Euler Enrichment Stage

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Chapter 1.
Problems I Like – *Michael Evans*

1. In magic squares, the sum of the numbers in each row, each column and each diagonal is constant. Find A, B, C, D, E in the following magic square.

15	A	35
50	B	C
25	D	E

2. (a) What two whole numbers, neither ending in zero, when multiplied together equal exactly 1 000 000 000?

(b) Repeat for 1 000 000 000 000 000 000.

3. Find three consecutive numbers such that the sum of the first and third is 18.

4. A *palindromic number* is a number which remains the same when the digits are reversed. For example, 14941 is a palindromic number. What is the next largest palindromic number ?

5. What two-digit number is twice the product of its digits ?

6. Find the values of the letters, each of which stands for a particular but different digit.

$$\begin{array}{cccccc} & F & O & R & T & Y \\ + & & & T & E & N \\ + & & & T & E & N \\ \hline & S & I & X & T & Y \end{array}$$

7. By replacing the asterisks with a selection of the four operational symbols $(+, -, \times, \div)$, complete this equation to make a true statement:

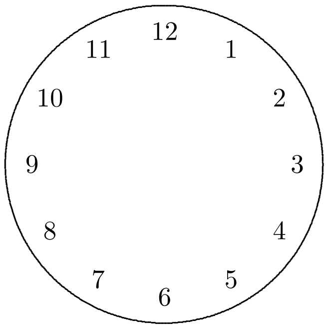
$$6 * 6 * 6 * 6 = 13.$$

8. A prime number is an integer greater than 1 whose only divisors are itself and 1. 1993 is a prime. What is the next year that is a prime?

9. What is the largest three-digit prime each of whose digits is a prime?

10. A perfect number equals the sum of its factors, excluding the number itself. Since $6 = 1+2+3$, 6 is a perfect number. Find another.

11. Divide the face of the clock into 3 parts with 2 straight lines so that the sums of the numbers in the 3 parts are equal.



12. A total of 642 digits was used in numbering the pages of a book. How many pages did the book contain?

13. In the cells shown, place a ten-digit number such that the digit in the first cell indicates the total number of 0s in the entire number, the digit in cell 1 indicates the number of 1s in the number and so on, to the last cell.

0	1	2	3	4	5	6	7	8	9

Now try Problem 1 in the Euler Student Problems Book.

Chapter 2.
Primes and Composites

A code is devised by Secret Sam. Sam has decided that he will denote A by the smallest number with 1 factor, B the smallest number with 2 factors and so on. He sets out his code as shown in the table.

A	1
B	2
C	4
D	6
E	16
⋮	
Z	12288

(a) Complete the table for the code.

(b) Secret Sam decides that he would like to punctuate his code properly and use some mathematical symbols. For example he denotes '=' with the smallest number with 39 factors. What is this number?

Factors and Primes

The factors of 6 are: 1, 2, 3 and 6. The factors of 28 are: 1, 2, 4, 7, 14 and 28. The factors of 7 are: 1 and 7.

A natural number, a, is a *factor* of a natural number, b, if there exists a natural number, k, such that:

$$b = ak.$$

If a number greater than 1 has only factors 1 and itself it is said to be a *prime*. Among the first 100 numbers, the following 25 are prime: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

To consider whether a number, n, is prime one method is to consider all the numbers less than $\sqrt{n}$ and test if they are factors of n.

For example, if $n = 97$, $9 < \sqrt{n} < 10$. Therefore, we could test the numbers: 2, 3, 4, 5, 6, 7, 8 and 9. However, we do not need to test numbers which are not prime, so we only need to test 2, 3, 5 and 7.

It is found that none of these numbers is a factor of 97. We have found 97 to be prime.

There are infinitely many primes.

Composites

A number, m, is called a composite number if it can be written as a product:

$$m = a \times b$$

where a and b are numbers greater than 1 and less than m. For example,

$$\begin{aligned}
72 &= 8 \times 9 = 2^3 \times 3^2 \\
77 &= 7 \times 11 \\
1190 &= 34 \times 35 = 17 \times 2 \times 5 \times 7 \\
2000 &= 16 \times 125 = 2^4 \times 5^3.
\end{aligned}$$

This method of expressing a composite number in terms of its prime numbers is called *prime decomposition*. For each number there is a unique prime decomposition.

One method of determining the prime decomposition is the following. The process is repeated division. Consider the number 24192.

$$\begin{array}{r|r}
2 & 24192 \\ \hline
2 & 12096 \\ \hline
2 & 6048 \\ \hline
2 & 3024 \\ \hline
2 & 1512 \\ \hline
2 & 756 \\ \hline
2 & 378 \\ \hline
3 & 189 \\ \hline
3 & 63 \\ \hline
3 & 21 \\ \hline
7 & 7 \\ \hline
 & 1
\end{array}$$

Therefore $24192 = 2^7 \times 3^3 \times 7$.

The number of factors of a number can be determined through the prime decomposition of that number. They can be systematically listed as shown:

$2^0\times$	$2^1\times$	$2^2\times$	$2^3\times$	$2^4\times$	$2^5\times$	$2^6\times$	$2^7\times$
$3^0\times 7^0$	$3^0\times 7^0$	$3^0\times 7^0$	$3^0\times 7^0$	$3^0\times 7^0$	...	...	...
$3^0\times 7^1$	$3^0\times 7^1$	$3^0\times 7^1$	$3^0\times 7^1$	$3^0\times 7^1$	...	...	...
$3^1\times 7^0$	$3^1\times 7^0$	$3^1\times 7^0$	$3^1\times 7^0$	$3^1\times 7^0$	...	...	...
$3^1\times 7^1$	$3^1\times 7^1$	$3^1\times 7^1$	$3^1\times 7^1$	$3^1\times 7^1$	...	...	...
$3^2\times 7^0$	$3^2\times 7^0$	$3^2\times 7^0$	$3^2\times 7^0$	$3^2\times 7^0$	...	...	...
$3^2\times 7^1$	$3^2\times 7^1$	$3^2\times 7^1$	$3^2\times 7^1$	$3^2\times 7^1$	...	...	...
$3^3\times 7^0$	$3^3\times 7^0$	$3^3\times 7^0$	$3^3\times 7^0$	$3^3\times 7^0$	...	...	...
$3^3\times 7^1$	$3^3\times 7^1$	$3^3\times 7^1$	$3^3\times 7^1$	$3^3\times 7^1$	...	...	...

This covers all combinations. Thus there are $8 \times 4 \times 2 = 64$ factors of 24192.
Alternatively, we note that the prime decomposition of 24192 is $2^7 \times 3^3 \times 7$ and that 2^7 has 8 factors, 3^3 has 4 factors and 7^1 has 2 factors. Hence 24192 has $8 \times 4 \times 2$ factors. This is investigated further in the **Exercises** of this chapter.

Exercises

1. Give the prime decomposition of 362880.

2. Give the prime decomposition of 135135.

3. (a) State the factors of $16 = 2^4$ (note: include 1 and 16).
(b) How many factors are there of 16?
(c) State the factors of $256 = 2^8$.
(d) How many factors are there of 256?
(e) State the number of factors of 2^n?

4. (a) State the factors of $125 = 5^3$.
(b) How many factors are there of 5^n?

5. (a) Give the prime decomposition of 128000.
(b) How many factors does this number have?

6. A number has a prime decomposition of $2^3 \times 3^4 \times 5^3 \times 7^2$. How many factors does this number have?

7. Two and three differ by one and are both prime numbers. What is the next such pair?

8. Find a prime number which is one less than a perfect square. Is there another such number?