is the largest positive integer that is 100-discerning?

Using powers of 2 it is easy to show that 7 is 100-discerning. A standard argument shows that 10 is not 100-discerning. Indeed, there are $2^{10} = 1024$ subsets. The sums of the elements in those subsets are non-negative integers lying within the range from 0 to 945. So by the pigeonhole principle two subsets have equal sums.

Getting an example to show 8 is 100-discerning is already a little tricky. To show that 9 is not 100-discerning is quite hard. The solution given here uses the idea of playing off different estimates against each other. But even finding a combination of such estimates that solves the problem is highly nontrivial.
5. **Solution 1** (Angelo Di Pasquale, Leader of the 2014 Australian IMO team)

Let $X = AB \cap \ell_P$, $Y = AB \cap \ell_Q$, $Z = \ell_P \cap \ell_Q$ and $F = MP \cap AB$.

Let $R$ be the second point where line $PQ$ intersects $\Omega$ and let $D$ be the second point where line $XR$ intersects $\Omega$. Let $\ell_M$ be the tangent to $\omega$ at $M$ and let $\ell$ be the tangent to $\Omega$ at $D$. It suffices to prove that

1. $D$ lies on circle $XYZ$; and
2. $\ell$ is tangent to circle $XYDZ$ at $D$.

Since $M$ is the midpoint of minor arc $AB$ of circle $\omega$, we have $\ell_M \parallel AB$. Furthermore, since $XP$ and $\ell_M$ are tangents to $\omega$, it follows that $\angle FPX = \angle MPX = \angle(\ell_M, MP) = \angle XFP$.\(^2\)

Using power of a point in circles $\omega$ and $\Omega$ we deduce

$$XF^2 = XP^2 = XA \cdot XB = XD \cdot XR.$$\

\(^2\)The notation $\angle(\ell_M, MP)$ stands for the angle between the two lines $\ell_M$ and $MP$. More specifically, it is the *directed angle* between the two lines. That is, the angle through which one may rotate $\ell_M$ anticlockwise so that it becomes parallel to $MP$. 

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Therefore, circle $DFR$ is tangent to $XF$ at $F$ and circle $DPR$ is tangent to $XP$ at $P$. Hence using the alternate segment theorem, we have

$$\angle DFX = \angle DRF = \angle DRP = \angle DPX.$$ 

Also since $ZQ$ is tangent to $\Omega$ at $Q$ we have $\angle DQZ = \angle DRQ$. Thus the four angles as marked in the diagram are equal.

From this it follows that $QFDY$ and $PQDZ$ are cyclic.

Therefore, $\angle XZD = \angle PZD = \angle FQD = \angle FYD$ and so $XYDZ$ is cyclic, establishing (1).

Finally, using the alternate segment theorem we have

$$\angle(XR, \ell) = \angle RQD = \angle PZD = \angle XZD$$

and so $\ell$ is tangent to circle $XYDZ$ at $D$, establishing (2). □

\[\text{3Alternatively, it is a known theorem that given any four lines each three of which determine a triangle, then the four circumcircles of the four triangles are concurrent at a point called the Miquel point of the four lines. (The proof is done by angle chasing.) In our setting, the four lines are } PZ, PF, XF \text{ and } QZ \text{ and the Miquel point for these four lines is } D.\]
Solution 2 (Alex Gunning, year 11, Glen Waverley Secondary College, VIC)

Define the points $X, Y, Z, F$ and $R$ as in solution 1. We prove that $XP = XF$ in the same way as solution 1. Define the point $U$ as the second point of intersection of circle $PFX$ with line $XR$.

It follows that $\angle BFR = \angle XFP = \angle XPF = \angle FUR$. Hence by the alternative segment theorem we have that circle $UFR$ is tangent to line $XB$ at point $F$. It then follows from power of a point in circle $UFR$ that

$$XU \cdot XR = XF^2 = XP^2 = XA \cdot XB,$$

where the last equality follows from power of a point in $\omega$. Hence points $U, R, A$ and $B$ are concyclic. Therefore, $U$ lies on $\Omega$.

Furthermore, from the alternate segment theorem applied to circles $QUR$ and $FUR$ we have

$$\angle QUF = \angle QUR - \angle FUR = \angle PQY - \angle QFY = \angle QYF.$$

Hence $QFUY$ is cyclic. Since circles $QFY$ and $PFX$ both pass through point $U$, it follows that $U$ is the Miquel point associated

---

4 We will shortly see that the point $U$ is in fact the same as the point $D$ of solution 1.

5 See footnote 3 in solution 1.
with the four lines $PZ$, $PF$, $XF$ and $QZ$. Hence circles $XYZ$ and $PQZ$ also pass through $U$, making $XYUZ$ and $PQUZ$ cyclic too.

Let $S$ denote the second intersection point of line $YU$ with circle $Ω$. Since $∠UXZ = ∠UYZ$ from $XYUZ$ being cyclic, it follows that $∠PXU = ∠QYU$. Also $∠XPU = ∠ZPU = ∠ZQU$ from $PQUZ$ being cyclic. Thus triangles $PXU$ and $QYU$ are similar (AA). We earlier found that

$$XP^2 = XU \cdot XR. \quad (1)$$

From power of a point in circle $Ω$ we also have

$$YQ^2 = YU \cdot YS. \quad (2)$$

Therefore,

$$\frac{XP^2}{XU^2} = \frac{YQ^2}{YU^2} \quad (△PXU \sim △QYU)$$

$$\Rightarrow \frac{XU \cdot XR}{XU^2} = \frac{YU \cdot YS}{YU^2} \quad \text{(from (1) and (2))}$$

$$\Rightarrow \frac{XR}{XU} = \frac{YS}{YU}$$

$$\Rightarrow \frac{XU + UR}{XU} = \frac{YU + US}{YU}$$

$$\Rightarrow \frac{UR}{XU} = \frac{US}{YU}$$

Since $XUR$ and $YUS$ are straight lines it follows that triangles $XYU$ and $RSU$ are similar (PAP). Furthermore, these two triangles are related by a dilation with centre $U$. Therefore, this dilation sends circle $XYU$ to circle $RSU$. Since the centre of dilation lies on circle $XYU$ it follows that the two circles are tangent at $U$, as required. □
Solution 3 (Seyoon Ragavan, year 10, Knox Grammar School, NSW)
Define the points $X, Y, Z, F, D$ and $R$ as in solution 1.

Note that $\angle PAF = \angle PAB = \angle PMB$ from circle $\omega$. Since $M$ is the midpoint of arc $AB$ we have $FPA = \angle MPA = \angle BPM$. It follows that triangles $PAF$ and $PMB$ are similar (AA). Therefore,

$$\angle PFX = \angle PFA$$
$$= \angle PBM \quad (\triangle PAF \sim \triangle PMB)$$
$$= \angle XPM \quad \text{(alternate segment theorem)}$$
$$= \angle XPF.$$

If follows that $XP = XF$.

From this we prove that $PQDZ$ and $QFDY$ are cyclic as in solution 1. Applying Miquel’s theorem\textsuperscript{6} to the four lines $PZ, PF, XF$ and $QZ$ we deduce that circles $XYZ, QFY, PXF$ and $PQZ$ are concurrent. This point of concurrency must be $D$ because circles $PQZ$ and $QFY$ intersect for the second time at $D$. In particular, $XYDZ$ is cyclic.

\textsuperscript{6}See footnote 3 in solution 1.
**Lemma.** Let two circles $\Gamma_1$ and $\Gamma_2$ intersect at points $K$ and $L$. Suppose that $U$ and $V$ are points on $\Gamma_1$ and $\Gamma_2$, respectively, such that $U$, $V$ and $K$ are collinear. Then the spiral symmetry with centre $L$ sending $\Gamma_1$ to $\Gamma_2$ also sends $U$ to $V$.

![Diagram showing circles and points](image)

**Proof.** Let $U' \in \Gamma_1$ and $V' \in \Gamma_2$ be any two points such that $U'$, $V'$ and $K$ are collinear. We have $\angle LUU' = \angle LKU = \angle LVV'$ and $\angle LU'U = \angle LKV = \angle LV'V$. Thus triangles $LUU'$ and $LVV'$ are similar (AA), establishing the lemma.  

Let us apply the lemma to circles $XYDZ$ and $PQDZ$ with line $PX$. Then the spiral symmetry, $f$ say, centred at $D$ that sends circle $XYDZ$ to circle $PQDZ$ satisfies $f(X) = P$. Let us also apply the lemma to circles $PQDZ$ and $\Omega$. Then the spiral symmetry, $g$ say, centred at $D$ that sends circle $PQDZ$ to $\Omega$ satisfies $g(P) = R$.

Consider the composition $h(x) = g(f(x))$. Then $h$ is a spiral symmetry centred at $D$ that sends circle $XYDZ$ to $\Omega$. Furthermore, we have $h(X) = g(f(X)) = g(P) = R$. But $X$, $D$ and $R = h(X)$ are collinear and so $h$ is in fact a dilation. Since the centre of dilation lies on circle $XYDZ$, and $\Omega$ is the image of circle $XYDZ$ under $h$, it follows that the two circles are tangent, as desired.

---

7The diagram shows the scenario of $K$ between $U$ and $V$. The lemma is still true if $K$ is not between $U$ and $V$ and the same proof works if directed angles are used.
## 26TH ASIAN PACIFIC MATHEMATICS OLYMPIAD
### RESULTS

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**RESULTS OF THE AUSTRALIAN STUDENTS**

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The 2014 AMOC Selection School was held 13-22 April at Dunmore Lang College, Macquarie University, Sydney. The main qualifying exams to attend are the AMO and the APMO from which 25 students are selected for the school.

The routine is similar to that for the December School of Excellence, however, there is the added interest of the actual selection of the Australian IMO team. This year the IMO would be held in Cape Town, South Africa.

The students are divided into a junior group and a senior group. This year there were 10 juniors, 7 of whom were attending for the first time. The remaining 15 students were seniors, 5 of whom were attending for the first time as seniors. It is from the seniors that the team of six for the IMO plus one reserve team member is selected. The AMO, the APMO and the final three senior exams at the school are the official selection exams.

Most students who attend for the first time as a junior find the experience very challenging compared to what they have previously experienced. The progress from junior to senior is similarly challenging. We generally try to arrange it so that as many new juniors as possible have an opportunity to attend, enabling us to see who best can make the transition to the senior group.

My thanks go to Andrew Elvey Price, Ivan Guo, Konrad Pilch and Sampson Wong, who assisted me as live-in staff members. Also to Paul Cheung, Mike Clapper, Nancy Fu, Declan Gorey, David Hunt, Victor Khou, Andrew Kwok, Jason Kwong, Tim Large, Vickie Lee, Marshall Ma, Peter McNamara, John Papantoniou, Christopher Ryba, David Vasak, Gareth White, Rachel Wong, and Jonathan Zheng, all of whom came in to give lectures or to help with the marking of exams.

*Angelo Di Pasquale*

*Director of Training, AMOC*
## PARTICIPANTS AT THE IMO TEAM SELECTION SCHOOL

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<td>Nicholas Pizzino</td>
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<tr>
<td>Austin Zhang</td>
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## AUSTRALIAN IMO TEAM

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<thead>
<tr>
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<tr>
<td>Alexander Gunning</td>
<td>Glen Waverley Secondary College VIC</td>
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<td>Seyoon Ragavan</td>
<td>Knox Grammar School NSW</td>
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<td>Mel Shu</td>
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<td>Yang Song</td>
<td>James Ruse Agricultural High School NSW</td>
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<tr>
<td>Andy Tran</td>
<td>Baulkham Hills High School NSW</td>
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The Australian IMO Team at the Announcement Ceremony at Parliament House on 16 June 2014 with Adam Spencer, University of Sydney’s Ambassador for Maths and Science. From left, Alexander Gunning, Seyoon Ragavan, Mel Shu, Adam Spencer, Yang Song, Praveen Wijerathna and Damon Zhong.
The IMO team preparation school, or pre-IMO school, for the Australian team was held in Cape Town between the 1st and 7th of July this year. Our journey there was fairly straightforward with only brief stops in Perth and Johannesburg.

As usual we were joined by the UK team for the duration of the school. Our two teams, as well as the teams from Ireland and Trinidad and Tobago were accommodated at a lovely hotel called the Little Scotia, situated at the bottom of the slopes of Table Mountain just a few hundred metres down the road from the IMO location. Apparently someone decided that we could all use a bit of exercise as our exam rooms were near the top of the slopes of Table Mountain. As a result, our competition with the British started earlier than usual each morning, with their team easily beating us up the hill before our daily exam.

The fifth and final exam was designated the Mathematical Ashes, an annual competition between Australia and the UK for the urn containing the ashes of the burnt scripts of the 2008 UK IMO team. Unfortunately we lost 50-59 this year despite a characteristic strong effort by Alex Gunning, the only student on either team to achieve a perfect score.

The training was only one of the many purposes of the pre-IMO school, some others being to recover from jet lag and acclimatise to local conditions. The South African winter is really just a rainier version of an Australian winter, so we were fairly well acclimatised already. That being said, the students from north of the Murray may have found it a little colder than desired.

The teams got plenty of free time too, which they generally spent playing various card games in their rooms. We also went on a number of excursions, including one memorable trip up Table Mountain, where we had a beautiful view of the surrounding cloud.

The pre-IMO school was the final part of the Australian team’s training program, leading up to the IMO. The school was extremely successful for us, as shown by our outstanding performance in the actual IMO\(^1\). Many thanks go to the organisers of the IMO for arranging our accommodation and exam venues, and the UK team and their leaders for a thoroughly enjoyable pre-IMO school.

\[\text{Andrew Elvey Price} \]
\[\text{IMO Deputy Leader} \]

---

\(^1\) Most importantly, we got our revenge against the UK team, finishing 9 places ahead of them in the country results.
Exam

1. Let $D$ be the point on side $BC$ of triangle $ABC$ such that $AD$ bisects $\angle BAC$. Let $E$ and $F$ be the incentres of triangles $ADC$ and $ADB$, respectively. Let $\omega$ be the circumcircle of triangle $DEF$. Let $Q$ be the point of intersection of the lines $BE$ and $CF$. Let $H$, $J$, $K$ and $M$ be the second points of intersection of $\omega$ with the lines $CE$, $CF$, $BE$ and $BF$, respectively. Circles $HQJ$ and $KQM$ intersect at the two points $Q$ and $T$. Prove that $T$ lies on line $AD$.

2. A crazy physicist discovered a new kind of particle which he called an *imon*, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be *entangled*, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

   (i) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.

   (ii) At any moment, he may double the whole family of imons in his lab by creating a copy $I'$ of each imon $I$. During this procedure, the two copies $I'$ and $J'$ become entangled if and only if the original imons $I$ and $J$ are entangled, and each copy $I'$ becomes entangled with its original imon $I$; no other entanglements occur or disappear at this moment.

   Prove that the physicist may apply a sequence of such operations resulting in a family of imons, no two of which are entangled.

3. Fix an integer $k \geq 2$. Two players, called Ana and Banana, play the following *game of numbers*: Initially, some integer $n \geq k$ gets written on the blackboard. Then they take moves in turn, with Ana beginning. A player making a move erases the number $m$ just written on the blackboard and replaces it by some number $m'$ with $k \leq m' < m$ that is coprime to $m$. The first player who cannot move anymore loses.

   An integer $n \geq k$ is called *good* if Banana has a winning strategy when the initial number is $n$, and *bad* otherwise.

   Consider two integers $n, n' \geq k$ with the property that each prime number $p \leq k$ divides $n$ if and only if it divides $n'$. Prove that either both $n$ and $n'$ are good or both are bad.
# THE MATHEMATICS ASHES RESULTS

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The 55th International Mathematical Olympiad (IMO) was held from 3 to 13 July in Cape Town, South Africa. This was the first time the IMO has been held on the African continent. Four countries, all from the African continent, participated in the IMO for the first time this year. They were Burkina Faso, Gambia, Ghana and Tanzania.

A total of 560 high school students from 101 countries participated. Of these, 56 were girls.

Each participating country may send a team of up to six students, a Team Leader and a Deputy Team Leader. At the IMO the Team Leaders, as an international collective, form what is called the Jury. This Jury was most ably chaired by Sizwe Mabizela.

The first major task facing the Jury is to set the two competition papers. During this period the Leaders and their observers are trusted to keep all information about the contest problems completely confidential. The local Problem Selection Committee had already shortlisted 30 problems, from 141 problem proposals submitted by 43 of the participating countries from around the world. During the Jury meetings four of the shortlisted problems had to be discarded from consideration due to being too similar to material already in the public domain. Eventually, the Jury finalised the exam questions and then made translations into all the more than 50 languages required by the contestants.

The six questions are described as follows.

1. An easy sequence problem based on a discrete version of the intermediate value theorem. It was proposed by Austria.

2. A medium minimax combinatorics problem proposed by Croatia. It is about placing a set of mutually non-attacking rooks on a chessboard so as to minimise the size of the largest square that contains no rook.

3. A difficult classical geometry problem with awkward angle conditions proposed by Iran.

4. A very easy classical geometry problem proposed by Georgia.

5. A medium bin-packing number theory problem proposed by Luxembourg.

6. A difficult inequality from combinatorial geometry. Originally proposed by Austria, the concluding inequality was strengthened by the Problem Selection Committee.

The asymptotic behaviour of the inequality in question 6 was further strengthened by Po-Shen Loh, the Leader from the USA. However, at the time it was not known how far the asymptotic behaviour could be pushed. So as an experiment, the wording of question 6 was phrased in such a way that it encouraged an open-ended investigation. Consequently, with the time restriction, question 6 is a genuine contemporary mathematical research problem although; full marks would be given to an asymptotic bound that matched the bound found by the Problem Selection Committee. As it turned out, no contestant was able to reach the asymptotic bound found by Po-Shen.

These six questions were posed in two exam papers held on Tuesday 8 July and Wednesday 9 July. Each paper had three problems. The contestants worked individually. They were allowed 4½ hours per paper to write their attempted proofs. Each problem was scored out of a maximum of seven points.
For many years now there has been an opening ceremony prior to the first day of competition. Following the formal speeches there was the parade of the teams. Starting with Romania, the contestants came out in the order of the year their country first participated at the IMO. At the conclusion of the opening ceremony the 2014 IMO was declared open.

After the exams the Leaders and their Deputies spent about two days assessing the work of the students from their own countries, guided by marking schemes, which had been discussed earlier. A local team of markers called Coordinators also assessed the papers. They too were guided by the marking schemes but are allowed some flexibility if, for example, a Leader brings something to their attention in a contestant’s exam script that is not covered by the marking scheme. The Team Leader and Coordinators have to agree on scores for each student of the Leader’s country in order to finalise scores. Any disagreements that cannot be resolved in this way are ultimately referred to the Jury.

Questions 1 and 4 turned out to be very easy as expected. Both averaged in excess of 5 points. As expected, question 6 was very difficult averaging just 0.3 points. Only 15 students scored full marks on this question.

The medal cuts were set at 29 for Gold, 22 for Silver and 16 for Bronze. Consequently, there were 295 (52.7%) medals awarded, a little more generous than usual. The medal distributions were 49 (8.8%) Gold, 113 (20.2%) Silver and 133 (23.8%) Bronze. These awards were presented at the closing ceremony. Of those who did not get a medal, a further 151 contestants received an Honourable Mention for solving at least one question perfectly. Three students achieved the most excellent feat of a perfect score of 42. They were Alex Gunning of Australia, Jiayang Gao of China and Po-Sheng Wu of Taiwan. They were given a standing ovation during the presentation of medals at the closing ceremony.

Congratulations to the Australian IMO team on their extraordinary performance this year. They finished equal 11th in the unofficial country rankings with a clean sweep of medals. Only on one occasion has our ranking been higher. Their solid performance gained one Gold medal, three Silver medals and two Bronze medals.

Of particular note is the extraordinary Gold performance of Alex Gunning, year 11, Glen Waverley Secondary College, Victoria. He is the first Australian ever to write a perfect paper at the IMO. He now has one Bronze and two Gold medals at the IMO putting him in equal top position for the most decorated Australian at the IMO.

The three Silver medallists were Mel Shu, year 12, Melbourne Grammar School, Victoria; Praveen Wijerathna, year 12, James Ruse Agricultural High School, NSW; and Damon Zhong, year 12, Shore School, NSW. These results are outstanding given that none of these three students had ever had the experience of competing in an IMO before.

The two Bronze Medalists were Seyoon Ragavan, year 10, Knox Grammar School, NSW and Yang Song, year 11, James Ruse Agricultural High School, NSW.

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1 The total number of medals must be approved by the Jury and should not normally exceed half the total number of contestants. The numbers of Gold, Silver and Bronze medals must be approximately in the ratio 1:2:3.

2 Fifty-five contestants managed the feat of what might be called a “double Honourable Mention”: They did not get a medal, but solved two questions perfectly.

3 This was in 1997 when the Australian team ranked 9th out of 82 countries.

4 Peter McNamara also won these medals at the IMO in the years 1999-2001.
With three members of this year's team eligible for selection for the 2015 IMO team, things are looking very promising.

Congratulations also to Australia’s Deputy Leader, Andrew Elvey Price. A well decorated exolympian himself\(^5\), being the Deputy Leader was a new experience for him. He handled his role superbly.

The 2014 IMO was organized by the South African Mathematics Foundation in partnership with the University of Cape Town.

Venues for future IMOs have been secured up to 2019 as follows.

2015 Chiang Mai, Thailand
2016 Hong Kong
2017 Brazil
2018 Romania
2019 United Kingdom

Much of the statistical information found in this report can also be found at the official website of the IMO.

[www.imo-official.org](http://www.imo-official.org)

*Angelo Di Pasquale*

*IMO Team Leader, Australia*

---

\(^5\) Andrew won a Silver medal at the 2008 IMO and a Gold medal at the 2009 IMO.
Problem 1. Let $a_0 < a_1 < a_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geq 1$ such that
\[ a_n < \frac{a_0 + a_1 + \cdots + a_n}{n} \leq a_{n+1}. \]

Problem 2. Let $n \geq 2$ be an integer. Consider an $n \times n$ chessboard consisting of $n^2$ unit squares. A configuration of $n$ rooks on this board is *peaceful* if every row and every column contains exactly one rook. Find the greatest positive integer $k$ such that, for each peaceful configuration of $n$ rooks, there is a $k \times k$ square which does not contain a rook on any of its $k^2$ unit squares.

Problem 3. Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90^\circ$. Point $H$ is the foot of the perpendicular from $A$ to $BD$. Points $S$ and $T$ lie on sides $AB$ and $AD$, respectively, such that $H$ lies inside triangle $SCT$ and
\[ \angle CHS - \angle CSB = 90^\circ, \quad \angle THC - \angle DTC = 90^\circ. \]
Prove that line $BD$ is tangent to the circumcircle of triangle $TSH$. 

Language: English Time: 4 hours and 30 minutes Each problem is worth 7 points.
Problem 4. Points $P$ and $Q$ lie on side $BC$ of acute-angled triangle $ABC$ so that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Points $M$ and $N$ lie on lines $AP$ and $AQ$, respectively, such that $P$ is the midpoint of $AM$, and $Q$ is the midpoint of $AN$. Prove that lines $BM$ and $CN$ intersect on the circumcircle of triangle $ABC$.

Problem 5. For each positive integer $n$, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

Problem 6. A set of lines in the plane is in general position if no two are parallel and no three pass through the same point. A set of lines in general position cuts the plane into regions, some of which have finite area; we call these its finite regions. Prove that for all sufficiently large $n$, in any set of $n$ lines in general position it is possible to colour at least $\sqrt{n}$ of the lines blue in such a way that none of its finite regions has a completely blue boundary.

Note: Results with $\sqrt{n}$ replaced by $c \sqrt{n}$ will be awarded points depending on the value of the constant $c$.

Language: English 
Time: 4 hours and 30 minutes 
Each problem is worth 7 points
1. **Solution 1** (Found independently by Alex Gunning, year 11, Glen Waverley Secondary College, VIC; and Mel Shu, year 12, Melbourne Grammar School, VIC. Alex was a Gold medallist and Mel was a Silver medallist with the 2014 Australian IMO team.)

For a given sequence \( a_0 < a_1 < a_2 < \cdots \) satisfying the conditions of the problem, let us define the function \( f: \mathbb{N}^+ \to \mathbb{Z} \) given by

\[
f(n) = na_n - a_0 - a_1 - \cdots - a_n.
\]

Note that

\[
f(n) < 0 \iff a_n < \frac{a_0 + a_1 + \cdots + a_n}{n}
\]

and

\[
f(n + 1) \geq 0 \iff a_{n+1} \geq \frac{a_0 + a_1 + \cdots + a_n}{n}.
\]

Note that \( f(1) = -a_0 < 0 \), and

\[
f(n + 1) - f(n) = na_{n+1} - a_n - (n - 1)a_n
= n(a_{n+1} - a_n)
> 0.
\]

Thus \( f(1), f(2), f(3), \ldots \) is a strictly increasing sequence of integers, whose first term, \( f(1) \), is negative.

Therefore, when we consider the graph of this function, and join the dots, it will cross the \( x \)-axis exactly once. That is, there is a unique positive integer \( n \) such that \( f(n) < 0 \) and \( f(n + 1) \geq 0 \). This is the unique value of \( n \) that satisfies the problem. \( \square \)
Solution 2 (Damon Zhong, year 12, Shore School, NSW. Damon was a Silver medallist with the 2014 Australian IMO team.)

For each integer $n \geq 0$, let $L_n$ and $R_n$ denote the following inequalities.

\[
L_n : \quad na_n < a_0 + a_1 + \cdots + a_n \\
R_n : \quad a_0 + a_1 + \cdots + a_n \leq na_{n+1}
\]

Lemma 1. If $R_n$ is true, then $L_{n+1}$ is false.

Proof. This follows from adding $a_{n+1}$ to both sides of $R_n$. \hfill \Box

Lemma 2. If $R_n$ is true, then $R_{n+1}$ is true.

Proof. If $R_n$ is true, then adding $a_{n+1}$ to both sides of $R_n$ yields

\[
a_0 + a_1 + \cdots + a_n + a_{n+1} \leq (n+1)a_{n+1} < (n+1)a_{n+2},
\]

since $a_{n+1} < a_{n+2}$. Hence $R_{n+1}$ is true. \hfill \Box

Lemma 3. $L_n$ and $R_n$ are both true for at most one value of $n$.

Proof. Suppose $n$ is the smallest value for which $L_n$ and $R_n$ are both true. Then lemma 2 tells us that $R_m$ is true for all $m \geq n$, while lemma 1 tells us that $L_m$ is false for all $m \geq n+1$. Thus $L_m$ and $R_m$ cannot both be true for any $m \geq n+1$. \hfill \Box

Lemma 4. If $R_n$ is false, then $L_{n+1}$ is true.

Proof. This follows from adding $a_{n+1}$ to both sides of $R_n$. \hfill \Box

Lemma 5. There is a value of $n$ for which $L_n$ and $R_n$ are both true.

Proof. Assume there is no such $n$. Clearly, $L_0$ is true. But whenever $L_n$ is true, then by assumption, $R_n$ is false, and then by lemma 4, $L_{n+1}$ is true. Hence inductively, $L_n$ is true and $R_n$ is false for all $n$.

Since the sequence is increasing, there exists a value of $n$ ($n \geq 1$) for which $a_{n+1} > a_0 + a_1$. For this value of $n$ we have

\[
na_{n+1} = a_{n+1} + (n-1)a_{n+1} > a_0 + a_1 + (n-1)a_{n+1} > a_0 + a_1 + a_2 + a_3 + \cdots + a_n,
\]

and so $R_n$ is true. This contradiction establishes the lemma. \hfill \Box

Combining lemmas 3 and 5 completes the solution. \hfill \Box
Solution 3 (Angelo Di Pasquale, Leader of the 2014 Australian IMO team)

Let \( d_0 = a_0 \) and for each positive integer \( n \) let \( d_n = a_n - a_{n-1} \). Thus \( d_n > 0 \) for each non-negative integer \( n \). In addition, for each positive integer \( n \), we have

\[
a_n = \sum_{i=0}^{n} d_i.
\]

We also have

\[
\sum_{j=0}^{n} a_j = \sum_{j=0}^{n} \sum_{i=0}^{j} d_i = \sum_{i=0}^{n} (n + 1 - i)d_i,
\]

because \( d_i \) occurs in the sum for \( a_j \) whenever \( i \leq j \leq n \).

The given inequality is equivalent to

\[
na_n < a_0 + a_1 + \cdots + a_n \leq a_{n+1}
\]

\[
\iff n \sum_{i=0}^{n} d_i < \sum_{i=0}^{n} (n + 1 - i)d_i \leq n \sum_{i=0}^{n+1} d_i
\]

\[
\iff \sum_{i=1}^{n} (i-1)d_i < d_0 \leq \sum_{i=1}^{n+1} (i-1)d_i
\]

\[
\iff S_n < d_0 \leq S_{n+1},
\]

where \( S_n = \sum_{i=1}^{n} (i-1)d_i \).

Note that \( S_1 = 0 \) and that \( S_{n+1} = S_n + nd_{n+1} > S_n \) for \( n \geq 1 \). Hence \( S_1 < S_2 < \cdots \) is a strictly increasing sequence of integers that starts from 0. Since \( d_0 > 0 \), there must exist a unique \( n \) for which \( S_n < d_0 \leq S_{n+1} \), as desired. \( \square \)

Comment The fact that the \( a_i \) are all integers is crucial. Without it the conclusion of the problem is not necessarily true. For example, if \( a_n = 2 - \frac{1}{2^n} \), then

\[
\frac{a_0 + a_1 + \cdots + a_n}{n} > a_{n+1},
\]

for all positive integers \( n \).
2. We shall use the following terminology in the solutions that follow.

We assign coordinates \((i, j)\) to a unit square of a chessboard if it lies in the \(i\)th column from the left and the \(j\)th row from the bottom.

A configuration of rooks shall be called \(k\)-open if there is a \(k \times k\) sub-square that does not contain any rooks. Conversely, it shall be called \(k\)-closed if each \(k \times k\) sub-square contains at least one rook. The problem asks us to find the greatest positive integer \(k\) such that every peaceful configuration of rooks is \(k\)-open.

**Solution 1** (Praveen Wijerathna, year 12, James Ruse Agricultural High School, NSW. Praveen was a Silver medallist with the 2014 Australian IMO team.)

The answer is \(\lfloor \sqrt{n-1} \rfloor\).

Our proof splits naturally into two parts. We prove that \(k \geq \lfloor \sqrt{n-1} \rfloor\) and that \(k \leq \lfloor \sqrt{n-1} \rfloor\).

**Part 1.** Proof that \(k \geq \lfloor \sqrt{n-1} \rfloor\).

Let \(m = \lfloor \sqrt{n-1} \rfloor\). Note that \(m^2 < n\). We claim that any peaceful configuration is \(m\)-open. Assume, for the sake of contradiction, that we have found an \(m\)-closed peaceful configuration.

Consider the bottom-left \(m^2 \times m^2\) sub-board of such a configuration. It can be partitioned into \(m^2\) squares each of side length \(m\). Consequently, each of these squares contains a rook. However, the \(m^2\) rooks must be a peaceful configuration for the bottom-left \(m^2 \times m^2\) sub-board. It follows that the remaining \(n - m^2\) rooks must all be in the top-right \((n - m^2) \times (n - m^2)\) sub-board. Furthermore, they must be in peaceful configuration for that sub-board.

Similarly, the bottom-right \(m^2 \times m^2\) sub-board contains \(m^2\) rooks in peaceful configuration and the top-left \((n - m^2) \times (n - m^2)\) sub-board contains \(n - m^2\) rooks in peaceful configuration.

Consequently, the left column of the bottom-left \(m^2 \times m^2\) sub-board and the left column of the top-left \((n - m^2) \times (n - m^2)\) sub-board both contain a rook. Hence there are two rooks in the left column of the \(n \times n\) chessboard. This contradicts that the rooks are in peaceful configuration for the \(n \times n\) chessboard. Therefore, \(k \geq \lfloor \sqrt{n-1} \rfloor\). □

**Part 2.** Proof that \(k \leq \lfloor \sqrt{n-1} \rfloor\).
Consider the following two ways of dividing up the board into squares. The first way is simply the ordinary way, which results in $n^2$ unit squares. The second way is to divide the board into $m^2$ squares of size $m \times m$ by $m$ columns of width $m$ and $m$ rows of height $m$. Each such square shall be called a box. A box is given the coordinates $(i, j)$ if those are its coordinates according to the wider columns and rows. For example, in the diagram below, the shaded box has coordinates $(3, 2)$. The rook in it has coordinates $(2, 3)$ with respect to the box but coordinates $(8, 6)$ with respect to the whole chessboard.

We now describe a peaceful placement of rooks. Suppose the box $S$ has coordinates $(i, j)$. Now view $S$ as an $m \times m$ chessboard in its own right and place a rook in the unit square of $S$ whose coordinates are $(j, i)$ within $S$. This is illustrated in the diagram below.

Lemma 1. The configuration of rooks just described is peaceful.

Proof. Suppose that two different rooks, $R_1$ and $R_2$, are in the same column of the $n \times n$ array. If $S_1$ and $S_2$ are the boxes containing $R_1$ and $R_2$, respectively, then $S_1$ and $S_2$ are in the same column. This implies that $R_1$ is in the same row of $S_1$ as $R_2$ is in $S_2$. Hence the coordinates of $R_1$ within $S_1$ are the same as the coordinates of $R_2$ within $S_2$. This implies that the coordinates of $S_1$ are the same as the coordinates of $S_2$. Therefore, $S_1 = S_2$ and hence also $R_1 = R_2$, a contradiction. Similarly, no two rooks are in the same row.

Lemma 2. The described configuration of rooks has the property that every $m \times m$ square (not just the boxes) contains a rook.

Proof. Imagine a movable $m \times m$ square frame. When the frame is around the bottom-left box it certainly contains a rook. It suffices to show that whenever the frame is moved up one unit or right one unit then, provided it remains within the $n \times n$ square, it will still bound a square that contains a rook.
Suppose we are at a situation where the old position of the frame bounds an \( m \times m \) square, \( F \) say, that contains a rook \( R \). Consider the \( m \times m \) square \( F' \) that lies one unit to the right of \( F \). Let \( S \) be the box containing \( R \). Let the coordinates of \( S \) be \((i, j)\). Thus the coordinates of \( R \) within \( S \) are \((j, i)\). Three cases arise.

**Case 1.** \( R \) is not in the left column of \( F \).

Then \( R \) is also in \( F' \).

**Case 2.** \( R \) is in the leftmost column of \( F \) but not in the top row of \( F \).

Then since \( F' \) cannot overhang the rightmost edge of the \( n \times n \) board we must have \( i < n \). Let \( S' \) be the box to the right of \( S \). It has coordinates \((i+1, j)\). Hence the rook \( R' \) in \( S' \) has coordinates \((j, i+1)\) within \( S' \). Thus \( R' \) lies \( m \) units to the right and 1 unit above \( R \) in the \( n \times n \) board. Therefore, \( F' \) contains \( R' \).

**Case 3.** \( R \) is in the top-left square of \( F \).
For the same reason as in case 2 we must have $i < n$. Furthermore, since $F$ cannot overhang the bottom edge of the $n \times n$ board we must have $j > 1$. Let $S''$ be the box diagonally down and to the right of $S$. It has coordinates $(i + 1, j - 1)$. Hence the rook $R''$ in $S''$ has coordinates $(j - 1, i + 1)$ within $S''$. Thus $R''$ lies $m - 1$ units to the right and $m - 1$ units below $R$ in the $n \times n$ board. So $R''$ is in the bottom-right square of $F$. This implies that $F'$ contains $R''$.

A similar proof applies for the case when $F'$ is one unit above $F$. □

**Step 2.** There is a peaceful $m$-closed configuration whenever $n$ satisfies $(m - 1)^2 < n \leq m^2$.

From step 1 we can find a peaceful $m$-closed configuration for an $m^2 \times m^2$ chessboard. Consider the bottom-left $n \times n$ sub-board of this configuration. It is certainly $m$-closed because it inherits this property from the configuration of the $m^2 \times m^2$ chessboard. It also inherits the property of containing at most one rook in each row and at most one rook in each column. Thus our $n \times n$ sub-board contains at most $n$ rooks. If it contains less than $n$ rooks, we add them one at a time as follows. Find an empty row and an empty column and place a rook in the square at their intersection. Repeat until the $n \times n$ sub-board contains $n$ rooks. In this way, the final configuration will indeed be peaceful and $m$-closed.

This shows that we cannot have $k \geq m$. Since $m = \lceil \sqrt{n - 1} \rceil + 1$, it follows that $k \leq \lceil \sqrt{n - 1} \rceil$. □

Since we have established that $k \geq \lfloor \sqrt{n - 1} \rfloor$ and $k \leq \lfloor \sqrt{n - 1} \rfloor$, it follows that $k = \lfloor \sqrt{n - 1} \rfloor$. This completes the proof. □
Solution 2 (Alex Gunning, year 11, Glen Waverley Secondary College, VIC. Alex was a Gold medallist with the 2014 Australian IMO team.)

Let us call an ordered pair \((n, m)\) nice if there exists a peaceful configuration of rooks on an \(n \times n\) chessboard that is \(m\)-closed. A pair \((n, m)\) is called nasty if it is not nice. That is, every peaceful configuration of rooks on an \(n \times n\) chessboard is \(m\)-open. We are asked to find the greatest positive integer \(k\) such that \((n, k)\) is nasty.

Lemma 1. If \((n, m)\) is nice, then \((n - 1, m)\) is nice also.

Proof. Since \((n, m)\) is nice, there is a peaceful configuration of rooks on the \(n \times n\) chessboard that is \(m\)-closed.

If there is a rook in the top-left corner, we may cut off the top row and the left column of our configuration to arrive at a peaceful configuration of rooks on an \((n - 1) \times (n - 1)\) chessboard that is \(m\)-closed.

Otherwise, the top-left square is vacant and there is a rook \(R_1\) in the top row and a rook \(R_2\) in the left column. If we cut off the top row and the left column and then place a rook at the intersection of the row containing \(R_2\) and the column containing \(R_1\), we again arrive at a peaceful configuration of rooks on an \((n - 1) \times (n - 1)\) chessboard.

\(\square\)

Lemma 2. If \((n, m)\) is nasty, then \((n + 1, m)\) is also nasty.

Proof. This is an immediate corollary of lemma 1. \(\square\)

Lemma 3. \((m^2 + 1, m)\) is nasty.

Proof. Suppose to the contrary that there exists a peaceful configuration of rooks on an \((m^2 + 1) \times (m^2 + 1)\) chessboard that is \(m\)-closed.

The bottom-left \(m^2 \times m^2\) sub-board can be tiled neatly with \(m^2\) squares of size \(m \times m\). Since our configuration is \(m\)-closed, each of these \(m \times m\) squares must contain a rook and so there are \(m^2\) rooks in our sub-board. But each of the \(m^2\) rows contains at most one rook as do each of the \(m^2\) columns. Hence each row of the \(m^2 \times m^2\) sub-board contains exactly one rook as does each column of the sub-board. This implies that the original \((m^2 + 1) \times (m^2 + 1)\) chessboard must have a rook in its top-right square.

A similar argument shows that the original chessboard must have a rook in its top-left square. This is a contradiction because now we have two rooks in the top row. \(\square\)

Lemma 4. \((m^2, m)\) is nice.

Proof. We use the same construction as used in step 1 of part 2 of solution 1. The proof that such a configuration is peaceful is also
the same as that found in solution 1. However, we prove that the configuration is \( m \)-closed in a slightly different way.

Consider any \( m \times m \) square, \( S \) say, of the \( m^2 \times m^2 \) chessboard.

**Case 1.** \( S \) is one of the boxes.

Then it contains a rook by construction.

**Case 2.** \( S \) straddles exactly two boxes.

Let the boxes be \( S_1 \) and \( S_2 \). Suppose that \( S_1 \) lies to the left of \( S_2 \). Let \( R_i \) be the rook in \( S_i \) for \( i = 1, 2 \). If \( S \) contains no rook, then it must lie entirely to the right of \( R_1 \) and entirely to the left of \( R_2 \). But \( R_1 \) and \( R_2 \) have only \( m - 1 \) columns between them. Hence \( S \) has width at most \( m - 1 \), a contradiction.

We can deal with the case of when \( S_1 \) lies below \( S_2 \) similarly. Thus in both situations \( S \) contains a rook.

**Case 3.** \( S \) straddles exactly four boxes.

Let the boxes be \( S_1, S_2, S_3 \) and \( S_4 \), where \( S_1 \) is to the left of \( S_2 \) and below \( S_3 \). Let \( R_i \) be the rook in \( S_i \) for \( i = 1, 2, 3, 4 \). Consider the full row in the \( m^2 \times m^2 \) chessboard that contains \( R_1 \). If \( S \) straddles this row, then it must lie entirely to the right of \( R_1 \) and entirely to the left of \( R_2 \). As in case 2 this implies that \( S \) has width at most \( m - 1 \), a contradiction. Hence \( S \) does not straddle the row containing \( R_1 \).
Similarly, $S$ does not straddle the column containing $R_1$, nor the row containing $R_4$, nor the column containing $R_4$. This eliminates everything except for the unique $m \times m$ square that contains $R_2$ and $R_3$, in which case we still have that $S$ contains a rook.

Since cases 1–3 cover all scenarios for $S$, our lemma is proven. □

We may now complete our proof as follows. Let $k$ be the unique integer satisfying

$$k^2 < n \leq (k + 1)^2.$$

From lemma 3, $(k^2 + 1, k)$ is nasty and so from lemma 2, $(n, k)$ is also nasty. From lemma 4, $((k + 1)^2, k + 1)$ is nice and so from lemma 1, $(n, k + 1)$ is also nice. Since $k = \lfloor \sqrt{n - 1} \rfloor$, we have found our required answer. □
Solution 3 (Mel Shu, year 12, Melbourne Grammar School, VIC. Mel was a Silver medallist with the 2014 Australian IMO team.)

Only Mel’s proof that $k \geq \lfloor \sqrt{n-1} \rfloor$ is shown here.

Let $m = \lfloor \sqrt{n-1} \rfloor$. We claim that we can always find an $m \times m$ square that does not contain a rook. Note that $n \geq m^2 + 1$.

Any peaceful configuration certainly contains a rook in its left column. Take any $m$ consecutive rows that contain this rook. Their union, $U$ say, contains exactly $m$ rooks.

Now remove the leftmost column of $U$, thus removing at least one rook in the process, and consider the next $m^2$ columns of $U$. These form an $m^2 \times m$ rectangle that contains at most $m - 1$ rooks. This rectangle can be partitioned into $m$ squares of size $m \times m$. Therefore, by the pigeonhole principle at least one of these squares does not contain a rook. The diagram below illustrates the situation for the case $n = 10$.
3. In the solutions that follow, the concept of isogonal conjugate is used.

Given \( \angle BAC \), two lines \( \ell_1 \) and \( \ell_2 \) passing passing through \( A \) are said to be *isogonal conjugates* with respect to \( \angle BAC \) if they are symmetric with respect to the angle bisector of \( \angle BAC \).

Given a triangle \( ABC \) and a point \( P \) in the plane, it can be proven that the isogonal conjugates of the lines \( AP \), \( BP \) and \( CP \) with respect to the angles at \( A \), \( B \) and \( C \), respectively, are always concurrent at a point \( P' = f(P) \). The point \( f(P) \) is called the isogonal conjugate of \( P \) with respect to triangle \( ABC \). Furthermore, \( f(f(P)) = P \) so that the operation of taking the isogonal conjugate with respect to a fixed triangle is in fact an involution.¹

It is probably easiest to recognise isogonal conjugates when \( P \) lies inside the triangle. The diagram below shows a situation where \( P \) lies outside the triangle.

![Diagram of isogonal conjugates](image)

The two solutions we present involve diagrams where a point and its isogonal conjugate lie outside the triangle.

¹There are a couple of exceptions that should be noted here. First, these statements are only valid if \( P \) does not lie on any of the sidelines of the triangle. Second, if \( P \) lies on the circumcircle of the triangle, then the isogonal conjugates of \( AP \), \( BP \) and \( CP \) are parallel. In this instance \( f(P) \) is a point on the line at infinity of the projective plane containing the triangle.
Solution 1 (Alex Gunning, year 11, Glen Waverley Secondary College, VIC. Alex was a Gold medallist with the 2014 Australian IMO team.)

Note that $\angle CAB = \angle CDB = 90^\circ - \angle HDA = \angle DAH$. Hence the lines $AB$ and $AD$ are isogonal conjugates with respect to $\angle HAC$. This implies that the isogonal conjugate of $S$ with respect to triangle $ACH$ lies on the line $AD$. Let $T'$ be the isogonal conjugate of $S$ with respect to triangle $ACH$. We claim that $T'' = T$.

Let $Z$ be any point on the extension of line $T'H$ beyond $H$. Then

$$\angle T'H C - \angle DT'C = 180^\circ - \angle CHZ - (\angle T'AC + \angle ACT')$$
$$= 180^\circ - \angle SHA - \angle HAS - \angle SCH$$
$$= \angle ASH - \angle SCH$$
$$= 180^\circ - \angle HSC - \angle CSB - \angle SCH$$
$$= \angle CHS - \angle CSB$$
$$= 90^\circ$$
$$= \angle THC - \angle DTC.$$

If $T'$ were closer to $A$ than $T$, then we would have $\angle T'H C > \angle THC$ and $\angle DT'C < \angle DTC$ and so $\angle T'H C - \angle DT'C > \angle THC - \angle DTC$. 


which is a contradiction. A similar contradiction arises if $T'$ were further from $A$ than $T$. Therefore, we must have $T = T'$.

Next we calculate

$$\angle DTC = \angle THC - 90^\circ$$
$$= 180^\circ - \angle CHZ - 90^\circ$$
$$= 90^\circ - \angle SHA$$
$$= \angle BHS. \tag{1}$$

A similar calculation shows that

$$\angle CSB = \angle THD. \tag{2}$$

From (1) and (2) we find

$$\angle TCD + \angle BCS = 90^\circ - \angle DTC + 90^\circ - \angle CSB$$
$$= 180^\circ - \angle BHS - \angle THD$$
$$= \angle SHT. \tag{3}$$

Let $X$ be the unique point in the plane satisfying $\triangle CTD \sim \triangle HTX$.

We claim that $\triangle CSB \sim \triangle HSX$ also. If this were the case, it would follow that $\angle TXH = \angle CDT = 90^\circ$ and $\angle HXS = \angle SBC = 90^\circ$. 
Consequently, \(X\) would lie on \(ST\). Then it would follow from the aforesaid similar triangles and from (1) that

\[ \angle HTS = \angle HTX = \angle DTC = \angle BHS, \]

and so circle \(TSH\) would be tangent to line \(BD\) at \(H\) by the alternate segment theorem.

Thus it only remains to prove that \(\triangle CSB \sim \triangle HSX\).

Using (3) and the definition of \(X\) we know that \(\angle SHX = \angle BCS\). Hence it suffices to prove that

\[ \frac{HX}{BC} = \frac{HS}{CS}. \]

However, from \(\triangle CTD \sim \triangle HTX\) we already know that

\[ \frac{HX}{CD} = \frac{HT}{CT}. \]

Making \(HX\) the subject of the last two equations we see that it suffices to prove that

\[ \frac{HS \cdot BC}{CS} = \frac{HT \cdot CD}{CT} \]

\[ \Leftrightarrow \frac{HT}{CT} \cdot \frac{CS}{HS} \cdot \frac{CD}{BC} = 1. \quad (4) \]

We established this with a few applications of the sine rule as follows.

\[
\begin{align*}
\frac{HT}{CT} &= \frac{\sin \angle HCT}{\sin \angle THC} \quad \text{(sine rule \(\triangle THC\))} \\
\frac{CS}{SA} &= \frac{\sin \angle CAS}{\sin \angle SCA} \quad \text{(sine rule \(\triangle SCA\))} \\
\frac{SA}{HS} &= \frac{\sin \angle SHA}{\sin \angle SHA} \quad \text{(sine rule \(\triangle SHA\))} \\
\frac{CD}{BC} &= \frac{\sin \angle DAC}{\sin \angle CAB} \quad \text{(extended sine rule circle \(ABCD\))}
\end{align*}
\]

Clearly \(\angle CAS = \angle CAB\). Furthermore, from \(S\) and \(T\) being isogonal conjugates with respect to triangle \(ACH\), we have \(\angle HCT = \angle SCA\), \(\angle THC = 180^\circ - \angle SHA\) and \(\angle DAC = \angle HAS\). Therefore, if we multiply the above four equations together, exactly the right things cancel out to yield (4). This completes the proof. \(\square\)
Solution 2 (Andrew Elvey Price, Deputy Leader of the 2014 Australian IMO team)

Let $X$ and $Y$ be the reflections of $C$ in the lines $AB$ and $AD$, respectively. Thus $\triangle ABC \equiv \triangle ABX$, $\triangle SBC \equiv \triangle SBX$, $\triangle ADC \equiv \triangle ADY$ and $\triangle TDC \equiv \triangle TDY$. We also have that $C, B, X$ and $C, D, Y$ are triples of collinear points.

We have,

$$\angle CHS = 90^\circ + \angle CSB = 180^\circ - \angle BCS = 180^\circ - \angle SXC.$$ 

Therefore, $CHSX$ is a cyclic quadrilateral. Similarly, so is $CHTY$. We also have

$$\angle DAH = 90^\circ - \angle BDA = 90^\circ - \angle BCA = \angle CAB = \angle BAX.$$ 

Similarly, $\angle HAB = \angle Y AD$.

Therefore,

$$\angle YAH = \angle YAD + \angle DAH = \angle HAB + \angle BAX = \angle HAX.$$ 

Furthermore, since $AY = AC = AX$, and $AH = AH$, we deduce that triangles $HAY$ and $HAX$ are congruent (SAS). It follows that

$$\angle YHD = 90^\circ - \angle AHY = 90^\circ - \angle XHA = \angle BHX.$$
Next we perform the following angle computation.

\[ 2\angle CSB = \angle CSX \quad (\triangle SBC \equiv \triangle SBX) \]
\[ = \angle CHX \quad (CHSX \text{ cyclic}) \]
\[ = \angle CHB + \angle BHX \]
\[ = 180^\circ - \angle DHC + \angle YHD \quad (\angle YHD = \angle BHX) \]
\[ = 180^\circ - \angle YHC + 2\angle YHD \]
\[ = 180^\circ - \angle YTC + 2\angle YHD \quad (CHTY \text{ cyclic}) \]
\[ = \angle TCY + \angle CYT + 2\angle YHD \quad \text{(angle sum } \triangle CTY) \]
\[ = 2\angle TCY + 2\angle YHD \quad (TC = TY) \]
\[ = 2\angle THY + 2\angle YHD \]
\[ = 2\angle THD \]

Therefore, \( \angle BSX = \angle CSB = \angle THD \).

Furthermore,

\[ \angle AHT = 90^\circ - \angle THD = 90^\circ - \angle CSB = \angle XCS = \angle XHS. \]

So lines \( HT \) and \( HX \) are isogonal conjugates with respect to \( \angle SHA \).

Also, since \( \angle TAH = \angle DAH = \angle BAX = \angle SAX \), lines \( AT \) and \( AX \) are isogonal conjugates with respect to \( \angle HAS \). Therefore, points \( X \) and \( T \) are isogonal conjugates with respect to triangle \( ASH \). Hence lines \( ST \) and \( SX \) are isogonal conjugates with respect to \( \angle ASH \).

Thus, if line \( XS \) is extended beyond point \( S \) to any point \( Z \), we find

\[ \angle TSH = \angle ASZ = \angle BSX = \angle THD. \]

Hence, by the alternate segment theorem, circle \( THS \) is tangent to line \( BD \) at \( S \). \( \square \)
Solution 3 (Problem Selection Committee)

Let the perpendicular to line $SH$ at point $H$ intersect line $AB$ at $Q$. From the given angle condition, it follows that

$$\angle CSB = \angle CHS - 90^\circ = \angle CHQ.$$  

Therefore, $CHSQ$ is a cyclic quadrilateral. Since $\angle QHS = 90^\circ$, its circumcentre, $K$ say, is the midpoint of $QS$. Thus $K$, which is the centre of circle $CHS$, lies on line $AB$.

Similarly, the centre, $L$ say, of circle $CHT$ lies on line $AD$.

We want to prove that circle $TSH$ is tangent to $BD$ at $H$. Since $AH$ is perpendicular to $HD$, it is enough to show that the centre of circle $TSH$ lies on line $AH$. This is equivalent to showing that the perpendicular bisectors of $HS$ and $HT$ intersect on the line $AH$. However, since $KH = KS$ and $LH = LT$, these two perpendicular bisectors are the angle bisectors of $\angle AKH$ and $\angle ALH$, respectively. Therefore, by the angle bisector theorem it is enough to show that

$$\frac{AK}{KH} = \frac{AL}{LH} \iff \frac{AK}{AL} = \frac{KH}{LH}. \tag{1}$$

Let $M$ be the intersection of lines $KL$ and $CH$. Since $KH = KC$ and $LH = LC$ it follows that, the points $C$ and $H$ are symmetric with respect to line $KL$. Therefore, $KL$ is the perpendicular bisector of $CH$ and $M$ is the midpoint of $CH$. 
The centre, $O$ say, of circle $ABCD$ is the midpoint of its diameter $AC$. Since $M$ is the midpoint of $CH$, it follows that $OM \parallel AH$ and hence $OM \perp BD$. Since $OB = OD$ it follows that $OM$ is the perpendicular bisector of $BD$ and so $BM = DM$.


The sine rule in triangle $AKL$ yields
\[
\frac{AK}{AL} = \frac{\sin \angle ALK}{\sin \angle AKL}. \tag{2}
\]

Applying the extended sine rule to circle $CHDL$ we find
\[
\frac{DM}{\sin \angle DLM} = LC.
\]

Therefore, since $LC = LH$ (because $L$ is the centre of circle $CHT$), we have
\[
\sin \angle ALK = \frac{DM}{LH}.
\]

Similarly, we have
\[
\sin \angle AKL = \frac{BM}{KH}.
\]

Substituting these results into (2) and using the fact that $BM = DM$ yields equation (1), which completes the proof. □
4. **Solution 1** (Yang Song, year 11, James Ruse Agricultural High School, NSW. Yang was a Bronze medallist with the 2014 Australian IMO team.)

Let $T$ be the intersection of $BM$ and $CN$, let $R$ be the intersection of $AN$ and $BM$, and let $S$ be the intersection of $AM$ and $CN$.

![Diagram with labeled points A, B, C, P, Q, R, S, T, N, M]

Since $\angle ACB = ACQ$ and $\angle QAC = \angle CBA$ we may deduce that $\triangle ABC \sim \triangle QAC$. Similarly, we have $\triangle ABC \sim \triangle PBA$. Therefore,

$$\triangle QAC \sim \triangle PBA. \quad (1)$$

Thus $\angle CQA = \angle APB$ and so $\angle NQC = \angle BPM$. From (1) along with the facts that $AQ = QN$ and $AP = PM$, we have

$$\frac{AQ}{QC} = \frac{PB}{AP} \Rightarrow \frac{QN}{QC} = \frac{PB}{PM}.$$

Hence $\triangle NQC \sim \triangle BPM$ (PAP), from which $\angle QCN = \angle PMB$. Thus $PCMT$ is a cyclic quadrilateral. But now it follows that

$$\angle MTC = \angle MPC = \angle APB = \angle BAC,$$

where the last angle equality follows from $\triangle PBA \sim \triangle ABC$.

Since $\angle MTC = \angle BAC$ we may conclude that $ABTC$ is cyclic and hence $T$ lies on the circumcircle of triangle $ABC$. \qed
Solution 2 (Damon Zhong, year 12, Shore School, NSW. Damon was a Silver medallist with the 2014 Australian IMO team.)

Points $R$, $S$ and $T$ are as in solution 1. Furthermore, as in solution 1, we deduce that

$$\triangle ABC \sim \triangle QAC \sim \triangle PBA.$$

Let $D$ be the reflection of $C$ about point $Q$. Note that $ACND$ is a parallelogram because its diagonals bisect each other and so

$$\triangle AQD \equiv \triangle NQC.$$

Since $\triangle QAC \sim \triangle PBA$, and $M$ is the reflection of $A$ about $P$, and $D$ is the reflection of $C$ about $Q$, it follows that quadrilaterals $PBAM$ and $QACD$ are similar.\(^2\) Therefore,

$$\triangle BPM \sim \triangle AQD \equiv \triangle NQC.$$

Hence $\angle QCN = \angle PMB$ and so $PCMT$ is cyclic. The rest is a simple angle chase as found in solution 1.

\[\square\]

\(^2\)In the current situation the two quadrilaterals are obviously degenerate. What we are really saying is that the two sets of four points are related by a similarity.
Solution 3 (Alex Gunning, year 11, Glen Waverley Secondary College, VIC. Alex was a Gold medallist with the 2014 Australian IMO team.)

Let \( B' \) be the reflection of \( B \) about point \( A \) and let \( C' \) be the reflection of \( C \) about point \( A \). Note that \( BCB'C' \) is a parallelogram because its diagonals bisect each other.

As in solution 1, we deduce that \( \triangle ABC \sim \triangle QAC \). Furthermore, since \( AQ : QN = BA : AB' \), it follows that quadrilaterals \( NQAC \) and \( B'ABC \) are similar.\(^3\) Therefore, \( \angle ACN = \angle B'CB \). Similarly, we find that \( \angle MBA = \angle CBC' \). But since \( BCB'C' \) is a parallelogram, we have

\[
\angle ACN + \angle MBA = \angle B'CB + \angle CBC' = 180^\circ.
\]

Therefore, \( BM \) and \( CN \) intersect on circle \( ABC \).

\(^3\)See footnote 2 from solution 2.
Solution 4 (Mel Shu, year 12, Melbourne Grammar School, VIC. Mel was a Silver medallist with the 2014 Australian IMO team.)

For any two lines $\ell_1$ and $\ell_2$, let $\angle(\ell_1, \ell_2)$ denote the directed angle between the two lines. That is, the angle by which one may rotate $\ell_1$ anticlockwise so that it becomes parallel to $\ell_2$.

As in solution 1 we deduce that triangles $QAC$ and $PBA$ are similar. Furthermore, they are directly similar, that is, in the same orientation, and hence related by a spiral symmetry. Let $\theta$ be the angle of rotation of the spiral symmetry that sends triangle $QAC$ to triangle $PBA$.

Let $F$ and $E$ be the midpoints of $AC$ and $AB$, respectively. Note that $FQ \parallel CN$ and $EP \parallel BM$. Furthermore, the spiral symmetry also sends $F$ to $E$. Since the directed angle between any line and its image under the spiral symmetry is equal to $\theta$, we have

$$\angle(AC, BA) = \angle(QF, PE)$$
$$= \angle(CN, BM).$$

Therefore, lines $BM$ and $CN$ intersect on circle $ABC$.  

Solution 5 (Seyoon Ragavan, year 10, Knox Grammar School, NSW. Seyoon was a Bronze medallist with the 2014 Australian IMO team.)

We solve the problem using inversion. Recall that an inversion $f$, of radius $r > 0$ about a point $O$ is defined by

$$OX \cdot OX' = r^2,$$

where $X$ is any point ($X \neq O$) in the plane and $X' = f(X)$ lies on the ray $OX$. Inversion has the following properties.

- Circle $OXY$ becomes line $X'Y'$.
- Line $XY$ becomes circle $OX'Y'$.
- $\triangle OXY \sim \triangle OY'X'$ in opposite orientation, which implies that

$$\angle OXY = \angle OY'X' \quad \text{and} \quad \angle OYX = \angle OX'Y'.$$

For the problem at hand, consider an inversion $f$ of arbitrary radius $r > 0$ about point $A$. As usual, for any point $X$ let $X' = f(X)$. We have the following properties.

- $BC$ becomes the arc $B'C'$ of circle $AB'C'$ not containing $A$.
- Rays $AP'$ and $AQ'$ lie between rays $AB'$ and $AC'$.
- $P'$ and $Q'$ lie on circle $AB'C'$.
- $\angle AB'C' = \angle ACB$ and $\angle AC'B' = \angle ABC$.

The problem’s angle condition tells us that $\angle PAB = BCA$. In the inverted diagram this becomes $\angle P'AB' = \angle C'B'A$. However, from
circle $AB'C'$ we have $\angle C'B'A = \angle C'P'A$. Consequently, we have $\angle P'AB' = \angle C'P'A$, implying that $AB' \parallel C'P'$. Therefore, $AC'P'B'$ is an isosceles trapezium. Similarly, $AB'Q'C$ is an isosceles trapezium.

We are required to prove that lines $BM$ and $CN$ intersect on circle $ABC$. In the inverted diagram, this is equivalent to proving that circles $AB'M'$ and $AC'N'$ intersect for a second time on line $B'C''$. We claim that this common second point of intersection is the midpoint, $K$, say, of line $B'C''$.

Note that $AM' \cdot AM = r^2 = AP \cdot AP'$. Since $M$ is the midpoint of $AP$ we know that $AP = 2AM$. It follows that $AP' = 2AM'$ and so $M'$ is the midpoint of $AP'$. Analogously, $N'$ is the midpoint of $AQ'$. Consider isosceles trapezium $AC''P'B'$. Its sides $AB'$ and $C'P'$ are parallel and so have the same perpendicular bisector, $\ell$ say. Thus points $A$ and $B'$ are symmetric with respect to $\ell$, as are points $P'$ and $C'$. Therefore, segments $AP'$ and $B'C''$ are symmetric with respect to $\ell$ and hence so are their respective midpoints $M'$ and $K$. It follows that $AM'KB'$ is an isosceles trapezium and hence it is cyclic. Therefore, circle $AB'M'$ passes through $K$. Analogously circle $AC''N'$ also passes through $K$. This establishes our earlier claim and hence completes the proof. □

Comment Alex Gunning noted the following interesting property of the original diagram. If $T$ is the intersection of lines $BM$ and $CN$, then $ABTC$ is a harmonic quadrilateral.\(^4\) This can be proven using similar triangles $BTC$ and $MTN$, addendo and elements of his proof for the original problem.

\(^4\)A cyclic quadrilateral is harmonic if the products of its opposite sides are equal. They have many interesting and useful properties.
5. In both solutions, *coins* shall refer to coins issued by the Bank of Cape Town. Both solutions prove the following more general statement.

For any positive integer \( n \), given a finite collection of coins with total value at most \( n - \frac{1}{2} \), it is possible to split this collection into \( n \) or fewer groups, such that each group has total value at most 1.

**Solution 1** (Alex Gunning, year 11, Glen Waverley Secondary College, VIC. A similar idea was carried out by Praveen Wijerathna, year 12, James Ruse Agricultural High School, NSW. Alex was a Gold medallist and Praveen was a Silver medallist with the 2014 Australian IMO team.)

*Case 1.* No sub-collection of coins in the collection has total value 1.

We say that the value of a coin is *large* if its denomination is at least \( \frac{1}{2n} \) and *small* otherwise. The plan is to distribute all the large coins first and the small coins after this.

For each positive integer \( i \) \( (1 \leq i \leq n) \) let \( T_i \) be the group of all large coins whose denominations take the form \( \frac{1}{2^k(2i-1)} \). We claim that the total value of the coins in each \( T_i \) is less than 1.

Suppose that the coins in \( T_i \) have total value at least 1 for some \( i \). If \( T_i \) has \( t \) coins, let us list their values in weakly decreasing order,

\[
\frac{1}{2^{a_1}(2i-1)}, \frac{1}{2^{a_2}(2i-1)}, \ldots, \frac{1}{2^{a_t}(2i-1)},
\]

where \( a_1 \leq a_2 \leq \cdots \leq a_t \).

For each \( j \), \( (1 \leq j \leq t) \) let

\[
S_j = \frac{1}{2^{a_1}(2i-1)} + \frac{1}{2^{a_2}(2i-1)} + \cdots + \frac{1}{2^{a_j}(2i-1)}.
\]

Since \( S_1 < 1 \leq S_t \) and \( S_1 < S_2 < \cdots < S_t \), and we do not have \( S_j = 1 \) for any \( j \), there is a unique index \( k \) such that \( S_{k-1} < 1 < S_k \). But \( S_k = S_{k-1} + \frac{1}{2^{a_k}(2i-1)} \), and so we have

\[
S_{k-1} < 1 < S_{k-1} + \frac{1}{2^{a_k}(2i-1)} \Rightarrow 0 < 2^{a_k}(2i-1)(1-S_{k-1}) < 1.
\]

This is impossible because \( 2^{a_k}(2i-1)(1-S_{k-1}) \) is an integer.

Hence \( T_1, T_2, \ldots, T_n \) form \( n \) groups each of total value less than 1.
It remains to distribute the small coins. Let us add the small coins one at a time to the \( n \) groups. If we reach a point where this is no longer possible, then all the groups have total value at least \( 1 - \frac{1}{2n+1} \). But then the total value of all the coins is at least

\[
n \left( 1 - \frac{1}{2n+1} \right) > n - \frac{1}{2},
\]

a contradiction. This concludes the proof for case 1.

**Case 2.** A sub-collection of coins has total value 1.

Let us remove sub-collections of coins of total value 1 and put them aside into their own group until there are no sub-collections of total value 1 among the remaining coins. If this occurs \( d \) times, we end up with \( d \) groups of coins, each of which has total value 1. What remains is a collection of coins whose total value is \( n - d - \frac{1}{2} \) which does not possess a sub-collection of coins that have total value 1. Hence we may apply the result of case 1 to distribute these remaining coins into \( n - d \) groups, each of which has total value at most 1. Combining these \( n - d \) groups with the \( d \) groups from earlier in this paragraph concludes the proof for case 2. \( \square \)
Solution 2 (Based on the solution by Damon Zhong, year 12, Shore School, NSW. Damon was a Silver medallist with the 2014 Australian IMO team.)

If several coins together have total value of the form $\frac{1}{k}$ for some positive integer $k$, then we may merge them into a single coin of value $\frac{1}{k}$. Clearly, if the resulting collection can be split in the required way, then the initial collection can also be split.

Each time a merge is performed the number of coins is reduced. Hence merging can only be applied a finite number of times to a given collection of coins. If it is no longer possible to merge anymore coins, we shall call the collection unmergeable.

Now, suppose there are exactly $d$ coins of denomination 1 in an unmergeable collection. These can be put into their own groups. The coins remaining are unmergeable and have total value at most $n-d-\frac{1}{2}$. This is the same situation where we have $n-d$ instead of $n$ but with the added restriction that no coin has denomination 1. Hence it suffices to prove the following.

For any positive integer $n$, suppose we are given a finite collection of coins of total value at most $n-\frac{1}{2}$. If the collection is unmergeable and contains no coin of denomination 1, then it is possible to split this collection into $n$ or fewer groups, such that each group has total value at most 1.

For $i = 1, 2, \ldots, n$ let us place all the coins of denominations $\frac{1}{2i-1}$ and $\frac{1}{2i}$ into group $G_i$. Since our collection is unmergeable, there are at most $2i-2$ coins of denomination $\frac{1}{2i-1}$ and at most one coin of denomination $\frac{1}{2i}$. Hence the total value of coins in $G_i$ is at most

$$\frac{2i-2}{2i-1} + \frac{1}{2i} < 1.$$ 

In this way, all coins of denominations at least $\frac{1}{2n}$ can be placed.

It remains to place the “small” coins of denominations less than $\frac{1}{2n}$. We add them one by one. In each step, take any remaining small coin. If all groups have total values exceeding $1-\frac{1}{2n}$, then the total value of all coins is more than $n \left(1-\frac{1}{2n}\right) = n-\frac{1}{2}$, a contradiction. Hence at each step it is possible to distribute one more small coin into one of the groups. In this way, all coins are eventually distributed. \(\Box\)
6. **Solution 1** (Alex Gunning, year 11, Glen Waverley Secondary College, VIC. Alex was a Gold medallist with the 2014 Australian IMO team.)

Let us colour the lines blue one by one and only stop when it is no longer possible to colour any further line blue without forming a finite region whose perimeter is completely blue. Such a colouring shall be called a *maximal blue colouring*.

Before proceeding, it is convenient to introduce some terminology.

**Polygon.** A finite region as described in the problem statement.

**Triangle.** A polygon as per above, bounded by three lines.

**Useless polygon.** A polygon sharing a common edge with a triangle.

**Useful polygon.** A polygon that is not useless.

**Lemma 1.** A triangle never shares an edge with another triangle.

**Proof.** If a triangle shares an edge with another triangle, then the lines defining the first triangle also define consecutive sides of any polygon that is adjacent to it. In particular these lines must define the second triangle. This is a contradiction because the same three lines cannot define two different triangles.

Suppose we have a maximal blue colouring where \( k \) lines are blue. Let us colour the remaining \( n - k \) lines red. The \( k \) blue lines define \( \binom{k}{2} \) blue intersection points. Each such intersection point defines four blue angles. For each such blue angle let us place labels as follows.

- The angle gets the label 1 if it is part of a triangle.
- The angle gets the label \( \frac{1}{2} \) if it is part of a useful \( n \)-gon \( (n \geq 4) \).
- The angle gets the label 0 in all other cases.

Note that a blue angle gets the label 0 if it is part of a useless polygon or it is part of an unbounded region.

**Lemma 2.** If \( P \) is the intersection of any two blue lines, then the sum of the labels of the four angles around \( P \) is at most 2.

**Proof.** If none of the four regions around \( P \) is a triangle, then each of the four angles at \( P \) has label equal to either \( \frac{1}{2} \) or 0. Hence the sum of the labels of the angles at \( P \) is at most 2.

If one of the four regions around \( P \) is a triangle, let it be \( T \). Then from lemma 1 each region at \( P \) that shares a common edge with \( T \) is either infinite or is a useless polygon.
Either way, the angle at $P$ in each of these adjacent regions gets the label 0. Since the other two angles get label at most 1 we again find that the sum of the labels of the angles at $P$ is at most 2. □

Next we define the score of any polygon to be the sum of the labels of its interior angles. (Note that some angles do not have labels.)

**Lemma 3.** The sum of the scores of all polygons is at most $k(k-1)$.

**Proof.** There are $\binom{k}{2}$ blue intersection points—one for each pair of blue lines. Each label contributes to the score of at most one polygon. Therefore, using lemma 2, it follows that the sum of the scores is at most $2\binom{k}{2} = k(k - 1)$. □

Let us call a polygon almost blue if all but one of its edges are blue. Define the special red line of an almost blue polygon to be the red line that contains its unique red edge.

**Lemma 4.** Each red line $\ell$ is the special red line of an almost blue polygon whose score is at least 1.

**Proof.** Since we have a maximal blue colouring, if we were to change the colour of $\ell$ from red to blue, we would end up with a polygon all of whose sides are blue. Hence there must be an almost blue polygon, $Q$ say, that has $\ell$ as its special line.

If $Q$ is not adjacent to a triangle, then $Q$ is a useful polygon. If $Q$ is a triangle, it has score 1. If $Q$ is an $n$-gon ($n \geq 4$), it has score $\frac{n-2}{2} \geq 1$.

If $Q$ is adjacent to a triangle, $T$ say, then all of the sides of $T$ must come from extending the sides of $Q$. 
Since the sides of $T$ are not all blue, $\ell$ must be the red side of $T$ while the other two sides of $T$ are blue. Hence $T$ is almost blue and has $\ell$ as its special line. Furthermore, the score of $T$ is 1. $\square$

Lemma 4 allows us to define a function $f$ from the set $\mathcal{L}$ of red lines to the set of almost blue polygons whose score is at least 1. Specifically, $f(\ell)$ is an almost blue polygon of score at least 1 for which $\ell$ is a special red line. Since each almost blue polygon has exactly one special line, $f$ is injective. It follows that

$$n - k = \sum_{\ell \in \mathcal{L}} 1 \leq \sum_{\ell \in \mathcal{L}} \text{score}(f(\ell)) \quad \text{(from lemma 4)}$$

$$\leq k(k - 1). \quad \text{(from lemma 3)}$$

Rearranging the final inequality yields $k \geq \sqrt{n}$, as required. $\square$

**Comments** Suppose we tried to carry out the above proof without making the distinction between useful and useless polygons. Lemmas 1 and 4 would remain unchanged. Lemma 2 would be weakened to yield that the sum of the labels of the four angles around $P$ is at most 3. The estimate from lemma 3 would then be weakened to $\frac{3}{2}k(k - 1)$. The concluding argument would be the same except that the computation yields $n - k \leq \frac{3}{2}k(k - 1)$, which implies $k \geq c\sqrt{n}$ where $c = \sqrt{\frac{2}{3}}$.

If we were to carry out the above proof but instead give all blue angles a label equal to 1, we would end up with $k \geq c\sqrt{n}$ where $c = \sqrt{\frac{1}{2}}$. 
Solution 2 (Problem Selection Committee)

Suppose, as in solution 1, that we have a maximal blue colouring where \( k \) lines are coloured blue and the remaining \( n - k \) lines are coloured red. As in solution 1 we define an almost blue polygon to be a finite region with exactly one of its sides red.

Let us define a function \( f \) from the set \( \mathcal{L} \) of red lines to the set of blue points. For any red line \( \ell \) pick any almost blue polygon \( A \) that has part of \( \ell \) as its only red side. Suppose the vertices of \( A \) enumerated in clockwise order are \( R', R, B_1, B_2, \ldots, B_k \), where \( R' \) and \( R \) are the endpoints of its red side. We define \( f(\ell) = B_1 \). Note that \( B_1 \) is the intersection of two blue lines.

We claim that \( f \) is at most two-to-one. That is, for any intersection point \( B \) of two blue lines, there are at most two red lines \( \ell \) for which \( f(\ell) = B \). If this claim were true, then since the number of blue intersection points is \( \binom{k}{2} \) it would follow that

\[
    n - k = |\mathcal{L}| \leq 2|f(\mathcal{L})| \leq 2 \binom{k}{2} = k^2 - k,
\]

from which \( k \geq \sqrt{n} \) immediately follows.

It only remains to prove our claim. Suppose to the contrary that three different red lines \( \ell_1, \ell_2, \ell_3 \) are mapped by \( f \) to the same blue point \( B \).

Let \( R'_1, R_1, B \) be the three clockwise consecutive vertices of an almost blue region \( A_1 \) used to generate \( B \) from \( \ell_1 \). Note that \( \ell_1 \) contains the red edge \( R'_1 R_1 \). Similarly, \( R'_2, R_2, B \) are three clockwise consecutive
vertices for an almost blue region $A_2$ used to generate $B$ from $\ell_2$ and $R'_3, R_3, B$ are three clockwise consecutive vertices for an almost blue region $A_3$ used to generate $B$ from $\ell_3$.

Note that if $\ell_1 \neq \ell_2$, it follows that $A_1 \neq A_2$ and so, because of the clockwise ordering, we must also have $R_1 \neq R_2$. Similarly, $R_1 \neq R_3$ and $R_2 \neq R_3$. Thus $R_1, R_2, R_3$ are three different points.

The edges $BR_1, BR_2, BR_3$ all emanate from $B$. For any $i \in \{1, 2, 3\}$, since $BR_i$ is an edge of polygon $A_i$, there can be no other intersection point of the $n$ lines that lies on the segment $BR_i$. Hence the edges $BR_1, BR_2, BR_3$ must lie on three of the four blue rays emanating from $B$. Furthermore, these are the points closest to $B$ on these rays.

Without loss of generality $R_2$ and $R_3$ are on the same blue line through $B$ where $B$ is in between $R_2$ and $R_3$, and $R_1$ is on the other blue line through $B$.

Consider the almost blue region $A_1$ used to define $f(\ell_1) = B$. Three of its clockwise consecutive vertices are $R_1, B$ and either $R_2$ or $R_3$. Without loss of generality, they are $R_1, B, R_2$. Part of $\ell_1$ forms the red side of $A_1$. Since $\ell_2$ passes through $R_2$, part of $\ell_2$ also forms a red side of $A_1$. However, since $\ell_1 \neq \ell_2$ this means that $A_1$ has at least two red sides, a contradiction. Hence we have proven our claim, as required. \hfill \Box

**Comment** The wording of question 6 suggests investigating what happens when $c > 1$. Indeed, by using more sophisticated methods, Po-Shen Loh\(^5\) was able to show that the bound $\sqrt{n}$ could be further improved to $O(\sqrt{n \log n})$. Consequently, the inequality is true for any real number $c > 1$! It is currently unknown if Po-Shen’s bound can be pushed further.

\(^5\)Po-Shen was the Leader of the USA team at IMO 2014.
**INTERNATIONAL MATHEMATICAL OLYMPIAD RESULTS**

Leading Country Scores

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### AUSTRALIAN IMO TEAM SCORES

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ORIGIN OF SOME QUESTIONS

Senior Contest
Questions 1 and 2 were submitted by Dr Angelo Di Pasquale.
Question 5 was submitted by Dr Norman Do.

Australian Mathematical Olympiad
Questions 3 and 4 were submitted by Dr Angelo Di Pasquale.
Question 8 originated from Ian Wanless.

Asian Pacific Mathematical Olympiad
Question 4 was submitted by the AMOC Senior Problems committee.
# HONOUR ROLL

Because of changing titles and affiliations, the most senior title achieved and later affiliations are generally used, except for the Interim committee, where they are listed as they were at the time.

## Interim Committee 1979–1980

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## Australian Mathematical Olympiad Committee

The Australian Mathematical Olympiad Committee was founded at a meeting of the Australian Academy of Science at its meeting of 2–3 April 1980.

* denotes Executive Position

### Chair*

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### Deputy Chair*

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<thead>
<tr>
<th>Name</th>
<th>Affiliation</th>
<th>Years</th>
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<tbody>
<tr>
<td>Prof P J O'Halloran</td>
<td>University of Canberra, ACT</td>
<td>15 years; 1980–1994</td>
</tr>
<tr>
<td>Prof A P Street</td>
<td>University of Queensland</td>
<td>1 year; 1995</td>
</tr>
<tr>
<td>Prof C Praeger</td>
<td>University of Western Australia</td>
<td>6 years; 1996–2001</td>
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<tr>
<td>Assoc Prof D Hunt</td>
<td>University of New South Wales</td>
<td>13 years; 2002–2014</td>
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### Executive Director*

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<tr>
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<th>Years</th>
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<tr>
<td>Prof P J O'Halloran</td>
<td>University of Canberra, ACT</td>
<td>15 years; 1980–1994</td>
</tr>
<tr>
<td>Prof P J Taylor</td>
<td>University of Canberra, ACT</td>
<td>18 years; 1994–2012</td>
</tr>
<tr>
<td>Adj Prof M G Clapper</td>
<td>University of Canberra, ACT</td>
<td>2 years; 2013–2014</td>
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### Secretary

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<tr>
<th>Name</th>
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<tr>
<td>Prof J C Burns</td>
<td>Australian Defence Force Academy, ACT</td>
<td>9 years; 1980–1988</td>
</tr>
<tr>
<td>Vacant</td>
<td></td>
<td>4 years; 1989–1992</td>
</tr>
<tr>
<td>Mrs K Doolan</td>
<td>Victorian Chamber of Mines, VIC</td>
<td>6 years; 1993–1998</td>
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### Treasurer*

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<tr>
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<tr>
<td>Prof J C Burns</td>
<td>Australian Defence Force Academy, ACT</td>
<td>8 years; 1981–1988</td>
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<tr>
<td>Prof P J O'Halloran</td>
<td>University of Canberra, ACT</td>
<td>2 years; 1989–1990</td>
</tr>
<tr>
<td>Ms J Downes</td>
<td>CPA</td>
<td>5 years; 1991–1995</td>
</tr>
<tr>
<td>Dr P Edwards</td>
<td>Monash University, VIC</td>
<td>8 years; 1995–2002</td>
</tr>
<tr>
<td>Prof M Newman</td>
<td>Australian National University, ACT</td>
<td>6 years; 2003–2008</td>
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<tr>
<td>Dr P Swedosh</td>
<td>The King David School, VIC</td>
<td>6 years; 2009–2014</td>
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### Director of Mathematics Challenge for Young Australians*

<table>
<thead>
<tr>
<th>Name</th>
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<tbody>
<tr>
<td>Mr J B Henry</td>
<td>Deakin University, VIC</td>
<td>17 years; 1990–2006</td>
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<tr>
<td>Dr K McAvaney</td>
<td>Deakin University, VIC</td>
<td>9 years; 2006–2014</td>
</tr>
<tr>
<td>Chair, Senior Problems Committee</td>
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<tr>
<td>Prof B C Rennie</td>
<td>James Cook University, QLD</td>
<td>1 year; 1980</td>
</tr>
<tr>
<td>Mr J L Williams</td>
<td>University of Sydney, NSW</td>
<td>6 years; 1981–1986</td>
</tr>
<tr>
<td>Assoc Prof H Lausch</td>
<td>Monash University, VIC</td>
<td>27 years; 1987–2013</td>
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<tr>
<td>Dr N Do</td>
<td>Monash University, VIC</td>
<td>1 year; 2014</td>
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<tr>
<th>Director of Training*</th>
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<tbody>
<tr>
<td>Mr J L Williams</td>
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<tr>
<td>Mr G Ball</td>
</tr>
<tr>
<td>Dr D Paget</td>
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<tr>
<td>Dr M Evans</td>
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<td>Assoc Prof D Hunt</td>
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<tr>
<td>Dr A Di Pasquale</td>
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<tr>
<th>Team Leader</th>
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<tbody>
<tr>
<td>Mr J L Williams</td>
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<tr>
<td>Dr E Strzelecki</td>
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<tr>
<td>Dr D Paget</td>
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<tr>
<td>Dr A Di Pasquale</td>
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<td>Dr I Guo</td>
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<th>Deputy Team Leader</th>
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<tr>
<td>Prof G Szekeres</td>
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<tr>
<td>Mr G Ball</td>
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<tr>
<td>Dr D Paget</td>
</tr>
<tr>
<td>Dr J Graham</td>
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<tr>
<td>Dr M Evans</td>
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<tr>
<td>Dr A Di Pasquale</td>
</tr>
<tr>
<td>Dr D Mathews</td>
</tr>
<tr>
<td>Dr N Do</td>
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<tr>
<td>Dr I Guo</td>
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<tr>
<td>Mr G White</td>
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<td>Mr A Elvey Price</td>
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<th>State Directors</th>
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<tr>
<td><strong>Australian Capital Territory</strong></td>
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<tr>
<td>Prof M Newman</td>
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<tr>
<td>Mr D Thorpe</td>
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<tr>
<td>Dr R A Bryce</td>
</tr>
<tr>
<td>Mr R Welsh</td>
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<tr>
<td>Mrs J Kain</td>
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<tr>
<td>Mr J Carty</td>
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<td>Mr J Hassall</td>
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<td>Dr C Wetherell</td>
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<th><strong>New South Wales</strong></th>
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<tr>
<td>Dr M Hirschhorn</td>
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<tr>
<td>Mr G Ball</td>
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<tr>
<td>Dr W Palmer</td>
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Northern Territory
Dr I Roberts Charles Darwin University 1 year; 2014

Queensland
Dr N H Williams University of Queensland 21 years; 1980–2000
Dr G Carter Queensland University of Technology 10 years; 2001–2010
Dr V Scharaschkin University of Queensland 4 years; 2011–2014

South Australia/Northern Territory
Mr V Treilibs SA Department of Education 8 years; 1983–1990
Dr M Peake Adelaide 8 years; 2006–2013
Dr D Martin Adelaide 1 year; 2014

Tasmania
Mr J Kelly Tasmanian Department of Education 8 years; 1980–1987
Dr D Paget University of Tasmania 8 years; 1988–1995
Mr W Evers St Michael’s Collegiate School 9 years; 1995–2003
Dr K Dharmadasa University of Tasmania 11 years; 2004–2014

Victoria
Dr D Holton University of Melbourne 3 years; 1980–1982
Mr B Harridge Melbourne High School 1 year; 1982
Ms J Downes CPA 6 years; 1983–1988
Mr L Doolan Melbourne Grammar School 9 years; 1989–1998
Dr P Swedosh The King David School 17 years; 1998–2014

Western Australia
Dr N Hoffman WA Department of Education 3 years; 1980–1982
Assoc Prof W Bloom Murdoch University 2 years; 1989–1990
Dr E Stoyanova WA Department of Education 7 years; 1995, 2000–2005
Dr G Gamble University of Western Australia 9 years; 2006–2014

Editor
Prof P J O’Halloran University of Canberra, ACT 1 year; 1983
Dr A W Plank University of Southern Queensland 11 years; 1984–1994
Dr A Storozhev Australian Mathematics Trust, ACT 15 years; 1994–2008

Editorial Consultant
Dr O Yevdokimov University of Southern Queensland 6 years; 2009–2014

Other Members of AMOC (showing organisations represented where applicable)
Mr W J Atkins Australian Mathematics Foundation 18 years; 1995–2012
Dr S Britton University of Sydney, NSW 8 years; 1990–1998
Prof G Brown Australian Academy of Science, ACT 10 years; 1980, 1986–1994
Dr R A Bryce Australian Mathematical Society, ACT 10 years; 1991–1998
Mathematics Challenge for Young Australians 13 years; 1999–2012
Mr G Cristofani Department of Education and Training 2 years; 1993–1994
Ms L Davis IBM Australia 4 years; 1991–1994
Dr W Franzsen Australian Catholic University, ACT 9 years; 1990–1998
Dr J Gani Australian Mathematical Society, ACT 1980
Assoc Prof T Gagen ANU AAMT Summer School 6 years; 1993–1998
Ms P Gould Department of Education and Training 2 years; 1995–1996
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<tr>
<th>Name</th>
<th>Institution</th>
<th>Years</th>
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<tr>
<td>Prof G M Kelly</td>
<td>University of Sydney, NSW</td>
<td>6 years; 1982–1987</td>
</tr>
<tr>
<td>Prof R B Mitchell</td>
<td>University of Canberra, ACT</td>
<td>5 years; 1991–1995</td>
</tr>
<tr>
<td>Ms Anna Nakos</td>
<td>Mathematics Challenge for Young Australians</td>
<td>12 years; 2003–2014</td>
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<tr>
<td>Mr S Neal</td>
<td>Department of Education and Training</td>
<td>4 years; 1990–1993</td>
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<tr>
<td>Prof M Newman</td>
<td>Australian National University, ACT</td>
<td>15 years; 1986–1998</td>
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<td>(Mathematics Challenge for Young Australians)</td>
<td>10 years; 1999–2002, 2009–2014</td>
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<tr>
<td>Prof R B Potts</td>
<td>University of Adelaide, SA</td>
<td>1 year; 1980</td>
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<tr>
<td>Mr H Reeves</td>
<td>Australian Association of Maths Teachers</td>
<td>11 years; 1988–1998</td>
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<td>Australian Mathematics Foundation</td>
<td>2014</td>
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<tr>
<td>Mr N Reid</td>
<td>IBM Australia</td>
<td>3 years; 1988–1990</td>
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<tr>
<td>Mr R Smith</td>
<td>Telecom Australia</td>
<td>5 years; 1990–1994</td>
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<tr>
<td>Prof N S Trudinger</td>
<td>Australian Mathematical Society, ACT</td>
<td>3 years; 1986–1988</td>
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<tr>
<td>Assoc Prof I F Vivian</td>
<td>University of Canberra, ACT</td>
<td>1 year; 1990</td>
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<tr>
<td>Dr M W White</td>
<td>IBM Australia</td>
<td>9 years; 1980–1988</td>
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### Associate Membership (inaugurated in 2000)

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<tr>
<td>Ms S Britton</td>
<td>15 years; 2000–2014</td>
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<tr>
<td>Dr M Evans</td>
<td>15 years; 2000–2014</td>
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<tr>
<td>Dr W Franzsen</td>
<td>15 years; 2000–2014</td>
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<tr>
<td>Prof T Gagen</td>
<td>15 years; 2000–2014</td>
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<tr>
<td>Mr H Reeves</td>
<td>15 years; 2000–2014</td>
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<tr>
<td>Mr G Ball</td>
<td>15 years; 2004–2014</td>
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### AMOC Senior Problems Committee

#### Current members

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<thead>
<tr>
<th>Name</th>
<th>Institution</th>
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<tbody>
<tr>
<td>Dr N Do</td>
<td>Monash University, VIC (Chair)</td>
<td>1 year; 2014</td>
</tr>
<tr>
<td>(member)</td>
<td></td>
<td>11 years; 2003–2013</td>
</tr>
<tr>
<td>Dr A Di Pasquale</td>
<td>University of Melbourne, VIC</td>
<td>14 years; 2001–2014</td>
</tr>
<tr>
<td>Dr M Evans</td>
<td>Australian Mathematical Sciences Institute, VIC</td>
<td>26 years; 1990–2014</td>
</tr>
<tr>
<td>Dr I Guo</td>
<td>University of Sydney, NSW</td>
<td>7 years; 2008–2014</td>
</tr>
<tr>
<td>Assoc Prof D Hunt</td>
<td>University of New South Wales</td>
<td>29 years; 1986–2014</td>
</tr>
<tr>
<td>Dr J Kupka</td>
<td>Monash University, VIC</td>
<td>12 years; 2003–2014</td>
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<tr>
<td>Assoc Prof H Lausch</td>
<td>Monash University, VIC (Chair)</td>
<td>27 years; 1987–2013</td>
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<tr>
<td>(member)</td>
<td></td>
<td>1 year; 2014</td>
</tr>
<tr>
<td>Dr K McAvaney</td>
<td>Deakin University, VIC</td>
<td>19 years; 1996–2014</td>
</tr>
<tr>
<td>Dr D Mathews</td>
<td>Monash University, VIC</td>
<td>15 years; 2001–2014</td>
</tr>
<tr>
<td>Dr C Rao</td>
<td>NEC Australia</td>
<td>15 years; 2000–2014</td>
</tr>
<tr>
<td>Dr B B Saad</td>
<td>Monash University, VIC</td>
<td>21 years; 1994–2014</td>
</tr>
<tr>
<td>Dr J Simpson</td>
<td>Curtin University, WA</td>
<td>16 years; 1999–2014</td>
</tr>
<tr>
<td>Dr I Wanless</td>
<td>Monash University, VIC</td>
<td>15 years; 2000–2014</td>
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#### Previous members

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<th>Years</th>
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<tbody>
<tr>
<td>Mr G Ball</td>
<td>University of Sydney, NSW</td>
<td>16 years; 1982–1997</td>
</tr>
<tr>
<td>Mr M Brazil</td>
<td>LaTrobe University, VIC</td>
<td>5 years; 1990–1994</td>
</tr>
<tr>
<td>Dr M S Brooks</td>
<td>University of Canberra, ACT</td>
<td>8 years; 1983–1990</td>
</tr>
<tr>
<td>Dr G Carter</td>
<td>Queensland University of Technology</td>
<td>10 years; 2001–2010</td>
</tr>
<tr>
<td>Dr J Graham</td>
<td>University of Sydney, NSW</td>
<td>1 year; 1992</td>
</tr>
<tr>
<td>Dr M Herzberg</td>
<td>Telecom Australia</td>
<td>1 year; 1990</td>
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<tr>
<td>Dr L Kovacs</td>
<td>Australian National University, ACT</td>
<td>5 years; 1981–1985</td>
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<tr>
<td>Dr D Paget</td>
<td>University of Tasmania</td>
<td>7 years; 1989–1995</td>
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<tr>
<td>Prof P Schultz</td>
<td>University of Western Australia</td>
<td>8 years; 1993–2000</td>
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Dr L Stoyanov  University of Western Australia  5 years; 2001–2005
Dr E Strzelecki  Monash University, VIC  5 years; 1986–1990
Dr E Szekeres  University of New South Wales  7 years; 1981–1987
Prof G Szekeres  University of New South Wales  7 years; 1981–1987
Em Prof P J Taylor  Australian Capital Territory  1 year; 2013
Dr N H Williams  University of Queensland  20 years; 1981–2000

Mathematics School of Excellence
Dr S Britton  University of Sydney, NSW (Coordinator)  2 years; 1990–1991
Mr L Doolan  Melbourne Grammar, VIC (Coordinator)  6 years; 1992, 1993–1997
Mr W Franzsen  Australian Catholic University, ACT (Coordinator)  2 years; 1990–1991
Dr D Paget  University of Tasmania (Director)  6 years; 1990–1995
Assoc Prof D Hunt  University of New South Wales (Director)  5 years; 1996–2000
Dr A Di Pasquale  University of Melbourne, VIC (Director)  14 years; 2001–2014

International Mathematical Olympiad Selection School
Mr J L Williams  University of Sydney, NSW (Director)  2 years; 1982–1983
Mr G Ball  University of Sydney, NSW (Director)  6 years; 1984–1989
Mr L Doolan  Melbourne Grammar, VIC (Coordinator)  3 years; 1989–1991
Dr S Britton  University of Sydney, NSW (Coordinator)  7 years; 1992–1998
Mr W Franzsen  Australian Catholic University, ACT (Coordinator)  8 years; 1992–1996, 1999–2001
Dr D Paget  University of Tasmania (Director)  6 years; 1990–1995
Assoc Prof D Hunt  University of New South Wales (Director)  5 years; 1996–2000
Dr A Di Pasquale  University of Melbourne, VIC (Director)  14 years; 2001–2014