AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE

A DEPARTMENT OF THE AUSTRALIAN MATHEMATICS TRUST

# Australian Intermediate Mathematics Olympiad 2018

## Questions

**1.** Let $x$ denote a single digit. The tens digit in the product of $2x7$ and 39 is 9. Find $x$.

[**2 marks**]

**2.** If $234_{b+1} - 234_{b-1} = 70_{10}$, what is $234_b$ in base 10?

[**3 marks**]

**3.** The circumcircle of a square $ABCD$ has radius 10. A semicircle is drawn on $AB$ outside the square. Find the area of the region inside the semicircle but outside the circumcircle.

[**3 marks**]

**4.** Find the last non-zero digit of $50! = 1 \times 2 \times 3 \times \cdots \times 50$.

[**3 marks**]

**5.** Each edge of a cube is marked with its trisection points. Each vertex $v$ of the cube is cut off by a plane that passes through the three trisection points closest to $v$. The resulting polyhedron has 24 vertices. How many diagonals joining pairs of these vertices lie entirely inside the polyhedron?

[**3 marks**]

**6.** Let $ABCD$ be a parallelogram. Point $P$ is on AB produced such that $DP$ bisects $BC$ at $N$. Point $Q$ is on $BA$ produced such that $CQ$ bisects $AD$ at $M$. Lines $DP$ and $CQ$ meet at $O$. If the area of parallelogram $ABCD$ is 192, find the area of triangle $POQ$.

[**4 marks**]

PLEASE TURN OVER THE PAGE FOR QUESTIONS 7, 8, 9 AND 10

**7.** Two different positive integers $a$ and $b$ satisfy the equation $a^2 - b^2 = 2018 - 2a$. What is the value of $a + b$?

[**4 marks**]

**8.** The area of triangle $ABC$ is 300. In triangle $ABC$, $Q$ is the midpoint of $BC$, $P$ is a point on $AC$ between $C$ and $A$ such that $CP = 3PA$, $R$ is a point on side $AB$ such that the area of $\triangle PQR$ is twice the area of $\triangle RBQ$. Find the area of $\triangle PQR$.

[**4 marks**]

**9.** Prove that 38 is the largest even integer that is *not* the sum of two positive odd composite numbers.

[**4 marks**]

**10.** A pair of positive integers is called *compatible* if one of the numbers equals the sum of all digits in the pair and the other number equals the product of all digits in the pair. Find all pairs of positive compatible numbers less than 100.

[**5 marks**]

*Investigation*

Find all pairs of positive compatible numbers less than 1000 with at least one number greater than 99.

[**3 bonus marks**]

# Australian Intermediate Mathematics Olympiad 2018

## Solutions

**1.** *Method 1*

The table shows the product $2x7 \times 39$ for all values of $x$.

| $x$ | $2x7 \times 39$ |
|---|---|
| 0 | 8073 |
| 1 | 8463 |
| 2 | 8853 |
| 3 | 9243 |
| 4 | 9633 |

| $x$ | $2x7 \times 39$ |
|---|---|
| 5 | 10023 |
| 6 | 10413 |
| 7 | 10803 |
| 8 | 11193 |
| 9 | 11583 |

Thus $x = \mathbf{8}$.

*Method 2*

We have $2x7 \times 39 = 2x7 \times 30 + 2x7 \times 9$.
The units digit in $2x7 \times 30$ is 0, and its tens digit is 1.
The tens digit of $2x7 \times 9$ is the units digit of $6 + 9 \times x$.

Hence $1 + 6 + 9 \times x \equiv 9 \pmod{10}$, $9 \times x \equiv 2 \pmod{10}$, $x = \mathbf{8}$.

*Method 3*

We have $2x7 \times 39 = 207 \times 39 + 390 \times x$.
The units digit in $207 \times 39$ is 3, and its tens digit is 7.
The tens digit of $390 \times x$ is the units digit of $9 \times x$.

Hence $7 + 9 \times x \equiv 9 \pmod{10}$, $9 \times x \equiv 2 \pmod{10}$, $x = \mathbf{8}$.

*Method 4*

We have $2x7 \times 39 = 2x7 \times 40 - 2x7$.
The units digit in $2x7 \times 40$ is 0, and its tens digit is 8.
So the tens digit of $2x7 \times 39$ is the units digit of $8 - x - 1$ or $18 - x - 1$.

Since $x$ is non-negative, $17 - x = 9$ and $x = \mathbf{8}$.

**2.** *Method 1*

We have

$$\begin{aligned} 70_{10} &= 234_{b+1} - 234_{b-1} \\ &= 2(b+1)^2 + 3(b+1) + 4 - 2(b-1)^2 - 3(b-1) - 4 \\ &= 2(b^2+2b+1) + 3(b+1) - 2(b^2-2b+1) - 3(b-1) \\ &= 8b + 6 \\ b &= 8 \end{aligned}$$

So $234_b = 234_8 = 2 \times 64 + 3 \times 8 + 4 = \mathbf{156}$.

*Method 2*

The largest digit on the left side of the given equation is 4. Hence $b - 1$ is at least 5. So $b \geq 6$.

If $b = 6$, then the left side in base 10 is $234_7 - 234_5 = (2 \times 49 + 3 \times 7 + 4) - (2 \times 25 + 3 \times 5 + 4) = (98 + 21) - (50 + 15) = 119 - 65 = 54 \neq 70$.

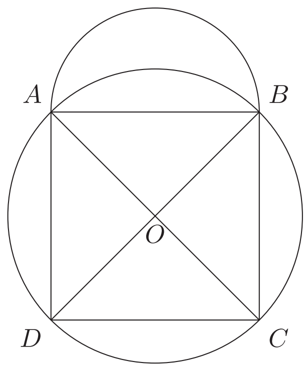
If $b = 7$, then the left side in base 10 is $234_8 - 234_6 = (2 \times 64 + 3 \times 8 + 4) - (2 \times 36 + 3 \times 6 + 4) = (128 + 24) - (72 + 18) = 152 - 90 = 62 \neq 70$.

If $b = 8$, then the left side in base 10 is $234_9 - 234_7 = (2 \times 81 + 3 \times 9 + 4) - (2 \times 49 + 3 \times 7 + 4) = (162 + 27) - (98 + 21) = 189 - 119 = 70$.

Each time $b$ increases by 1, $4_b$ remains the same, $30_b$ increases by 3, but $200_b$ increases by $2(b+1)^2 - 2b^2 = 4b + 2$. So the increase in $234_{b+1}$ is greater than the increase in $234_{b-1}$. Hence $234_{b+1} - 234_{b-1}$ increases with increasing $b$. This means $234_{b+1} - 234_{b-1} > 70_{10}$ for $b > 8$.

So $234_b = 234_8 = 2 \times 64 + 3 \times 8 + 4 = \mathbf{156}$.

**3.** Let $O$ be the centre of the circumcircle.




Since $OA = OB = OC = OD$ and $AB = BC = CD = DA$, triangles $AOB$, $BOC$, $COD$, $DOA$ are isosceles and congruent. So $\angle AOB = 360/4 = 90°$. Hence the area of $\triangle AOB$ is $\frac{1}{2} \times 10 \times 10 = 50$ and the area of the sector $AOB = \frac{1}{4}\pi 100 = 25\pi$.

By Pythagoras, $AB^2 = AO^2 + OB^2 = 200$. Hence the area of the semicircle on $AB$ is $\frac{1}{2}\pi(AB/2)^2 = AB^2\pi/8 = 25\pi$.

So the required area is $25\pi - (25\pi - 50) = \mathbf{50}$.

**4.** *Method 1*

We first arrange the factors 1, 2, 3, ... , 50 in a table:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |

From this we see that in the prime factorisation of 50!, 5 occurs exactly 12 times. Then $50!/(2^{12}5^{12})$ is the product of these factors:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 3 | 4 | 1 | 6 | 7 | 1 | 9 | 1 |
| 11 | 12 | 13 | 14 | 3 | 16 | 17 | 18 | 19 | 1 |
| 21 | 22 | 23 | 24 | 1 | 26 | 27 | 28 | 29 | 3 |
| 31 | 32 | 33 | 34 | 7 | 36 | 37 | 38 | 39 | 1 |
| 41 | 42 | 43 | 44 | 9 | 46 | 47 | 48 | 49 | 1 |

So the last digit of $50!/(2^{12}5^{12})$ is the last digit in the product

$$1^{13}.2^4.3^7.4^5.6^5.7^6.8^4.9^6 = (2.3.4.6.7.8.9)^4 \times (3.7.9)^2 \times (3.4.6)$$

The last digit of 2.3.4.6.7.8.9 is 6. So the last digit of $(2.3.4.6.7.8.9)^4$ is 6.
The last digit of 3.7.9 is 9. So the last digit of $(3.7.9)^2$ is 1.
The last digit of 3.4.6 is 2.
So the last non-zero digit of 50! is the last digit of $6 \times 1 \times 2$, which is **2**.

*Comment*

There are many other workable groupings of factors.

*Method 2*

In the prime factorisation of 50!, 5 occurs exactly 12 times and 2 occurs more than 12 times. So the last non-zero digit of 50! is the last digit of $50!/2^{12}5^{12}$ and it is even.

The remainders of 2, 4, 6, 8 when divided by 5 are respectively 2, 4, 1, 3. So the remainder of $50!/(2^{12}5^{12})$ when divided by 5 will reveal its last digit.

Since the remainder of $2^{12}$ is 1 when divided by 5, the remainder for $50!/2^{12}5^{12}$ is the same as the remainder for $50!/5^{12}$. So we want the remainder of the product of the following factors.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 1 | 6 | 7 | 8 | 9 | 2 |
| 11 | 12 | 13 | 14 | 3 | 16 | 17 | 18 | 19 | 4 |
| 21 | 22 | 23 | 24 | 1 | 26 | 27 | 28 | 29 | 6 |
| 31 | 32 | 33 | 34 | 7 | 36 | 37 | 38 | 39 | 8 |
| 41 | 42 | 43 | 44 | 9 | 46 | 47 | 48 | 49 | 2 |

Before we multiply these factors we may subtract from each any multiple of 5. So we need only multiply these numbers.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 1 | 1 | 2 | 3 | 4 | 2 |
| 1 | 2 | 3 | 4 | 3 | 1 | 2 | 3 | 4 | 4 |
| 1 | 2 | 3 | 4 | 1 | 1 | 2 | 3 | 4 | 1 |
| 1 | 2 | 3 | 4 | 2 | 1 | 2 | 3 | 4 | 3 |
| 1 | 2 | 3 | 4 | 4 | 1 | 2 | 3 | 4 | 2 |

After multiplying any two of these numbers, we may subtract any multiple of 5. Since $2 \times 3$ and $4 \times 4$ have remainder 1 when divided by 5, we need only multiply 3, 2, 4, 2, 4, 3, 2. Hence the required remainder is 2. So the last non-zero digit of 50! is **2**.

**5.** *Method 1*

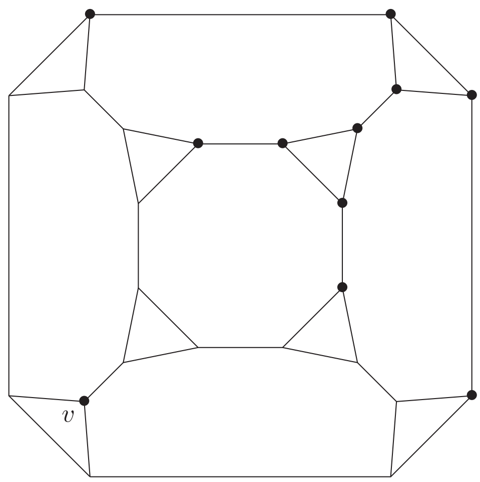
Each vertex is on three faces: a triangular face and two octagonal faces. So each vertex shares a face with 7 other vertices from one octagonal face, and 6 other vertices from the other octagonal face (the two octagons share an edge and 2 vertices). The vertices from the triangular face have already been counted.

A vertex can be joined to 23 other vertices. Of these, $7 + 6 = 13$ lie on the faces of the polyhedron. So each vertex joins to $23 - 13 = 10$ vertices by diagonals that are internal to the polyhedron.

Since each of these diagonals joins two vertices, multiplying 10 by 24 counts each diagonal exactly twice. So the number of diagonals inside the polyhedron is $10 \times 24/2 = \mathbf{120}$.

*Method 2*

The diagram is a projection of the polyhedron.




As indicated by dots, there are exactly 10 vertices that are *not* on a face containing $v$. So there are exactly 10 internal diagonals joined to $v$.

By rotating the polyhedron about one or more of its axes of symmetry, $v$ represents any of its 24 vertices. Since each of these diagonals joins two vertices, multiplying 10 by 24 counts each diagonal exactly twice. So the number of diagonals inside the polyhedron is $10 \times 24/2 = \mathbf{120}$.

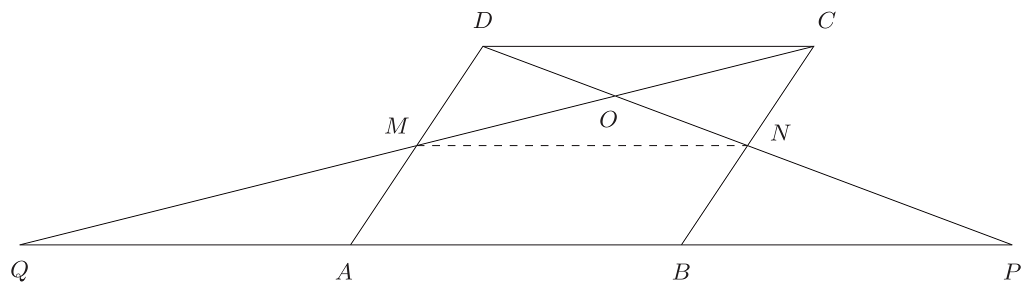
*Method 3*

The polyhedron has 8 triangular faces and 6 octagonal faces. Since each edge of the polyhedron is shared by two faces, its total number of edges is $(8 \times 3 + 6 \times 8)/2 = 36$.

Each octagonal face has 20 diagonals. So the number of diagonals of the polyhedron on its faces is $6 \times 20 = 120$.

The number of pairs of vertices of the polyhedron is $\binom{24}{2} = 276$. So the number of internal diagonals of the polyhedron is $276 - 36 - 120 = \mathbf{120}$.

**6.** Draw $MN$.




*Method 1*

Since $BP$ and $CD$ are parallel and $BN = NC$, triangles $BNP$ and $CND$ are congruent (ASA). Similarly, triangles $AMQ$ and $DMC$ are congruent.

Since $AM$ and $BN$ are parallel and equal, $MN$ and $AB$ are parallel. So $ABNM$ and $MNCD$ are congruent parallelograms and their areas are half the area of $ABCD$, that is, $192/2 = 96$.

Since $MNCD$ is a parallelogram, its area is twice the area of triangle $CND$, twice the area of triangle $DMC$, and 4 times the area of triangle $MNO$.

So the area of triangle $POQ$ is $96 + 2(96/2) + (96/4) = \mathbf{216}$.

*Method 2*

Since $BP$ and $CD$ are parallel and $BN = NC$, triangles $BNP$ and $CND$ are congruent (ASA). Similarly, triangles $AMQ$ and $DMC$ are congruent. So $QP = 3 \times DC$.

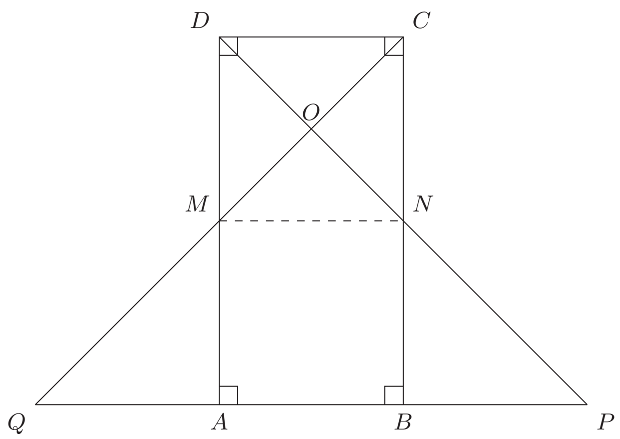
Since $AM$ and $BN$ are parallel and equal, $MN$ and $AB$ are parallel. So $ABNM$ and $MNCD$ are congruent parallelograms and their areas are half the area of $ABCD$, that is, $192/2 = 96$.

Since $MNCD$ is a parallelogram, the area of triangle $COD$ is $96/4 = 24$.

Since $PQ$ and $CD$ are parallel, triangles $POQ$ and $DOC$ are similar. So the area of triangle $POQ$ is $9 \times 24 = \mathbf{216}$.

*Method 3*

Parallelogram $ABCD$ is unspecified so we may take it to be a rectangle with $AD = 2AB$.




Thus $ABNM$ and $MNCD$ are congruent squares, triangles $NBP$ and $NCD$ are congruent, and triangles $MAQ$ and $MDC$ are congruent.

Let $|\ |$ denote area. Then

$$\begin{aligned} |ABNM| = |MNCD| &= \frac{1}{2}|ABCD| = \frac{192}{2} = 96 \\ |NBP| = |NDC| &= \frac{1}{2}|MNCD| = 48 \\ |MAQ| = |MDC| &= \frac{1}{2}|MNCD| = 48 \\ |MNO| &= \frac{1}{4}|MNCD| = 24 \end{aligned}$$

So $|POQ| = 96 + 48 + 48 + 24 = \mathbf{216}$.

**7.** *Method 1*

We have

$$\begin{aligned} 2018 &= a^2 + 2a - b^2 \\ 2019 &= a^2 + 2a + 1 - b^2 \\ &= (a+1)^2 - b^2 \\ &= (a+1-b)(a+1+b) \end{aligned}$$

We know $a$ and $b$ are positive, so $a + 1 + b$ is positive, hence $a + 1 - b$ is positive.
Since $a$ and $b$ are integers, both factors are integers.
Since $2019 = 3 \times 673$ and 673 is prime, the only positive integer factor pairs are (1, 2019) and (3, 673).
Since $b$ is positive, $a + 1 + b > a + 1 - b$. So $a + 1 - b$ equals 1 or 3.

If $a + 1 - b = 1$, then $a = b$, which contradicts the requirement that $a$ and $b$ are different.

So $a + 1 - b = 3$ and $a + 1 + b = 673$. Hence $a + b = \mathbf{672}$.

*Method 2*

Solving $a^2 + 2a - b^2 - 2018 = 0$ using the quadratic formula yields

$$a = \frac{-2 \pm \sqrt{4 + 4(b^2 + 2018)}}{2} = -1 \pm \sqrt{b^2 + 2019}$$

Since $a$ is positive, $a = \sqrt{b^2 + 2019} - 1$.

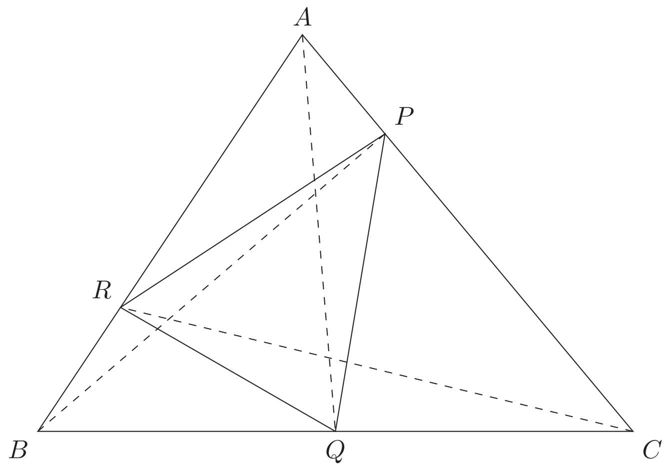
Since $a + 1$ is an integer, we have $b^2 + 2019 = c^2$ for some positive integer $c$.
So $a = c - 1$ and $(c + b)(c - b) = 2019$.

The only positive factorisations of 2019 are $1 \times 2019$ and $3 \times 673$.
Since $b$ is positive, we have $c + b = 2019$ and $c - b = 1$, or $c + b = 673$ and $c - b = 3$.

So $2b = 2018$ or 670.
Hence, respectively, $b = 1009$ or 335, $c = 1010$ or 338, and $a = 1009$ or 337.
Since $a$ and $b$ are different, $a + b = 337 + 335 = \mathbf{672}$.

**8.** *Method 1*

Draw $AQ$, $BP$, and $CR$. Let $|\ |$ denote area.




Let $BR = 1$ and $RA = p$. Noting that $BQ = QC$ and $CP = 3PA$, we have:

$$\begin{aligned}
|BQR| &= \frac{1}{2}|BCR| = \frac{1}{2} \times \frac{1}{1+p} \times |ABC| = \frac{1}{2} \times \frac{300}{1+p} \\
|APR| &= \frac{1}{4}|ACR| = \frac{1}{4} \times \frac{p}{1+p} \times |ABC| = \frac{1}{4} \times \frac{300p}{1+p} \\
|CPQ| &= \frac{1}{2}|BCP| = \frac{1}{2} \times \frac{3}{4} \times |ABC| = \frac{3}{8} \times 300 \\
|PQR| &= 2|BQR| = \frac{300}{1+p}
\end{aligned}$$

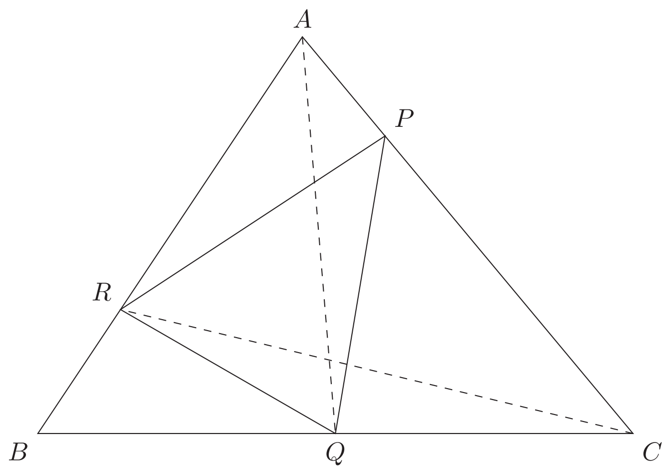
Adding these areas gives

$$\begin{aligned}
300 &= \frac{4}{8}\left(\frac{300}{1+p}\right) + \frac{2}{8}\left(\frac{300p}{1+p}\right) + \frac{3}{8}\left(\frac{300(p+1)}{p+1}\right) + \frac{8}{8}\left(\frac{300}{1+p}\right) \\
&= \frac{1}{8}\left(\frac{300}{p+1}\right)(4 + 2p + 3(p+1) + 8) \\
8(p+1) &= 15 + 5p \\
3p &= 7
\end{aligned}$$

Finally $\qquad |PQR| = \frac{300}{1+p} = \frac{300 \times 3}{3+3p} = \frac{900}{10} = \mathbf{90}.$

*Method 2*

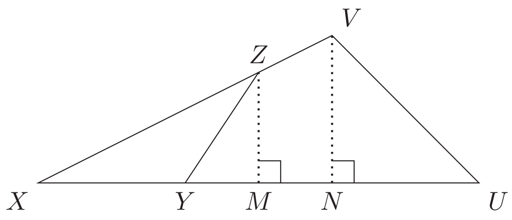



Let | | denote area. Noting that $BQ = QC$ and $CP = 3PA$, we have:

$$\begin{aligned}
300 &= |APR| + |PQR| + |BQR| + |CQP| \\
&= \frac{1}{4}|ACR| + |PQR| + \frac{1}{2}|PQR| + \frac{3}{4}|AQC| \\
1200 &= |ACR| + 6|PQR| + 3|AQC| \\
&= (300 - |BCR|) + 6|PQR| + \frac{3}{2}|ABC| \\
&= 300 - 2|BQR| + 6|PQR| + 450 \\
&= 750 - |PQR| + 6|PQR| \\
&= 750 + 5|PQR| \\
|PQR| &= \frac{1}{5}(1200 - 750) = \frac{1}{5}(450) = \mathbf{90}.
\end{aligned}$$

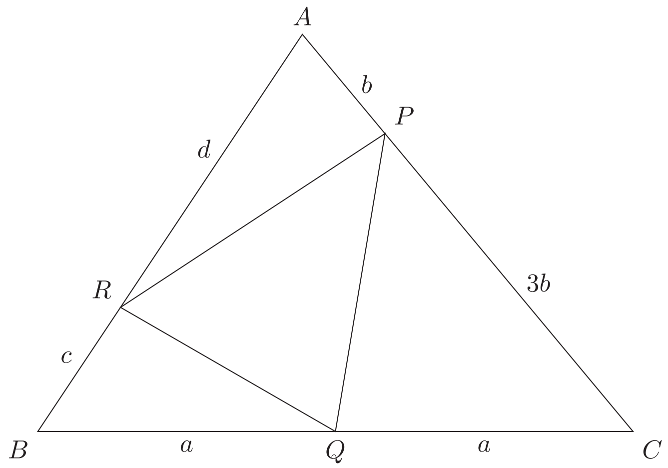
*Method 3*

First consider the following diagram and let | | denote area.




$$\frac{|XYZ|}{|XUV|} = \frac{MZ \times XY}{NV \times XU} = \frac{XZ}{XV} \times \frac{XY}{XU} = \frac{XY \times XZ}{XU \times XV} \qquad (1)$$

Next consider the given diagram with distances as shown.




From (1) we have

$$\frac{|BQR|}{300} = \frac{ac}{2a(c+d)} = \frac{c}{2(c+d)}$$
$$\frac{|APR|}{300} = \frac{bd}{4b(c+d)} = \frac{d}{4(c+d)}$$
$$\frac{|CPQ|}{300} = \frac{3ab}{(4b)(2a)} = \frac{3}{8}$$

Combining these gives

$$\begin{aligned} 300 &= |BQR| + |APR| + |CPQ| + |PQR| \\ &= 3|BQR| + |APR| + |CPQ| \\ 1 &= \frac{3c}{2(c+d)} + \frac{d}{4(c+d)} + \frac{3}{8} \\ 8(c+d) &= 12c + 2d + 3(c+d) = 15c + 5d \\ d &= 7c/3 \end{aligned}$$

So $|BQR| = \dfrac{300c}{2(c+d)} = \dfrac{300c}{20c/3} = \dfrac{90}{2} = 45$. Hence $|PQR| = 2|BQR| = \mathbf{90}$.

**9.** The odd composites less than 38 are 9, 15, 21, 25, 27, 33, 35. No two of these have their sum equal to 38.

*Method 1*

Now $40 = 15 + 25$, $42 = 9 + 33$, $44 = 9 + 35$, where each summand is an odd composite and at least one summand is a multiple of 3.

Adding 6 to an odd multiple of 3 gives an odd number that is also a multiple of 3. So we can express the next three even integers as the sum of two odd composites at least one of which is a multiple of 3: $46 = 21 + 25$, $48 = 15 + 33$, $50 = 15 + 35$.

We can therefore add 6 to express the next three even integers after these as the sum of two odd composites at least one of which is a multiple of 3, and repeat indefinitely.

So every even integer greater than 38 is the sum of two odd composites.

*Method 2*

Now $40 = 15+25$, $42 = 15+27$, $44 = 9+35$, $46 = 21+25$, $48 = 15+33$, where each summand is an odd composite and at least one summand is a multiple of 5.

Adding 10 to an odd multiple of 5 gives an odd number that is also a multiple of 5. So we can express the next five even integers as the sum of two odd composites at least one of which is a multiple of 5: $50 = 25 + 25$, $52 = 25 + 33$, $54 = 9 + 45$, $56 = 21 + 35$, $58 = 25 + 33$.

We can therefore add 10 to express the next five even integers after these as the sum of two odd composites at least one of which is a multiple of 5, and repeat indefinitely.

So every even integer greater than 38 is the sum of two odd composites.

*Comment*

There are similar arguments adding 12, 18, 20, 24, 30, etc. at a time.

**10.** Let $s$ and $p$ be compatible numbers, where $s$ is the sum of their digits and $p$ is the product of their digits. Note that all of the digits are non-zero since $p \neq 0$. We consider three cases.

Case 1. $s$ has a single digit.

Since $p$ has a positive digit, $s \geq s+1$, a contradiction. So $s$ has two digits.

Case 2. $s$ has two digits and $p$ has a single digit.

We have $s = 10a + b$, where $a$ and $b$ are positive digits. Then

$$10a + b = a + b + p \quad \text{and} \quad p = abp.$$

The second equation gives $a = b = 1$. Hence $p = 9$ from the first equation. So the only pair of compatible numbers in this case are $s = 11$ and $p = 9$.

Case 3. $s$ and $p$ both have two digits.

*Method 1*

Let $s = 10a + b$ and $p = 10c + d$, where $a$, $b$, $c$, and $d$ are positive digits. Then

$$10a + b = a + b + c + d \quad \text{and} \quad 10c + d = abcd.$$

The first equation gives $9a = c + d$. So $a = 1$ or 2.

If $a = 2$, then $c + d = 18$, $c = d = 9$, and the second equation gives $99 = 162b$, which is impossible.

If $a = 1$, then $c + d = 9$ and the second equation gives

$$9c + 9 = bc(9 - c) \quad \text{and} \quad 9 = c(b(9 - c) - 9).$$

So $c = 1$, 3, or 9. If $c = 1$, then $18 = 8b$, which is impossible. If $c = 9$, then $1 = 0 - 9$, which is a contradiction. If $c = 3$, then $3 = 6b - 9$, $b = 2$, $d = 6$ and we have the compatible numbers $s = 12$ and $p = 36$.

Hence there are only 2 pairs of compatible numbers less than 100, namely $\{9, 11\}$ and $\{12, 36\}$.

*Method 2*

Let $s = 10a + b$ and $p = 10c + d$, where $a$, $b$, $c$, and $d$ are positive digits. Then

$$10a + b = a + b + c + d \quad \text{and} \quad 10c + d = abcd.$$

The second equation gives $d = c(abd - 10)$. Since $d$ is a positive digit, $10 < abd < 20$. Since $a$, $b$, $d$ are digits and 11, 13, 17, 19 are primes, $abd = 12, 14, 15, 16$, or 18.

If $abd = 18$, then $18c = 10c + d$, $d = 8c$, and $4abc = 9$, which is impossible.

If $abd = 16$, then $16c = 10c + d$, $d = 6c$, and $3abc = 8$, which is impossible.

If $abd = 15$, then $15c = 10c + d$, $d = 5c$. So $c = 1$, $d = 5$, and $9a = 6$, which is impossible.

If $abd = 14$, then $14c = 10c + d$, $d = 4c$, and $2abc = 7$, which is impossible.

If $abd = 12$, then $12c = 10c + d$, and $d = 2c$. So $d$ is even and divides 12. Hence $d = 2$, 4, or 6, and correspondingly $c = 1$, 2, or 3. Since $10a + b = a + b + c + d$, we have $9a = c + d$. So 9 divides $c + d$. Hence $d = 6$, $c = 3$, $a = 1$, $b = 2$, and we have the compatible numbers $s = 12$ and $p = 36$.

Hence there are only 2 pairs of compatible numbers less than 100, namely $\{9, 11\}$ and $\{12, 36\}$.

*Investigation*

Let $s$ and $p$ be compatible numbers, where $s$ is the sum of their digits and $p$ is the product of their digits. Note that all of the digits are non-zero since $p \neq 0$.

As in the solution above, $s$ has at least 2 digits. Since the sum of six digits is at most 54, $s$ has exactly 2 digits. So $p$ has exactly 3 digits. Let $s = 10a + b$ and $p = 100c + 10d + e$, where $a$, $b$, $c$, $d$, $e$ are positive digits. Then

$$10a + b = a + b + c + d + e \quad \text{and} \quad p = abcde.$$

The first of these equations gives $9a = c + d + e$, so 9 divides $c + d + e$.
Hence $c + d + e = 9$, 18, or 27.

If $c + d + e = 27$, then $a = 3$, $c = d = e = 9$, $p = 999 = 3b \times 9 \times 9 \times 9 = 2187b$, which is impossible.

*Method 1*

If $c + d + e = 18$, then $a = 2$ and $p = 2bcde$. The table shows for each combination of digits for $c$, $d$, $e$, the product $2cde$, the last 1, 2, or 3 digits of $2bcde$ for $b = 1, 2, 3, 4, 5, 6, 7, 8, 9$ and $2bcde < 1000$, then a check whether any $2bcde$ could be $p = 100c + 10d + e$.

| $\{c, d, e\}$ | $2cde$ | last few digits of $2bcde$ | any $p$? |
|---|---|---|---|
| 1, 8, 9 | 144 | 4, 88, 2, 6, 0, 4 | none |
| 2, 7, 9 | 252 | 52, 4, 6 | none |
| 2, 8, 8 | 256 | 6, 12, 68 | none |
| 3, 6, 9 | 324 | 4, 8, 2 | none |
| 3, 7, 8 | 336 | 6, 2 | none |
| 4, 5, 9 | 360 | 0, 0, 0, 0, 0, 0, 0, 0, 0 | none |
| 4, 6, 8 | 384 | 384, 768 | none |
| 4, 7, 7 | 392 | 2, 84 | none |
| 5, 5, 8 | 400 | 0, 0, 0, 0, 0, 0, 0, 0, 0 | none |
| 5, 6, 7 | 420 | 0, 0, 0, 0, 0, 0, 0, 0, 0 | none |
| 6, 6, 6 | 432 | 2, 4 | none |

If $c + d + e = 9$, then $a = 1$ and $p = bcde$. The table shows for each combination of digits for $c$, $d$, $e$, the product $cde$, the last 1, 2, or 3 digits of $bcde$ for $b = 1, 2, 3, 4, 5, 6, 7, 8, 9$, then a check whether any $bcde$ could be $p = 100c + 10d + e$.

| $\{c, d, e\}$ | $cde$ | last few digits of $bcde$ | any $p$? |
|---|---|---|---|
| 1, 1, 7 | 7 | 07, 4, 21, 8, 5, 2, 9, 6, 3 | none |
| 1, 2, 6 | 12 | 012, 4, 36, 8, 0, 72, 4, 96, 8 | none |
| 1, 3, 5 | 15 | 015, 0, 45, 0, 75, 0, 05, 0, 135 | 135 |
| 1, 4, 4 | 16 | 6, 2, 8, 64, 0, 6, 2, 8, 144 | 144 |
| 2, 2, 5 | 20 | 0, 0, 0, 0, 0, 0, 0, 0, 0 | none |
| 2, 3, 4 | 24 | 024, 8, 72, 6, 0, 44, 8, 92, 6 | none |
| 3, 3, 3 | 27 | 7, 4, 1, 8, 5, 2, 9, 6, 43 | none |

Hence the only compatible pairs are {135, 19} and {144, 19}.

*Method 2*

Since 9 divides $c + d + e$ and $c$, $d$, $e$ are the digits of $p$, 9 divides $p$.

If $c+d+e = 18$, then $a = 2$ and $p = 2bcde$. So $p$ is even, hence its last digit $e$ is even. Therefore 4 divides $p$, hence $p = 36n$. Since $100 \le p \le 999$, $3 \le n \le 27$. Since $p$ is a product of digits, all prime factors of $n$ are digits. So $n \ne 11$, 13, 17, 19, 22, 23, 26. Of the remainder, only $n = 8$, 16, 18, 21, 24, 27 give 18 for the sum of the digits of $36n$. None of these give an integer value for $b = p/2cde$.

If $c + d + e = 9$, then $a = 1$ and $p = bcde$. By inspection, the maximum value for $cde$ is 27. So $p = 9n$ with $12 \le n \le 27$. Again $n \ne 13$, 17, 19, 22, 23, 26. Of the remainder, only $n = 15$, and $n = 16$ give a digit value for $b = p/cde$. If $n = 15$, then $p = 135$ and $b = 9$. If $n = 16$, then $p = 144$ and $b = 9$.

Hence the only compatible pairs are $\{135, 19\}$ and $\{144, 19\}$.