AUSTRALIAN MATHEMATICAL OLYMPIAD 2017
Solutions

1. For which integers $n \geq 2$ is it possible to write the numbers $1, 2, 3, \ldots, n$ in a row in some order so that any two numbers written next to each other in the row differ by 2 or 3?

Solution 1

It is clear that the task is impossible for $n = 2$ and $n = 3$. We will proceed to prove that it is possible for all integers $n \geq 4$. Consider the following constructions, which show that the task is possible for $n = 4, 5, 6, 7$.

- $2, 4, 1, 3$
- $2, 4, 1, 3, 5$
- $1, 3, 5, 2, 4, 6$
- $1, 3, 6, 4, 2, 5, 7$

Suppose now that the task is possible for some n, where the final number is either $n-1$ or n. Then the task is possible for $n + 4$, by extending the sequence with the four numbers $n + 2, n + 4, n + 1, n + 3$. Therefore, the four constructions above can be extended to give constructions for any integer $n \geq 4$.

Solution 2 (Angelo Di Pasquale and Ian Wanless)

For $n = 4, 5, 6$, we have the following constructions.

- 2,4,1,3
- 5,2,4,1,3
- 5,2,4,1,3,6

Suppose that the task is possible for some n, where the first and last terms are $n - 1$ and n in some order. Then the task is possible for $n+1$ by appending the term $n+1$ adjacent to the term $n - 1$.

Solution 3 (Angelo Di Pasquale)

We have the following sporadic examples for $n = 4, 7, 9$.

- 2,4,1,3
- 1,3,6,4,7,5,2
- 1,3,5,2,4,7,9,6,8

We also have the following examples for $n = 5, 6, 8$.

- 1,4,2,5,3
- 1,3,6,4,2,5

- 1,3,6,8,5,2,4,7

Observe that we can link any number of the above three blocks together because only the gaps matter and the difference between the last term and the next lowest number not in the sequence is 2 or 3. For example, to obtain the $n = 11$ construction, we take the example for $n = 5$ (i.e., $1, 4, 2, 5, 3$) and the example for $n = 6$ that has been translated up by 5 (i.e., $6, 8, 11, 9, 7, 10$) to form $1, 4, 2, 5, 3, 6, 8, 11, 9, 7, 10$. Valid sequences of length n may be formed in this way for any n of the form $n = 5a + 6b + 8c$. Finally, one observes that all $n \geq 10$ can be represented in this way. The reason is because $10, 11, 12, 13, 14$ are all representable in this way, and any larger integer can be represented as a multiple of 5 plus one of these numbers.

(Note that this proves the following stronger result: For $n = 5, 6, 8$ and $n \geq 10$, valid sequences can be found that start with 1.)

2. Given five distinct integers, consider the ten differences formed by pairs of these numbers. (Note that some of these differences may be equal.)

   Determine the largest integer that is certain to divide the product of these ten differences, regardless of which five integers were originally given.

Solution 1

If the five integers are $1, 2, 3, 4, 5$, then the product of the ten differences is

$$1 \times 1 \times 1 \times 1 \times 2 \times 2 \times 2 \times 3 \times 3 \times 4 = 288.$$

Now suppose that we are given five distinct integers a, b, c, d, e. We will show that 288 is certain to divide the product P of the ten differences formed by pairs of these numbers.

- First, we will show that 2^5 is a divisor of P. By the pigeonhole principle, at least three of the five numbers — without loss of generality, a, b, c — are congruent modulo 2. By the pigeonhole principle again, at least two of these numbers — without loss of generality, a and b — are congruent modulo 4. Therefore, the product $|a-b| \times |a-c| \times |b-c|$ contributes a factor of 2^4 to P. If d and e are congruent modulo 2, then the factor $|d-e|$ contributes an extra factor of 2 to P. Otherwise, d and e are distinct modulo 2, so at least one of them — without loss of generality, d — is congruent to a, b, c modulo 2. So the difference $|a-d|$ contributes an extra factor of 2 to P. Therefore, P is divisible by 2^5.
- Next, we will show that 3^2 is a divisor of P. By the pigeonhole principle, at least two of the five numbers — without loss of generality, a and b — are congruent modulo 3. Therefore, the difference $|a-b|$ contributes a factor of 3 to P. If the remaining three numbers are distinct modulo 3, then at least one of them — without loss of generality, c — is congruent to a and b modulo 3. Therefore, the difference $|a-c|$ contributes an extra factor of 3 to P. Otherwise, the remaining three numbers are not distinct modulo 3, so the pigeonhole principle guarantees that two of them — without loss of generality, c and d — are congruent modulo 3. So the difference $|c-d|$ contributes an extra factor of 3 to P. Therefore, P is divisible by 3^2.

Since the product is divisible by both 2^5 and 3^2, which are relatively prime, it is divisible by $2^5 \times 3^2 = 288$.

Solution 2 (Kevin McAvaney)

As in Solution 1, if the five integers are $1, 2, 3, 4, 5$, then the product of the ten differences is

$$1 \times 1 \times 1 \times 1 \times 2 \times 2 \times 2 \times 3 \times 3 \times 4 = 288.$$

Now suppose that we are given five distinct integers a, b, c, d, e. We will show that 288 is guaranteed to divide the product P of the ten differences formed by pairs of these numbers.

The fact that 32 divides P follows from the observations below.

- If four or more of the five numbers are even then at least six of their differences are even, hence 32 divides P.

- If four or more of the five numbers are odd then at least six of their differences are even, hence 32 divides P.
- If three of the five numbers are even and two are odd, then they have the form $2x, 2y, 2z, 2u+1, 2v+1$. At least two of x, y, z are congruent mod 2, so the corresponding difference is divisible by 4. Hence 32 divides P.
- If three of the five numbers are odd and two are even, then they have the form $2x, 2y, 2u+1, 2v+1, 2w+1$. At least two of u, v, w are congruent mod 2, so the corresponding difference is divisible by 4. Hence 32 divides P.

The fact that 9 divides P follows from the observations below.

- If three or more of the five numbers are congruent mod 3, then at least three of their differences are divisible by 3, hence 9 divides P.
- If no three of the five numbers have the same remainder when divided by 3, then two have the same remainder r and two have the same remainder s. The difference of each pair is divisible by 3, hence 9 divides P.

Since the product is divisible by both 32 and 9, which are relatively prime, it is divisible by $32 \times 9 = 288$.

3. Determine all functions f defined for real numbers and taking real numbers as values such that
$$f(x^2 + f(y)) = f(xy)$$
for all real numbers x and y.

Solution (Angelo Di Pasquale)

The solutions are given by $f(x) = c$, where c is any real constant.

Let $f(0) = c$. Set $y = 0$ in the functional equation to find $f(x^2 + c) = c$. Since $x^2 + c$ ranges over all real numbers greater than or equal to c, we deduce that
$$f(x) = c, \qquad \text{for all } x \geq c.$$

Now let u be any real number such that $u \neq 0$ and $u \geq c$. Substitute $y = u$ into the functional equation to find
$$f(xu) = f(x^2 + f(u)) = f(x^2 + c) = c.$$

Since $u \neq 0$, it follows that xu ranges over $\mathbb{R}$ as x ranges over $\mathbb{R}$. Hence, $f(x) = c$ for all $x \in \mathbb{R}$. It is easy to verify that all such functions do indeed satisfy the functional equation.

4. Suppose that S is a set of 2017 points in the plane that are not all collinear.

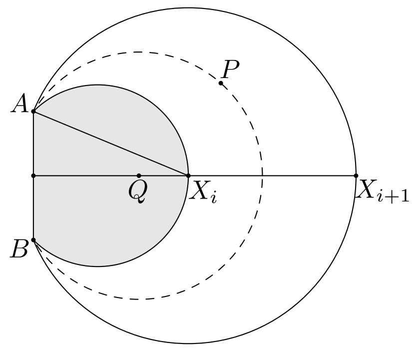
   Prove that S contains three points that form a triangle whose circumcentre is not a point in S.

Solution 1 (Angelo Di Pasquale)

To obtain a contradiction, we assume that the circumcentre of any triangle formed by three points of S is also a point in S.

There exist two points $A, B \in S$ such that all points of S lie on or to the right of the line AB. (This can be explained by taking a vertical line ℓ which lies to the left of all points in S and slowly moving to the right until it encounters a point of S, and then rotating it about that point until it encounters a second point of S. Alternatively, one can appeal to the convex hull of S.)

Without loss of generality, let A and B have coordinates $(0, 1)$ and $(0, -1)$, respectively. Construct the sequence of points $X_1, X_2, \ldots$ as follows. Let X_1 be the midpoint of AB. For $i = 1, 2, \ldots$, the point X_{i+1} is defined to be the intersection of the circle with centre X_i and passing through points A and B, with the positive x-axis. We call this circle C_i and we will prove by induction that no point of S lies in the interior of C_i.



For the base case, C_1 is the circle with diameter AB. Remember that no point of S lies to the left of the line AB. If a point $P \in S$ lies in the interior of C_1, then since $\angle APB > 90°$, the circumcentre of $\triangle ABP$ would lie to the left of AB, a contradiction.

For the inductive step, suppose that no point of S lies in the interior of C_{i-1} for some $i \geq 1$. Suppose that a point $P \in S$ lies in the interior of C_i. Then arc APB lies to the left of arc $AX_{i+1}B$. Thus, the centre of circle APB, which we will call Q, lies to the left of the centre X_i of circle C_i. But Q lies on the x-axis. Hence, Q lies in the interior of C_{i-1}. But since $P \in S$, we have $Q \in S$, which yields a contradiction. Thus, the induction is complete.

Let $X_i = (x_i, 0)$. Since X_i is the centre of C_i, we know that $X_iA = X_iX_{i+1} = x_{i+1} - x_i$. Pythagoras' theorem yields

$$(x_{i+1} - x_i)^2 = X_iA^2 = x_i^2 + 1 \qquad \Rightarrow \qquad x_{i+1} > 2x_i.$$

Hence, the circles C_i grow arbitrarily large. Since they all pass through A and B, it follows that no point of S can lie to the right of the line AB, which yields the desired contradiction.

Solution 2 (Angelo Di Pasquale)

To obtain a contradiction, we assume that the circumcentre of any triangle formed by three points of S is also a point in S.

Let $A, B \in S$ be as in the official solution, but with the additional property that no other point of S lies on the segment AB. Of the remaining points of S that are not on the line AB, let P be the point such that the anticlockwise oriented angle $\angle APB$ is maximal. Let Q be the circumcentre of $\triangle APB$. By assumption $Q \in S$ and the anticlockwise oriented angles $\angle APB$ and $\angle AQB$ do not exceed $180°$. However, $\angle AQB = 2\angle APB > \angle APB$. This contradicts the maximality of $\angle APB$.

Solution 3 (Ivan Guo)

We will prove a stronger statement — namely, that S contains three non-collinear points whose circumcircle does not contain a point of S in its interior. First, we construct a circle through two points of S that does not contain a point of S in its interior. This can be achieved in either of the following ways.

- Find a pair of points in S with the smallest distance between them and construct the circle on which these two points lie diametrically opposite each other.
- Start with a circle of small radius passing through one point of S and dilate about that point. Keep expanding the circle until it hits a second point of S. If the circle never hits a second point of S, then we simply make the circle very large and then rotate it about the centre of dilation until it hits a second point of S.

Now we have a circle through two points A and B of S. Since not all points in S are collinear, at least one side of the line AB contains at least one point in S. Continuously expand the circle towards that side, while making sure that it still passes through A and B. Eventually, it must hit a third point of S. At this stage, the circle meets three points of S but contains no points of S in its interior.

Solution 4 (Alan Offer)

Amongst all triangles whose vertices are in S, let ABC be one whose circumradius, R, is smallest. Let D be the circumcentre of triangle ABC. If D is not in S then we are done. Suppose then that D is in S.

Relabelling if necessary, let $\angle BAC = \alpha \le 60°$ be the smallest angle in triangle ABC. Let R_A be the circumradius of triangle BCD. By the sine rule, we have $2R = \frac{BC}{\sin\alpha}$. Also, since $\angle BDC = 2\alpha$, we have

$$2R_A = \frac{BC}{\sin 2\alpha} = \frac{BC}{2\sin\alpha\cos\alpha}.$$

Thus $\frac{R}{R_A} = 2\cos\alpha$. If $\alpha < 60°$ then $\cos\alpha > \frac{1}{2}$ and so $R > R_A$. By the minimality of R, it follows that the circumcentre of triangle BCD is not in S and we are done.

Suppose then that $\alpha = 60°$, in which case triangle ABC is equilateral. Let E be the reflection of D through the line AB. Then E is the circumcentre of triangle ABD. If E is not in S then we are done. Otherwise, E is in S and triangle ADE is an equaliteral triangle smaller than ABC, and so by the minimality of R, the circumcentre of ADE is not in S.

5. Determine the number of positive integers n less than one million for which the sum

$$\frac{1}{2 \times \lfloor\sqrt{1}\rfloor + 1} + \frac{1}{2 \times \lfloor\sqrt{2}\rfloor + 1} + \frac{1}{2 \times \lfloor\sqrt{3}\rfloor + 1} + \cdots + \frac{1}{2 \times \lfloor\sqrt{n}\rfloor + 1}$$

is an integer.

(Note that $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x.)

Solution

Observe that each term of the sequence is of the form $\frac{1}{2m+1}$ for some positive integer m and that the terms form a non-increasing sequence. The terms equal to $\frac{1}{2m+1}$ are

$$\frac{1}{2\lfloor\sqrt{m^2}\rfloor + 1}, \frac{1}{2\lfloor\sqrt{m^2+1}\rfloor + 1}, \frac{1}{2\lfloor\sqrt{m^2+2}\rfloor + 1}, \ldots, \frac{1}{2\lfloor\sqrt{(m+1)^2-1}\rfloor + 1}.$$

In particular, there are $(m+1)^2 - m^2 = 2m+1$ terms equal to $\frac{1}{2m+1}$. So the first 3 terms are equal to $\frac{1}{3}$, the next 5 terms are equal to $\frac{1}{5}$, the next 7 terms are equal to $\frac{1}{7}$, and so on. It follows that the sum of the series is an integer if and only if

$$n = 3 + 5 + 7 + \cdots + (2m+1) = (m+1)^2 - 1,$$

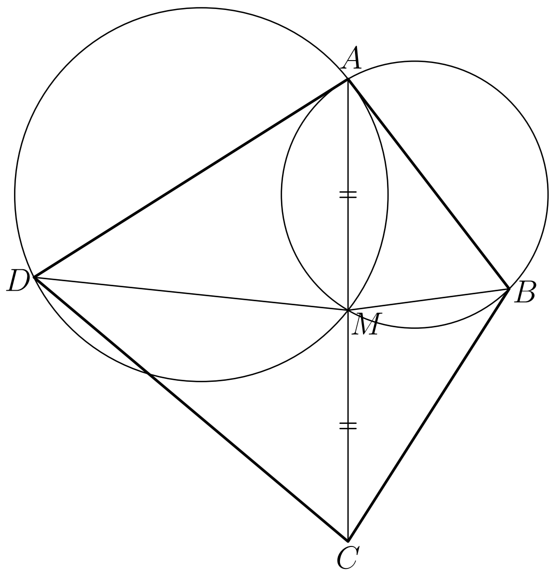
for some positive integer m. Since $1000^2 - 1$ is less than one million while $1001^2 - 1$ is more than one million, we deduce that the answer is $1000 - 1 = 999$.

6. The circles K_1 and K_2 intersect at two distinct points A and M. Let the tangent to K_1 at A meet K_2 again at B and let the tangent to K_2 at A meet K_1 again at D. Let C be the point such that M is the midpoint of AC.

   Prove that the quadrilateral $ABCD$ is cyclic.

Solution 1

Let $\angle MAD = x$ and $\angle MAB = y$. By the alternate segment theorem, we have $\angle ABM = \angle MAD = x$ and $\angle ADM = \angle MAB = y$. Therefore, we have $\triangle AMD \sim \triangle BMA$ and the equal ratios $\frac{MA}{MD} = \frac{MB}{MA}$. Since $MC = MA$, we also have the equal ratios $\frac{MC}{MD} = \frac{MB}{MC}$.



Now since $\angle CMD$ is an external angle of $\triangle AMD$, we have $\angle CMD = \angle MAD + \angle MDA = x+y$. Similarly, since $\angle CMB$ is an external angle of $\triangle AMB$, we have $\angle CMB = \angle MAB + \angle MBA = x + y$. It follows that $\triangle CMD \sim \triangle BMC$.

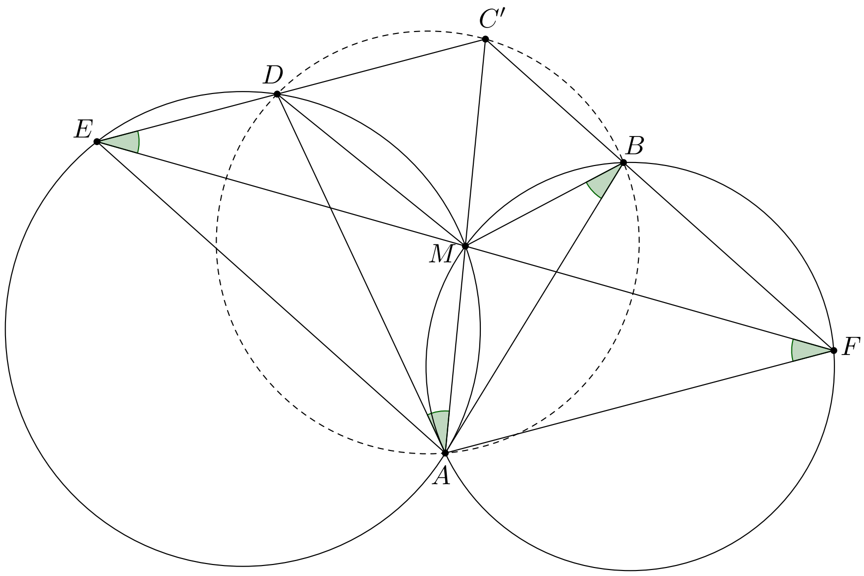
Therefore,

$$\begin{aligned}\angle BCD &= \angle BCM + \angle MCD = \angle BCM + \angle MBC \\ &= 180^\circ - \angle BMC = 180^\circ - (x+y) = 180^\circ - \angle BAD.\end{aligned}$$

So $\angle BAD + \angle BCD = 180^\circ$, from which conclude that $ABCD$ is a cyclic quadrilateral.

Solution 2 (Angelo Di Pasquale)

Let the line AM intersect the circumcircle of triangle ABD again at the point C'. It now suffices to prove that M is the midpoint of AC', as this will establish that $C = C'$.



Let $C'D$ intersect K_1 again at E and let $C'B$ intersect K_2 again at F. Using the cyclic quadrilaterals $AMDE$, $ABC'D$ and $AFBM$, we find that

$$\angle EMA = \angle EDA = \angle C'BA = 180^\circ - \angle ABF = 180^\circ - AMF.$$

Hence E, M and F are collinear. (Alternatively, one can invoke the pivot theorem to deduce that E, M and F are collinear.)

From this, we have

$$\begin{aligned} \angle MFA &= \angle MBA &&(AFMB \text{ cyclic})\\ &= \angle MAD &&(\text{alternate segment theorem})\\ &= \angle MED. &&(AMDE \text{ cyclic}) \end{aligned}$$

Hence, AF is parallel to ED. Similarly, we find that AE is parallel to FB. Hence, $AFC'E$ is a parallelogram. So its diagonals bisect each other and M is the midpoint of AC' as desired.

Solution 3 (Ivan Guo)

Let the midpoints of AD and AB be X and Y, respectively. Since the quadrilaterals $AYMX$ and $ABCD$ are related by a dilation with centre A, it suffices to prove that the quadrilateral $AYMX$ is cyclic. Applying the alternate segment theorem as in the official solution, we know that $\triangle AMD$ is similar to $\triangle BMA$. By the so-called “similar switch” argument, they are also similar to $\triangle XMY$. (This follows from the observation that the spiral symmetry that maps $\triangle AMD$ to $\triangle BMA$ must map X to Y.) Thus, $\angle XYM = \angle DAM = \angle XAM$. It now follows that the quadrilateral $AYMX$ is cyclic.

Solution 4 (Ivan Guo)

Construct X so that M is the midpoint of BX. So $ABCX$ is a parallelogram. By the alternate segment theorem, we have $\triangle AMD \sim \triangle BMA$. By construction $\triangle ACD \sim \triangle BXA$. Thus $\angle ADC = \angle BAX = 180^\circ - \angle ABC$, as required.

Solution 5 (Dan Mathews)

Let $\angle MAD = x$ and $\angle MAB = y$. By the alternate segment theorem, $\angle ADM = y$ and $\angle ABM = x$, so the triangles BAM and ADM are similar. It follows that $\angle AMB = \angle DMA$.

We now claim that the triangles ABD and MCD are similar. As an exterior angle of triangle AMD, we have $\angle CMD = \angle MAD + \angle MDA = x + y = \angle BAD$. Now $\frac{CM}{MD} = \frac{AM}{MD} = \frac{\sin \angle ADM}{\sin \angle DAM} = \frac{\sin y}{\sin x}$. On the other hand, $\frac{BA}{DA} = \frac{BA}{MA} \times \frac{MA}{DA} = \frac{\sin \angle AMB}{\sin \angle MBA} \times \frac{\sin \angle MDA}{\sin \angle DMA} = \frac{\sin \angle MDA}{\sin \angle MBA} = \frac{\sin y}{\sin x}$. So the triangles are similar as claimed.

Hence, $\angle ABD = \angle MCD = \angle ACD$, so the quadrilateral $ABCD$ is cyclic.

Solution 6 (Alan Offer and Chaitanya Rao)

Apply an inversion with centre A and radius AM, so that M is fixed. Let us indicate images under this inversion with a dash so, for instance, the image of D is D'.

The lines through A are fixed, so D' lies on AD and B' lies on AB. Since K_1 is a circle tangent to AB at A and passing through points D and M, the image K_1' is a line parallel to AB passing through D' and M. Similarly, K_2' is a line parallel to AD passing through B' and M. It follows that $AB'MD'$ is a parallelogram.

Since the fixed point M is the midpoint of AC, the point C' is the midpoint of AM. Since $AB'MD'$ is a parallelogram, C' is then also the midpoint of $B'D'$. In particular, B', C' and D' are collinear. Reversing the inversion, the line $B'C'D'$ maps to a circle through B, C, D and the centre A of the involution. Thus, the quadrilateral $ABCD$ is cyclic.

7. There are 1000 athletes standing equally spaced around a circular track of length 1 kilometre.

   (a) How many ways are there to divide the athletes into 500 pairs such that the two members of each pair are 335 metres apart around the track?

   (b) How many ways are there to divide the athletes into 500 pairs such that the two members of each pair are 336 metres apart around the track?

Solution (Ivan Guo)

More generally, we will prove the following result.

> Suppose that there are $2n$ points equally spaced around the circumference of a circle so that the arc length between adjacent points is 1. The number of ways to divide the points into n pairs such that, in each pair, the arc length between the two points is k is
>
> $$\begin{cases} 2^{\gcd(k,n)}, & \text{if } \frac{k}{\gcd(k,n)} \text{ is odd,} \\ 0, & \text{if } \frac{k}{\gcd(k,n)} \text{ is even.} \end{cases}$$

First, consider the case $\gcd(k, n) = 1$. If k is even, then n must be odd. Colour the points alternately red and blue around the circle and observe that a pair of points distance k apart are necessarily the same colour. Since it is impossible for the n blue points to be paired up, the required pairing is not possible. If k is odd, then we join each point with the points distance k away from it. Since $\gcd(k, 2n) = 1$, this produces a cycle of length $2n$. The required pairing consists of alternate edges from this cycle, so there are two such required pairings.

More generally, let $\gcd(k, n) = g$. By considering points which are distance k apart, the problem reduces to g independent problems of the type discussed above, each with $\frac{2n}{g}$ points around the circle that need to be divided into pairs of points distance $\frac{k}{g}$ away from each other. Thus, the required answer is given by the result above.

We may now return to the two specific examples from the problem statement.

(a) Since $\gcd(335, 500) = 5$ and $\frac{335}{5} = 67$ is odd, the answer is $2^5 = 32$.

(b) Since $\gcd(336, 500) = 4$ and $\frac{336}{4} = 84$ is even, the answer is 0.

8. Let $f(x) = x^2 - 45x + 2$.

Find all integers $n \geq 2$ such that exactly one of the numbers

$$f(1), f(2), \ldots, f(n)$$

is divisible by n.

Solution 1 (Angelo Di Pasquale)

The only answer is $n = 2017$.

Note that if $x \equiv y \pmod{n}$, then it follows that $f(x) \equiv f(y) \pmod{n}$. Therefore, we are seeking all n such that $f(x) \equiv 0 \pmod{n}$ has a unique solution modulo n.

Suppose that $f(a) = kn$ for some integer k. Using the quadratic formula, we find that

$$a = \frac{45 \pm \sqrt{2017 + 4kn}}{2}. \tag{1}$$

Hence, $2017 + 4kn$ is an odd perfect square. So if one root of the quadratic is an integer, then so is the other. By the condition of the problem, this implies that

$$\frac{45 + \sqrt{2017 + 4kn}}{2} \equiv \frac{45 - \sqrt{2017 + 4kn}}{2} \pmod{n} \qquad \Rightarrow \qquad 2017 \equiv 0 \pmod{n}.$$

Since 2017 is prime and $n \geq 2$, it follows that $n = 2017$.

Conversely, if $n = 2017$, then the quadratic formula (1) tells us that for a to be an integer, we require $1 + 4k = 2017j^2$ for some odd integer $j = 2i + 1$. Substituting this into the equation yields $a = 1031 + 2017i$ or $a = -986 - 2017i$. So the only such value of a in the required range is $a = 1031$, which corresponds to $i = 0$, $j = 1$ and $k = 504$.

Solution 2 (Angelo Di Pasquale)

Suppose that $f(a) \equiv 0 \pmod{n}$ and observe that $f(45 - a) = f(a)$. Hence, $f(45 - a) \equiv 0 \pmod{n}$ and it follows that

$$a \equiv 45 - a \pmod{n} \qquad \Leftrightarrow \qquad 2a \equiv 45 \pmod{n}.$$

However,

$$\begin{aligned} & a^2 - 45a + 2 \equiv 0 \pmod{n} \\ \Rightarrow \quad & 4a^2 - 180a + 8 \equiv 0 \pmod{n} \\ \Rightarrow \quad & 45^2 - 90 \cdot 45 + 8 \equiv 0 \pmod{n} \\ \Leftrightarrow \quad & 2017 \equiv 0 \pmod{n}. \end{aligned}$$

The third line follows from the second using $2a \equiv 45 \pmod{n}$. Since 2017 is prime and $n \geq 2$, it follows that $n = 2017$.

Let us verify that $n = 2017$ is indeed a valid solution. We do this by showing that the following congruence has exactly one solution modulo 2017.

$$\begin{aligned} & x^2 - 45x + 2 \equiv 0 \pmod{2017} \\ \Leftrightarrow \quad & 4x^2 - 180x + 8 \equiv 0 \pmod{2017} \\ \Leftrightarrow \quad & (2x - 45)^2 \equiv 0 \pmod{2017} \\ \Leftrightarrow \quad & 2x \equiv 45 \pmod{2017} \\ \Leftrightarrow \quad & \equiv 2062 \pmod{2017} \\ \Leftrightarrow \quad & x \equiv 1031 \pmod{2017}. \end{aligned}$$

The fourth line follows from the third since 2017 is square-free. Thus, $x = 1031$ is the unique value in $\{1, 2, \ldots, 2017\}$ such that $f(x)$ is divisible by 2017.