Australian Mathematical Olympiad Committee

A DEPARTMENT OF THE AUSTRALIAN MATHEMATICS TRUST

AUSTRALIAN MATHEMATICAL OLYMPIAD 2018

DAY 1
Tuesday, 6 February 2018
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

1. Find all pairs of positive integers (n, k) such that

$$n! + 8 = 2^k.$$

(If n is a positive integer, then $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.)

2. Consider a line with $\frac{1}{2}(3^{100} + 1)$ equally spaced points marked on it.

Prove that 2^{100} of these marked points can be coloured red so that no red point is at the same distance from two other red points.

3. Let $ABCDEFGHIJKLMN$ be a regular tetradecagon.

Prove that the three lines AE, BG and CK intersect at a point.

(A *regular tetradecagon* is a convex polygon with 14 sides, such that all sides have the same length and all angles are equal.)

4. Find all functions f defined for real numbers and taking real numbers as values such that

$$f(xy + f(y)) = yf(x)$$

for all real numbers x and y.

optiver
Official sponsor of the Olympiad program

The Mathematics/Informatics Olympiads are supported by the Australian Government through the National Innovation and Science Agenda.

AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

A DEPARTMENT OF THE AUSTRALIAN MATHEMATICS TRUST

AUSTRALIAN MATHEMATICAL OLYMPIAD 2018

DAY 2
Wednesday, 7 February 2018
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. The sequence $a_1, a_2, a_3, \dots$ is defined by $a_1 = 1$ and, for $n \geq 2$,

$$a_n = (a_1 + a_2 + \cdots + a_{n-1}) \times n.$$

Prove that a_{2018} is divisible by 2018^2.

6. Let P, Q and R be three points on a circle $\mathcal{C}$, such that $PQ = PR$ and $PQ > QR$. Let $\mathcal{D}$ be the circle with centre P that passes through Q and R. Suppose that the circle with centre Q and passing through R intersects $\mathcal{C}$ again at X and $\mathcal{D}$ again at Y.

Prove that P, X and Y lie on a line.

7. Let $b_1, b_2, b_3, \dots$ be a sequence of positive integers such that, for each positive integer n, b_{n+1} is the square of the number of positive factors of b_n (including 1 and b_n). For example, if $b_1 = 27$, then $b_2 = 4^2 = 16$, since 27 has four positive factors: 1, 3, 9 and 27.

Prove that if $b_1 > 1$, then the sequence contains a term that is equal to 9.

8. Amy has a number of rocks such that the mass of each rock, in kilograms, is a positive integer. The sum of the masses of the rocks is 2018 kilograms. Amy realises that it is impossible to divide the rocks into two piles of 1009 kilograms.

What is the maximum possible number of rocks that Amy could have?

AUSTRALIAN MATHEMATICAL OLYMPIAD 2018
Solutions

1. Find all pairs of positive integers (n, k) such that

$$n! + 8 = 2^k.$$

(If n is a positive integer, then $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.)

Solution (Angelo Di Pasquale)

If $n \geq 6$, then $n!$ is divisible by $6! = 720 = 16 \times 45$. It follows that $n! + 8$ is 8 more than a multiple of 16. Therefore, $n! + 8$ is divisible by 8, but not divisible by any larger power of 2. Since $n! + 8$ is larger than 8, it follows that there is no solution with $n \geq 6$.

One can now test all possible values $1 \leq n \leq 5$.

- For $n = 1$, we have $n! + 8 = 1 + 8 = 9$, which is not a perfect power of 2.
- For $n = 2$, we have $n! + 8 = 2 + 8 = 10$, which is not a perfect power of 2.
- For $n = 3$, we have $n! + 8 = 6 + 8 = 14$, which is not a perfect power of 2.
- For $n = 4$, we have $n! + 8 = 24 + 8 = 32$, which is equal to 2^5.
- For $n = 5$, we have $n! + 8 = 120 + 8 = 128$, which is equal to 2^7.

Therefore, the only pairs of positive integers that satisfy the conditions of the problem are $(n, k) = (4, 5)$ and $(5, 7)$.

The Mathematics/Informatics Olympiads are supported by the Australian Government through the National Innovation and Science Agenda.

2. Consider a line with $\frac{1}{2}(3^{100}+1)$ equally spaced points marked on it.

   Prove that 2^{100} of these marked points can be coloured red so that no red point is at the same distance from two other red points.

Solution 1 (Mike Clapper)

First, observe that if k points can be coloured red among m equally spaced points, then $2k$ points can be coloured red among $3m-1$ equally spaced points. This can be done by colouring k of the points among the leftmost m points and k of the points among the rightmost m points. The colouring is valid since among the $3m-1$ points, the leftmost m points are each closer to each other than to any of the rightmost m points.

Second, we use this fact to prove by induction that 2^n points can be coloured red among $\frac{1}{2}(3^n+1)$ equally spaced points, for all positive integers n. Observe that the base case $n=1$ is trivial.

Solution 2 (Alice Devillers, Angelo Di Pasquale, Chaitanya Rao and Jamie Simpson)

We prove a more general fact, replacing $\frac{1}{2}(3^{100}+1)$ with $\frac{1}{2}(3^n+1)$ and 2^{100} with 2^n in the original problem statement.

Take the points to be the integers $0, 1, 2, \ldots, \frac{3^n-1}{2}$ on the real line, so that the distance from x to y is just $|y-x|$. Now consider these integers written in base 3. They all have at most n digits, since the last one is $\frac{3^n-1}{2} = \frac{3^n-1}{3-1} = 1+3+3^2+\cdots+3^{n-1}$.

Take S to be the set of points whose corresponding numbers contain only 0s and 1s in base 3. All such numbers with at most n digits are in the set, so S has size 2^n. We colour the points in S red and show that no point in S is equidistant from two other points in S.

Suppose for a contradiction that there exist $x, a, b \in S$ such that x is equidistant from a and b with $a \neq b$. Then without loss of generality, we have $a < x < b$ and $x-a = b-x$, so $2x = a+b$. Now since $x \in S$, $2x$ only has digits 0 and 2 in base 3. On the other hand, a and b only have digits 0 and 1, so there is no carry anywhere when adding them up. Since $a \neq b$, there is a position where one has a digit 1 and the other has a digit 0, but then $a+b$ has a digit 1 in that position, a contradiction.

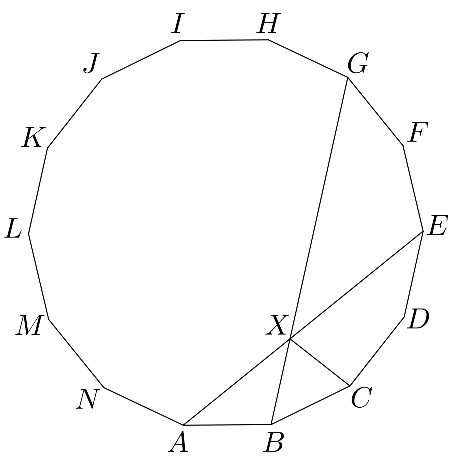
3. Let $ABCDEFGHIJKLMN$ be a regular tetradecagon.

   Prove that the three lines AE, BG and CK intersect at a point.

   (A *regular tetradecagon* is a convex polygon with 14 sides, such that all sides have the same length and all angles are equal.)

Solution 1 (Angelo Di Pasquale)

Let AE intersect BG at X. It suffices to prove that K, X and C are collinear. Let $\mathcal{P}$ denote the regular 14-sided polygon and set $\theta = \frac{1}{14} \times 180°$.



The angle subtended by a side of $\mathcal{P}$ with the centre of $\mathcal{P}$ is $\frac{1}{14} \times 360° = 2\theta$. It follows that the angle subtended by a side of $\mathcal{P}$ with any other vertex of $\mathcal{P}$ is equal to θ. From this information, we compute that

$$\angle KCB = 5\theta, \qquad \angle CBG = 4\theta, \qquad \angle BAE = 3\theta.$$

We also have

$$\angle AXB = \angle AEB + \angle EBG = \theta + 2\theta = 3\theta.$$

So $\triangle ABX$ is isosceles with $AB = BX$, so $BC = AB = BX$. Since $\angle CBX = \angle CBG = 4\theta$, and $14\theta = 180°$, we have

$$\angle BXC = \angle XCB = 5\theta.$$

Since $\angle XCB = \angle KCB$, it follows that K, X and C are collinear.

Solution 2 (Angelo Di Pasquale)

Another way to see that $\angle ACB = 3\theta$ is to note that $AX \parallel MG$, and so $\angle AXB = \angle MGB = 3\theta$.

Solution 3 (Ivan Guo)

Consider triangle ACG. Since

$$\angle KCA = \angle KCG = 5\theta, \quad \angle EAG = \angle EAC = 2\theta, \quad \angle BGC = \angle BGA = \theta,$$

the required concurrency follows from the trigonometric form of Ceva's theorem.

Solution 4 (Daniel Mathews)

First, we claim that AE is the angle bisector of AC and AG. To see this, note that CEG is isosceles, with $\angle ECG = \angle EGC$; and as $ACEG$ is cyclic, then $\angle EAG = \angle EAC$.

Second, we similarly claim that GB is the angle bisector of GC and GA. To see this, note that ABC is isosceles, so $\angle BAC = \angle BCA$; as $ABCG$ is cyclic, then $\angle BGC = \angle BGA$.

Third, we again similarly claim that CK is the angle bisector of CA and CG. To see this, note that GKA is isosceles, with $\angle KGA = \angle KAG$; as $ACGK$ is cyclic, then $\angle KCA = \angle KCG$.

Thus, AE, BG, CK are the angle bisectors of the triangle ACG. So they are concurrent at the incentre of this triangle.

Solution 5 (Alan Offer)

Since $AB = BC$, $CE = EG$ and $GK = KA$, this follows immediately from the theorem that the diagonals of a cyclic hexagon are concurrent if and only if the product of alternate sides is equal to the product of the other three sides.

Solution 6 (Alan Offer)

Using lower case for complex numbers representing the points with the corresponding upper case letter, let $a = 1$ and $b = e^{\pi i/7}$. Then $c = b^2$, $e = b^4$, $g = b^6$ and $k = b^{10} = -b^3$. Also, notice that $b^7 = -1$.

Since the points A, X and E are collinear, we have $(a-x)(\overline{a-e}) = (\overline{a-x})(a-e)$. With a little algebra, this is found to be equivalent to

$$x + b^4\bar{x} = 1 + b^4. \tag{1}$$

Similarly, since B, X and G are collinear, we have $(b-x)(\overline{b-g}) = (\overline{b-x})(b-g)$ which gives

$$x - \bar{x} = b + b^6. \tag{2}$$

To show that C, X and K are collinear, we must show that $(c-x)(\overline{c-k}) = (\overline{c-x})(c-k)$, or equivalently, that $x - \bar{x} = b^2 - b^3$.

Let $u = b + b^4$ and consider adding u times equation (2) to $(1-u)$ times equation (1). The resulting left hand side is

$$\begin{aligned} u(x-\bar{x}) + (1-u)(x + b^4\bar{x}) &= x - (u - b^4 + b^4u)\bar{x} \\ &= x - (b + b^4 - b^4 + b^5 + b^8)\bar{x} \\ &= x - b^5\bar{x}. \end{aligned}$$

The resulting right hand side is

$$\begin{aligned} u(b + b^6) + (1-u)(1 + b^4) &= u(b + b^6 - 1 - b^4) + 1 + b^4 \\ &= b^2 - 1 - b - b^5 + b^5 - b^3 - b^4 + b + 1 + b^4 \\ &= b^2 - b^3. \end{aligned}$$

Hence, $x - \bar{x} = b^2 - b^3$. As noted above, it follows that C, X and K are collinear.

4. Find all functions f defined for real numbers and taking real numbers as values such that

$$f(xy + f(y)) = yf(x)$$

for all real numbers x and y.

Solution 1 (Angelo Di Pasquale)

There are two such functions: $f(x) = 0$ for all $x \in \mathbb{R}$ and $f(x) = 1 - x$ for all $x \in \mathbb{R}$.

The function $f(x) = 0$ for all $x \in \mathbb{R}$ is obviously a solution. Now suppose that there exists a such that $f(a) \neq 0$. Then since $f(ay + f(y)) = yf(a)$, and $yf(a)$ covers all real numbers, we conclude that f is surjective.

Let $x = 0$ to find

$$f(f(y)) = yf(0). \tag{1}$$

Since f is surjective, so is $f \circ f = yf(0)$. Thus, $f(0) \neq 0$. Therefore, $f \circ f$ is injective and it follows that f is also injective.

Put $y = 1$ in (1) to get $f(f(1)) = f(0)$. Thus, $f(1) = 0$, since f is injective.

Put $x = 1$ in the original functional equation to find $f(y + f(y)) = 0 = f(1)$. Since f is injective, this implies $y + f(y) = 1$, and so $f(y) = 1 - y$.

Finally, we verify that $f(y) = 1 - y$ satisfies the given functional equation via the check

$$\text{LHS} = 1 - (xy + (1 - y)) = y(1 - x) = \text{RHS}.$$

Solution 2 (Angelo Di Pasquale)

As in Solution 1, we deduce equation (1) and the fact that f is bijective. Replacing y by $f(y)$ in the given functional equation, using (1), and then using symmetry yields

$$f(xf(y) + yf(0)) = f(y)f(x) = f(yf(x) + xf(0)).$$

Since f is injective, we deduce

$$xf(y) + yf(0) = yf(x) + xf(0). \tag{2}$$

Put $y = 1$ in (2) to find $f(x) = x(f(1) - f(0) + f(0)$. Hence, $f(x) = ax + b$ for real constants a and b. Putting this into the given functional equation yields

$$a(xy + ay + b) + b = y(ax + b) \quad \Leftrightarrow \quad (a^2 - b)y + b(a + 1) = 0,$$

for all real numbers y. Thus, $b = a^2$ and $b(a + 1) = 0$. Note that $b = 0$ implies $a = 0$, which yields $f(x) = 0$, a solution we have already considered. Otherwise, $a = -1$, and $b = 1$, which yields $f(x) = 1 - x$. As in Solution 1, we may verify this is also a solution to the functional equation.

Solution 3 (Alice Devillers and Angelo Di Pasquale)

Taking $y = 0$ yields

$$f(f(0)) = 0. \tag{3}$$

Taking $y = f(0)$ and $x = 0$ and using (3) yields $f(0) = f(0)^2$, so $f(0)$ is 0 or 1.

Suppose $f(0) = 0$. Taking $x = 0$ in the original equation, we get $f(f(y)) = 0$ for all y. Then applying f to the original equation, we get

$$0 = f(yf(x)). \tag{4}$$

If there exists a with $f(a) \neq 0$, take $x = a$ and $y = af(a)^{-1}$ in (4) to get that $f(a) = 0$, a contradiction. So f is 0 everywhere.

Suppose now $f(0) = 1$. Taking $x = 0$ in the original equation, we get

$$f(f(y)) = y. \tag{5}$$

Thus $0 = f(f(0)) = f(1)$. Taking $x = 1$ in the original equation, we get

$$f(y + f(y)) = yf(1) = 0. \tag{6}$$

Applying f to (6) and using (5), we get $y + f(y) = f(0) = 1$. Hence $f(y) = 1 - y$.

We can easily check that these two functions satisfy the functional equation.

Solution 4 (Chaitanya Rao)

We divide the problem into the two cases: $f(0) = 0$ and $f(0) \neq 0$.

- If $f(0) = 0$, setting $x = 0$ in the functional equation gives $f(f(y)) = 0$ for all y. Then applying f to both sides of the original equation gives $0 = f(yf(x))$. Assuming f is non-zero then x can be chosen so that $yf(x)$ spans all real values z and we conclude $f(z) = 0$, contradicting our assumption. We conclude $f = 0$ in this case and this indeed satisfies the functional equation.
- Now suppose $f(0) \neq 0$. Then setting $x = 0$, $f(f(y)) = yf(0)$. If $a \neq b$ then $af(0) \neq bf(0)$ from which $f(f(a)) \neq f(f(b))$. We conclude $f(a) \neq f(b)$ and so f is injective. The rest is as in the last part of Solution 1.

Solution 5 (Angelo Di Pasquale)

If f is not identically zero, let c satisfy $f(c) \neq 0$. Putting $x = c$ shows that the right side of the functional equation covers all of $\mathbb{R}$. Hence f is surjective.

Let $k = f(0)$. Putting $x = 0$ yields $f(f(y)) = ky$ for all $y \in \mathbb{R}$. If $k = 0$, then $f(f(y)) = 0$ for all $y \in \mathbb{R}$. But this impossible as f is surjective. So $k \neq 0$.

Replacing y with $f(y)$ in the given functional equation and using $f(f(y)) = ky$ yields

$$f(xf(y) + ky) = f(y)f(x).$$

Applying f to both sides of the above and using symmetry (swapping x and y) yields

$$kxf(y) + k^2y = f(f(y)f(x)) = f(f(x)f(y)) = kyf(x) + k^2x.$$

Dividing by k and setting $y = 1$ shows that f is linear. Thus, $f(x) = ax + b$ for constants a, b. Putting this into the given functional equation and equating coefficients yields the two solutions $f(x) = 0$ and $f(x) = 1 - x$.

5. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 1$ and, for $n \geq 2$,

$$a_n = (a_1 + a_2 + \cdots + a_{n-1}) \times n.$$

Prove that a_{2018} is divisible by 2018^2.

Solution 1 (Norman Do)

We will prove that $a_n = \frac{n}{2} \times n!$ for all $n \geq 2$. Let $b_n = a_1 + a_2 + \cdots + a_n$, so that the equation defining the sequence becomes

$$b_n - b_{n-1} = b_{n-1} \times n.$$

Rearranging, we obtain the equation $b_n = (n+1)b_{n-1}$ for $n \geq 2$.

We can repeatedly apply this equation to obtain

$$b_n = (n+1)b_{n-1} = (n+1)nb_{n-2} = (n+1)n(n-1)b_{n-3} = \cdots = (n+1)n(n-1)\cdots 3b_1 = \frac{1}{2}(n+1)!.$$

It follows that, for $n \geq 2$,

$$a_n = b_n - b_{n-1} = \frac{1}{2}(n+1)! - \frac{1}{2}n! = \frac{n}{2} \times n!.$$

In particular, $a_{2018} = \frac{2018}{2} \times 2018! = 2018^2 \times \frac{2017!}{2}$, which is divisible by 2018^2.

Solution 2 (Alice Devillers, Kevin McAvaney and Alan Offer)

By definition, n divides a_n. We need to show that n also divides $\frac{a_n}{n}$. For all $n > 2$, and in particular for $n = 2018$,

$$\begin{aligned}\frac{a_n}{n} &= a_1 + a_2 + \cdots + a_{n-2} + a_{n-1}\\ &= a_1 + a_2 + \cdots + a_{n-2} + (a_1 + a_2 + \cdots + a_{n-2}) \times (n-1)\\ &= (a_1 + a_2 + \cdots + a_{n-2}) \times n.\end{aligned}$$

Solution 3 (Angelo Di Pasquale)

We prove that $a_n = \frac{n}{2} \times n!$ by induction. The base case $n = 2$ is true. If it is true for $2, 3, \ldots, n-1$, then

$$\begin{aligned}a_n &= n\sum_{i=1}^{n-1} a_i = n\left(1 + \sum_{i=2}^{n-1} \frac{i \cdot i!}{2}\right)\\ &= n\left(1 + \frac{1}{2}\sum_{i=2}^{n-1}(i+1)! - i!\right) = n\left(1 + \frac{1}{2}(n! - 2!)\right) = \frac{n \cdot n!}{2}.\end{aligned}$$

Solution 4 (Angelo Di Pasquale)

We are given

$$a_n = (a_1 + a_2 + \cdots + a_{n-1}) \times n. \qquad (1)$$

Replacing n with $n+1$ in (1) yields

$$a_{n+1} = (a_1 + a_2 + \cdots + a_n) \times (n+1). \tag{2}$$

Computing $n \times (2) - (n+1) \times (1)$ yields for any $n \geq 2$,

$$\begin{aligned} na_{n+1} - (n+1)a_n &= n(n+1)a_n \\ \Rightarrow \qquad a_{n+1} &= \frac{(n+1)^2}{n} a_n. \end{aligned} \tag{3}$$

Applying (3) recursively yields, for any $n \geq 2$,

$$a_{n+1} = \frac{(n+1)^2}{n} \times \frac{n^2}{n-1} \times \cdots \times \frac{3^2}{2} \times a_2 = \frac{(n+1) \cdot (n+1)!}{2}. \tag{4}$$

Using formula (4), we finish as in Solution 1.

Solution 5 (Ivan Guo)

It is clear from induction that each term of the sequence $a_1, a_2, a_3, \ldots$ is an integer, which implies that $n \mid a_n$. Defining the integers $b_n = \frac{a_n}{n}$, we have the equation

$$b_n = b_1 + 2b_2 + \cdots + (n-2)b_{n-2} + (n-1)b_{n-1} = b_{n-1} + (n-1)b_{n-1} = nb_{n-1}.$$

Thus $n \mid b_n$, which implies that $n^2 \mid a_n$.

Solution 6 (Daniel Mathews and Ian Wanless)

Since $a_1 + a_2 + \cdots + a_{n-2} = \frac{a_{n-1}}{n-1}$ for $n \geq 3$, we may rearrange the given equation as

$$a_n = \left(\frac{a_{n-1}}{n-1} + a_{n-1} \right) \times n = \frac{n^2 a_{n-1}}{n-1}.$$

Since n and $n-1$ are relatively prime, we know that a_n is divisible by n^2 for all $n \geq 3$. So a_{2018} is divisible by 2018^2.

Solution 7 (Jamie Simpson)

Set $S_n = a_1 + a_2 + \cdots + a_n$ so that $a_n = nS_{n-1}$ for $n \geq 2$. Then $S_n = S_{n-1} + a_n = (n+1)S_{n-1}$, so that $n+1$ divides S_n for all $n \geq 2$. Therefore $(n+1)^2$ divides $a_{n+1} = (n+1)S_n$ for all $n \geq 2$. In particular 2018^2, divides S_{2018}.

Solution 8 (Mike Clapper)

For $n \geq 2$, $a_{n+1} = (a_1 + a_2 + \cdots + a_n) \times (n+1)$. So we also have $a_n = (a_1 + a_2 + \cdots + a_{n-1}) \times n$. Subtracting these two equations yields

$$a_{n+1} - a_n = (n+1)a_n + (a_1 + a_2 + \cdots + a_{n-1}).$$

So we have

$$a_{n+1} = \left(n + 2 + \frac{1}{n} \right) a_n = \frac{(n+1)^2}{n} a_n.$$

Since n and $n+1$ are relatively prime, we know that $(n+1)^2 \mid a_{n+1}$. In particular, $2018^2 \mid a_{2018}$.

6. Let P, Q and R be three points on a circle $\mathcal{C}$, such that $PQ = PR$ and $PQ > QR$. Let $\mathcal{D}$ be the circle with centre P that passes through Q and R. Suppose that the circle with centre Q and passing through R intersects $\mathcal{C}$ again at X and $\mathcal{D}$ again at Y.

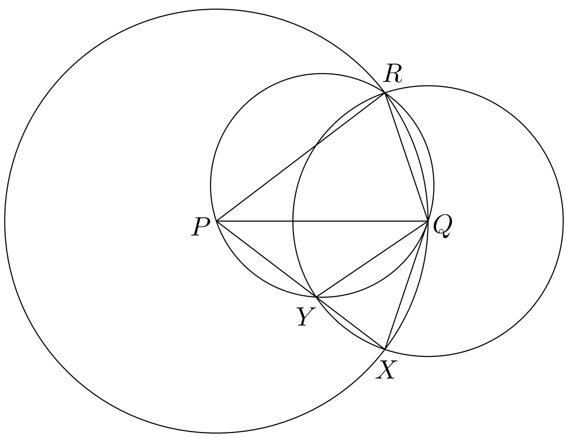
   Prove that P, X and Y lie on a line.

Solution 1 (Angelo Di Pasquale)

In the following, we consider all angles to be directed. The result follows immediately from

$$\angle YPQ = \angle QPR = \angle XPQ.$$

The first equality is due to $QR = QY$, and so these equal chords subtend equal angles in $\mathcal{C}$. The second is due to $QR = QX$, and so these equal chords subtend equal angles at the centre of $\mathcal{D}$.



Solution 2 (Alice Devillers and Daniel Mathews)

We need to prove that $\angle QPX = \angle QPY$.

- Since Q, Y, P, R are concyclic, $\angle QPY = \angle QRY$ and $\angle QPR = \angle QYR$.
- Since the triangle QRY is isoceles, $\angle QRY = \angle QYR$.
- Since the triangles QPR and QPX are congruent (SSS), $\angle QPX = \angle QPR$.

Putting it all together, we obtain

$$\angle QPY = \angle QRY = \angle QYR = \angle QPR = \angle QPX.$$

Solution 3 (Alan Offer)

We prove the claim in the more general situation that the centre of the third circle, which we call $\mathcal{K}$, need not be Q, but is free to be any point O on the circle $\mathcal{D}$.

Let Z and P' be the points distinct from X and Y where the line XY meets the circles $\mathcal{C}$ and $\mathcal{D}$, respectively.

Then

$$\begin{aligned}
\angle RP'Z &= \angle RP'Y && \text{since } P', Y, Z \text{ are collinear,}\\
&= \angle ROY && \text{since } R, Y, P', O \text{ are on } \mathcal{D},\\
&= 2\angle RXY && \text{since } O \text{ is the centre of } \mathcal{K},\\
&= 2\angle RXZ && \text{since } X, Y, Z \text{ are collinear,}\\
&= \angle RPZ && \text{since } P \text{ is the centre of } \mathcal{C}.
\end{aligned}$$

Therefore, R, Z, P and P' are concyclic, but since a circle through R and Z can meet $\mathcal{D}$ in at most one point distinct from R, it follows that $P = P'$. Hence X, P and Y are collinear.

Solution 4 (Alan Offer)

We prove the claim in the more general situation that the centre of the third circle, which we call $\mathcal{K}$, need not be Q, but is free to be any point O on the circle $\mathcal{D}$.

Apply an inversion about a circle with centre R and let P', O', X' and Y' be the images of P, O, X and Y, respectively. Since X is on $\mathcal{C}$, the point X' is on the perpendicular bisector of RP', and since X and Y are on $\mathcal{K}$, both X' and Y' lie on the perpendicular bisector of RO', with Y' also lying on $P'O'$, which is the image of $\mathcal{D}$.

Since X is on the perpendicular bisectors of RP' and RO', the triangles XRO' and XRP' are isosceles, so X' is the centre of the circle passing through R, O' and P'. Thus $\angle RP'O' = \frac{1}{2}\angle RX'O' = \angle RX'Y'$. It follows that R, P', X' and Y' are concyclic, which under the inversion, shows that P, X and Y are collinear.

7. Let $b_1, b_2, b_3, \ldots$ be a sequence of positive integers such that, for each positive integer n, b_{n+1} is the square of the number of positive factors of b_n (including 1 and b_n). For example, if $b_1 = 27$, then $b_2 = 4^2 = 16$, since 27 has four positive factors: 1, 3, 9 and 27.

   Prove that if $b_1 > 1$, then the sequence contains a term that is equal to 9.

Solution 1 (Angelo Di Pasquale)

For any perfect square m^2, there is a bijection $d \leftrightarrow \frac{m^2}{d}$ between factors strictly less than m and the factors strictly greater than m. Taking into acount that $m \mid m^2$ too, we deduce that all square numbers have an odd number of divisors. Thus b_i is an odd perfect square for $i \geq 3$.

Suppose that $i \geq 3$ and let $b_i = \prod p_i^{2k_i}$ be the prime factorisation of b_i. Then $b_{i+1} = \prod(2k_i + 1)^2 \geq 9$. For any prime $p \geq 3$, we have

$$p^{2k} \geq (2k+1)^2, \tag{1}$$

with equality if and only if $p = 3$ and $a = 1$ — we will prove this statement below. It follows that $b_i \geq b_{i+1}$ with equality if and only if $b_i = 9$. Thus, from b_3 onward, the sequence is strictly decreasing until it reaches the fixed point $b_i = 9$.

Finally, we present two different ways of justifying equation (1).

Way 1. If $a \geq 2$, then using the binomial theorem we have

$$\begin{aligned} p^{2k} \geq 3^{2k} = (1+8)^k &\geq 1 + \binom{k}{1} \cdot 8 + \binom{k}{2} \cdot 8^2 \\ &= 32k^2 - 24k + 1 > 4k^2 + 4k + 1 = (2k+1)^2. \end{aligned}$$

Moreover, the claim is trivial if $k = 1$.

Way 2. The claim is equivalent to $p^k \geq 2k+1$, with equality if and only if $p = 3$ and $k = 1$. This is obviously true for $k = 1$. Assume inductively that the claim is true for k. Then we have

$$p^{k+1} = p \cdot p^k \geq 3(2k+1) > 2(k+1) + 1.$$

Hence the result follows by induction on k.

Solution 2 (Ivan Guo)

Another way to prove that $p^{2k} \geq (2k+1)^2$ is to write it as $p^k \geq (2k+1)$, which is true since

$$p^k - 1 \geq 3^k - 1 = (3-1)(3^{k-1} + 3^{k-2} + \cdots + 1) \geq 2(1 + 1 + \cdots + 1) = 2k.$$

8. Amy has a number of rocks such that the mass of each rock, in kilograms, is a positive integer. The sum of the masses of the rocks is 2018 kilograms. Amy realises that it is impossible to divide the rocks into two piles of 1009 kilograms.

What is the maximum possible number of rocks that Amy could have?

Assumptions and terminology. In all of the following solutions, we will assume that masses are in kilograms and that rocks have positive integer mass. We call a collection of rocks *balanced* if it is possible to divide the rocks into two piles of equal mass.

Solution 1 (Angelo Di Pasquale)

The answer is 1009.

If one of the rocks has mass 1010 and the remaining 1008 rocks each have mass 1, then it is obvious that this collection of 1009 rocks is unbalanced, and yet the sum of the masses is 2018. (Another possibility is that Amy has 1009 rocks each of mass 2.)

We show that if Amy has 1010 rocks, then they must be balanced, by proving the following more general statement.

> Let n be a positive integer. Any collection of at least $n+1$ rocks with total mass $2n$ is balanced.

The proof is by strong induction on n. The result is easily verified for $n = 1$.

Suppose that the result is true for all positive integers m with $m < n$, and suppose we have a collection of at least $n+1$ rocks whose total mass is $2n$. Suppose that exactly k of the rocks have mass 1. Note that $k \geq 2$. If all of the rocks have mass 1, then the result is trivial. Suppose that there is a rock of mass $w > 1$. Note that $w \leq 2n-2$. The sum of all the rocks is at least $k \times 1 + w + (n-k) \times 2$. But the sum of the rocks is equal to $2n$. So we have

$$2n \geq k + w + 2(n-k) \qquad \Leftrightarrow \qquad k \geq w. \tag{1}$$

Case 1. Suppose $w = 2x$ for some positive integer x.

Remove the rock of mass w to leave at least n rocks whose total mass is $2n - 2x$. The inductive assumption with $m = n - x$ tells us that the remaining rocks can be divided into two piles, each of mass $n - x$. As $k \geq 2x$ from (1), one of those piles contains at least x unit rocks. Moving x of these unit rocks from one pile to the other yields piles of masses $n - 2x$ and n. Adding back the rock of mass w to the lighter pile yields the result.

Case 2. Suppose $w = 2x + 1$ for some positive integer x.

Remove the rocks of mass w and 1 to leave at least $n - 1$ rocks whose total mass is $2n - 2x - 2$. The inductive assumption with $m = n - x - 1$ tells us that the remaining rocks can be divided into two piles of mass $n - x - 1$. As $k \geq 2x + 1$ from (1), one of those piles has at least x unit masses. Moving x of these unit masses from one pile to the other yields piles of masses $n - 2x - 1$ and $n - 1$. Adding back the rock of mass w to the lighter pile, and the unit mass to the heavier pile yields the result.

Solution 2 (Angelo Di Pasquale)

As in Solution 1, we can find a collection of 1009 rocks that is unbalanced. We show that any collection of 1010 rocks is balanced.

Lemma. Any sequence of n positive integers contains a non-empty subsequence whose sum is a multiple of n.

Proof. (This result is folklore.) Let the integers be $a_1, a_2, \ldots, a_n$. Let $s_i = a_1+a_2+\cdots+a_i$ for $1 \le i \le n$. If $s_i \equiv 0 \pmod{n}$ for some i, then we are done. Otherwise, by the pigeonhole principle, $s_i \equiv s_j \pmod{n}$ for some $i < j$. Thus $s_j - s_i = a_{i+1}+a_{i+2}+\cdots+a_j$ is a multiple of n.

For the problem at hand, remove a rock from the collection. Now we have 1009 rocks. By the lemma, a non-empty subcollection of those rocks has mass equal to a multiple of 1009. Since the total mass of the sub-collection is strictly less than 2018, it is equal to 1009.

Solution 3 (Angelo Di Pasquale and Ian Wanless)

We prove that an unbalanced collection of total mass $2n$ has at most n rocks.

Note that if one rock has mass n, then the collection is balanced. And if any rock has mass greater than n, then the collection automatically has at most n rocks. Therefore we may restrict ourselves to the case where all rocks have masses at most $n-1$.

Let A and B be two buckets, each of which can contain a total mass of at most n. Suppose that Amy starts putting the rocks in the buckets as follows. At each stage she chooses the heaviest remaining rock, and if it can fit in one of the buckets, she puts it in.

Since Amy's collection is unbalanced she will eventually reach a situation where she is holding in her hand the heaviest remaining rock, say of mass w where $2 \le w \le n-1$, and it cannot be put in either bucket.

Suppose that A contains j rocks of masses $a_1, a_2, \ldots, a_j$, and B contains k rocks of masses $b_1, b_2, \ldots, b_k$. From Amy's algorithm, we have

$$a_1+a_2+\cdots+a_j+w \ge n+1 \quad \text{and} \quad b_1+b_2+\cdots+b_k+w \ge n+1.$$

Hence, the sum of the masses of the rocks in the buckets and in Amy's hand is at least $2n+2-w$. Since the total mass is $2n$, there are at most $w-2$ rocks remaining that are not in a bucket nor in Amy's hand. Hence, the total number of rocks in Amy's collection is at most $j+k+w-1$. We claim that this number is at most n.

From Amy's algorithm, we also have $jw \le a_1+a_2+\cdots+a_j \le n-1$. Similarly $kw \le n-1$. Hence $j+k \le \frac{2(n-1)}{w}$. Hence, it suffices to show

$$\frac{2(n-1)}{w}+w-1 \le n \quad \Leftrightarrow \quad (w-2)(w-(n-1)) \le 0.$$

Since the final inequality is true for $2 \le w \le n-1$, we are done.

Solution 4 (Ivan Guo)

The example for 1009 is as in Solution 1. We focus on showing that any collection of 1010 rocks is balanced. Let the heaviest rock be K. We must have $1 < K \le 1009$. If there are $n \le 1009$ unit mass rocks, then $2018-K-n \ge 2(1009-n)$, which implies $n \ge K$. Now consider the set S of rocks with mass greater than 1. Let us take rocks from S one by one and place them in a pile, while tracking the total mass of the pile. The total mass starts at 0 and finishes at $2018 - n \geq 1009$. Since no rock has more than K mass, the sequence of total mass must have at least one value in the range $r \in [1009 - K, 1009]$. If we pause when the pile reaches r, we can always add $1009 - r < K$ unit mass rocks to reach a pile of mass 1009, as required.

Solution 5 (Daniel Mathews)

We show that a collection of $k + 1$ rocks with total mass $2k$ must be balanced. Let the rocks have masses $w_1, w_2, \ldots, w_{k+1}$. Consider the $k + 1$ collections of rocks

$$\{w_1\}, \{w_1, w_2\}, \{w_1, w_2, w_3\}, \ldots, \{w_1, w_2, \ldots, w_{k+1}\}.$$

These collections all have distinct total mass lying between 1 and $2k$ inclusive. Partition the numbers $\{1, 2, \ldots, 2k\}$ into k pairs $\{1, k+1\}, \{2, k+2\}, \ldots, \{k, 2k\}$. By the pigeonhole principle, any $k + 1$ numbers from $1, 2, \ldots, 2k$ must contain both numbers from at least one of these pairs. The total masses of our $k + 1$ collections are $k + 1$ numbers from 1 to $2k$, hence there exist two collections of rocks $\{w_1, w_2, \ldots, w_i\}$ and $\{w_1, w_2, \ldots, w_j\}$ (with $i < j$) such that $w_1 + w_2 + \cdots + w_i = w_1 + w_2 + \cdots + w_j + k$. Thus, $w_{i+1} + w_{i+2} + \cdots + w_j = k$, and the collection of rocks $\{w_{i+1}, w_{i+2}, \ldots, w_j\}$ has total mass k, exactly half the total mass of all the rocks. So the rocks are balanced.