Pólya Enrichment Stage

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Chapter 1: Functions

> Let $f : [0,1] \to [0,1]$ be a continuous function. Prove that there exists a number c in $[0,1]$ such that $f(c) = c$.

What is a function?

You may have heard several possible definitions of a function. We'll discuss these, in order to clarify what we're talking about!

A function might be described in the following ways. But some of these descriptions are more accurate than others, and some are more correct than others!

A *function* $f : X \to Y$ is ...

(i) a rule which takes an element of X and gives you an element of Y

(ii) a 'machine' into which you can feed an input from X, and which gives an output from Y

(iii) a formula which, given an element x of X, tells you what $f(x)$ is, like $f(x) = x^2$

(iv) a graph of $y = f(x)$

(v) a set of pairs of elements (x, y) where $x \in X$ and $y \in Y$.

Some of these notions are more rigorous than others! Let's see how they stack up.

A function always has a *domain* and a *codomain*. Letting the function be f, the domain A and the codomain B, we write $f : A \to B$.

We can define a function with domain A and codomain B as follows.

> *A function $f : A \to B$ is a rule which associates, to each element a of A, a unique element of B.*

The element of B which is associated to $a \in A$ is denoted by $f(a)$.

In this way you can think of a function f as a 'machine' to which you can feed an input a from the set A, and the 'machine' gives you as output $f(a)$, which is an element of B. But it is a 'predictable' machine because if you feed it the input a you always get the same output $f(a)$!

Here's a function. Let $A = \{\text{Blue}, \text{Horse}, \pi\}$ and $B = \{0, \sqrt{2}, \text{Love}\}$. We can define a function $f : A \to B$ by declaring that

$$f(\text{Blue}) = 0, \quad f(\text{Horse}) = 0, \quad f(\pi) = \text{Love}.$$

This is not a very meaningful function—although we might love π! It's not likely to be useful in many applications. And there's no nice formula to write down for f. Nonetheless, it's a function, and we've defined it completely.

Here's another function. Let $M = \{\text{Mirzakhani}, \text{Tao}, \text{Villani}\}$ and let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers. We can define a function $f : M \to \mathbb{N}$ by

$$f(\text{Mirzakhani}) = 2014, \quad f(\text{Tao}) = 2006, \quad f(\text{Villani}) = 2010.$$

Again, this is a well-defined function. It's a bit more meaningful than the previous function, because the elements of M are all surnames of mathematicians who won the Fields medal. There's a 'formula' for the function f, given by

$$f(x) = \begin{array}{c}\text{year in which the mathematician} \\ \text{with surname } x \text{ won a Fields medal}\end{array}$$

This 'formula' doesn't involve any algebra, but it is a simple description of how to get the output from the input, and so it shows you what the 'machine' does.

More usually in school mathematics, the domain and codomain of a function are the *real numbers* $\mathbb{R}$ or subsets of the real numbers. And the functions usually have relatively simple formulas that you can write down with some algebra. But in fact the domain and codomain can be any sets whatsoever, and there doesn't have to be any nice 'formula' describing the function.

When a function does have a domain and codomain which consist of real numbers, we can represent it by *drawing its graph*. When we do this, we plot all the points (x, y) such that $y = f(x)$ in the Cartesian plane.

Actually, if we get into the really abstract, axiomatic pure mathematics of how functions are defined, all we start from are *sets*. It turns out that, at this level, the best way to describe a function is *as the set* $\{(x, y) : y = f(x)\}$. That is, from a pure mathematical point of view, a function is a *collection of ordered pairs*, where each pair (x, y) refers to one value of the function, $f(x) = y$. We'll discuss this idea further below.

In summary, a function $f : X \to Y$ can be described, accurately, in any of the following ways:

(i) a rule which associates to each element of X a *unique* element of Y

(ii) a '*predictable* machine' into which you can feed an input from X, and which gives a *definite* output from Y

(iii) a 'formula' which, given an element $x \in X$, tells you what $f(x)$ is. This 'formula' *might* be simple, like $f(x) = x^2$, but could also be complicated, and might even just be a listing of each possible x and $f(x)$.

(iv) a graph of $y = f(x)$, i.e. the set of points $\{(x, y) \ : \ y = f(x)\}$

(v) a set of ordered pairs $\{(x, y) \ : \ y = f(x)\}$ as described below.

Examples of functions

Most of the functions you will meet in life will be defined on $\mathbb{R}$, or subsets of $\mathbb{R}$. In this section we consider some examples of such functions.

1. A *constant* function is a function of the form $f(x) = c$, where $c \in \mathbb{R}$. Every x is sent to the same 'constant' value c.

For example, the function $f(x) = 7$ is a constant function.

The *zero* function $f(x) = 0$ is also a constant function.

2. A *linear* function is a function of the form $f(x) = ax + b$, where a, b are real numbers. We require that $a \neq 0$, because if $a = 0$ then the x term disappears, and the function becomes a constant function!

For example, the function $f(x) = -2x + 5$ is a linear function.

The *identity* function $f(x) = x$ is also a linear function.

3. A *quadratic* function is a function of the form $f(x) = ax^2 + bx + c$, where a, b, c are real numbers. We also require that $a \neq 0$, because if $a = 0$ then the x^2 term disappears, and the function becomes linear (or even constant)!

We'll say more about quadratic functions below.

4. A *polynomial* function is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $n \geq 0$ is an integer and $a_0, a_1, \ldots, a_n$ are all real numbers. The number n is called the *degree* of the polynomial. In order that the *leading term* $a_n x^n$ doesn't disappear, we require that $a_n \neq 0$.

For example, the function $f(x) = 2x^3 - x^2 + 8$ is a polynomial function of degree 3. A polynomial function of degree 3 is also known as a *cubic* function.

The function $f(x) = x^{100} + 3x^{14} - 15x^9 + 265$ is a polynomial function of degree 100.

Polynomial functions of degree 2, 1 and 0 are nothing but quadratic, linear, and constant functions.

Polynomial functions are a generalisation of constant, linear, and quadratic functions. See the *Symmetric Polynomials* chapter for more details.

5. A *rational* function is what you get when you *divide* two polynomials. That is, a rational function is a function of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials.

   For example, the function $f(x) = \frac{x^5+3x-2}{x^2-1}$ is a rational function.

   Note that a rational function is not defined for those x that make the denominator zero. For instance, $f(x) = \frac{x^5+3x-2}{x^2-1}$ is not defined when $x = -1$ or $x = 1$.

   If we take $q(x) = 1$, then we have $\frac{p(x)}{q(x)} = p(x)$. Thus, any polynomial is also a rational function. Rational functions are a generalisation of polynomials.

6. For any positive real number a, the function $f(x) = a^x$ is an *exponential* function. The *logarithm* function $g(y) = \log_a y$ is defined by the requirement that $\log_a y = x$ if and only if $y = a^x$. So $\log_a y$ is the power to which you have to raise a in order to obtain y.

   Note that $\log_a a^x = x$ and $a^{\log_a y} = y$. Exponential and logarithm functions are examples of *inverse* functions, which we'll describe further below.

7. The *trigonometric* functions include $\sin x$ and $\cos x$ and other functions related to them, such as:

$$\tan x = \frac{\sin x}{\cos x}, \quad \csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \cot x = \frac{1}{\tan x}.$$

8. Here is an example of a more exotic function $f : \mathbb{R} \to \mathbb{R}$. It is defined separately for *rational* numbers, i.e. fractions, and *irrational* numbers, i.e. numbers (like $\sqrt{2}$ and π) that cannot be expressed as fractions.

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x \text{ is rational, } x = \frac{p}{q} \text{ in simplest terms, } q > 0 \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

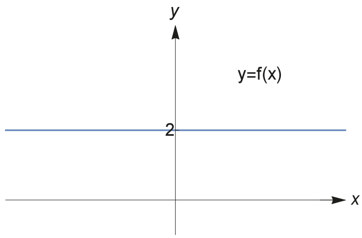
This function has some interesting properties; it is continuous precisely when x is irrational!

Graphs of functions

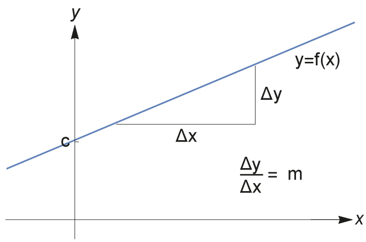
Let us suppose we have a function $f(x)$ whose inputs and outputs are real numbers. The *graph* of f is drawn by plotting all the points (x, y) in the Cartesian plane such that $y = f(x)$.

For example, we consider the graphs of some of the examples above.

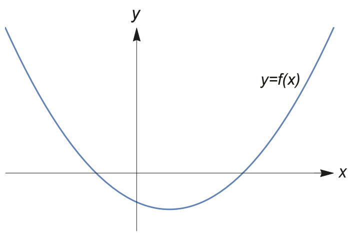
1. The graph of a *constant* function $f(x) = c$ is the *horizontal line* $y = c$. The graph consists of all the points (x, c), where x is a real number.
   For example, the graph of the function $f(x) = 2$ is a horizontal line at height 2.



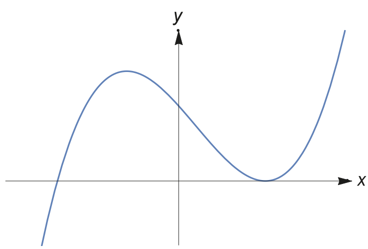
2. The graph of a *linear* function $f(x) = mx + c$ is, as the name suggests, a *line*. Moving along the line one unit to the right, you go up by m units; we say the line has *gradient* m. If $m \neq 0$, it intersects the x-axis (or indeed any horizontal line) exactly once.



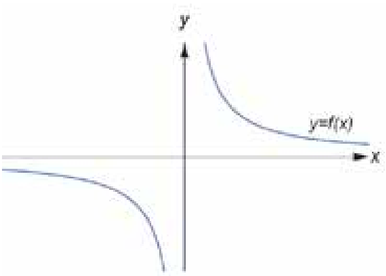
3. The graph of a *quadratic* function $f(x) = ax^2 + bx + c$ is a *parabola*. (We discuss quadratics in a later section.) It intersects the x-axis up to 2 times. It has exactly one turning point.



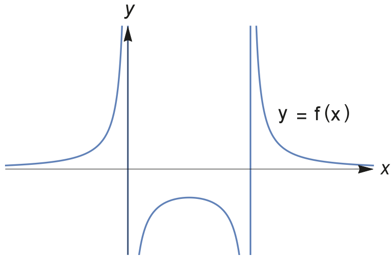
4. The graph of a *polynomial* of degree n is a smooth curve. It may intersect the x-axis (or indeed any horizontal line) up to n times. It may have up to $n - 1$ turning points.



5. A *rational* function $f(x) = \frac{p(x)}{q(x)}$, where $p(x), q(x)$ are polynomials, is not defined when $q(x) = 0$. If $p(x)$ have no factors in common, then at points where $q(x) = 0$, the graph of $f(x)$ has *vertical asymptotes*. If $p(x)$ has degree m and $q(x)$ has degree n, then the graph of $f(x)$ intersects the x-axis up to m times, and has at most n vertical asymptotes.

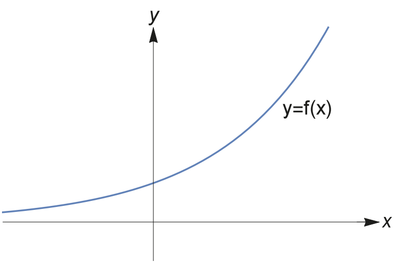


Graph of $f(x) = 1/x$

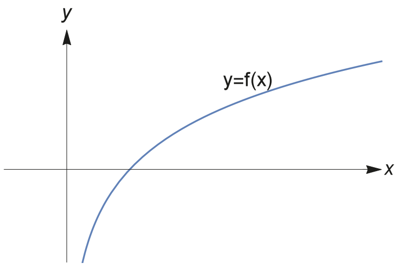


Graph of $f(x) = \frac{1}{x^2-2x}$

6. For a positive real number a, the graph of the exponential function $f(x) = a^x$, and the logarithm function $g(x) = \log_a x$, are as shown. One graph is obtained from the other by reflecting in the line $y = x$. (Actually, this is true of any two inverse functions, as we'll see below.)



Graph of $f(x) = a^x$



Graph of $f(x) = \log_a x$

Relations and functions

We mentioned earlier that a function $f : X \to Y$ can be described abstractly as a *set*

$$\{(x, y) \ : \ x \in X, y = f(x)\}.$$

This set can also be written as

$$\{(x, f(x)) \ : \ x \in X\}.$$

The elements of this set are *ordered pairs* (x, y), where $x \in X$ and $y \in Y$.

The set of *all* pairs (x, y), where x varies over all elements of X, and y varies over all elements of Y, is called the *product* $X \times Y$ of the sets X and Y.

$$X \times Y = \{(x, y) \; : \; x \in X, \; y \in Y\}$$

The set above describing f is a *subset* of $X \times Y$:

$$\{(x, y) \; : \; x \in X, \; y = f(x)\} \subset X \times Y.$$

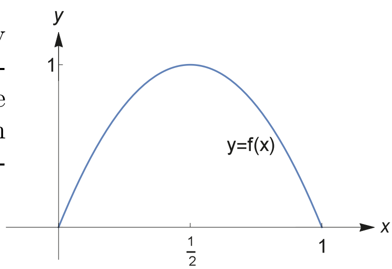
When we draw the graph $y = f(x)$, we plot the set of points (x, y) where $y = f(x)$, i.e. the points of the above set.

In general, any function $f : X \to Y$ is described by a subset of $X \times Y$. But not every subset of $X \times Y$ comes from a function! A subset of $X \times Y$ in general is called a *relation*.

Example 1
Consider the function $f : [0, 1] \to [0, 1]$, $f(x) = x$. It corresponds to the set

$$\{(x, x) \; : \; x \in [0, 1]\} \subset [0, 1] \times [0, 1].$$

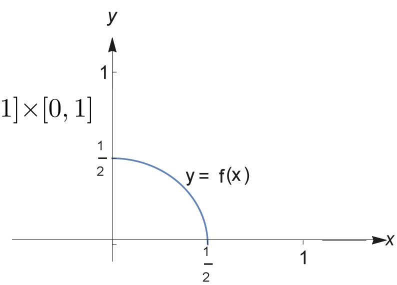
Example 2
Define the function $g : [0, 1] \to [0, 1]$ by $g(x) = x - x^2$. You can check g is well-defined: whenever $0 \le x \le 1$, the value of $x - x^2$ always lies between 0 and 1 (in fact between 0 and $\frac{1}{4}$). The corresponding set is



$$\{(x, x - x^2) \; : \; x \in [0, 1]\}$$
or equivalently
$$\{(x, y) \; : \; x \in [0, 1], \; y = x - x^2\}.$$

Example 3
However, consider the following set.

$$\left\{(x, y) \; : \; x \in [0, 1], \; y = \sqrt{\tfrac{1}{4} - x^2}\right\} \subset [0, 1] \times [0, 1]$$



The points of this set satisfy $x^2 + y^2 = \frac{1}{4}$, so lie on a circle centred at the origin with radius $1/2$. This set is a quadrant of the circle.