It is not in the stars to hold our destiny but in ourselves

William Shakespeare
SUPPORT FOR THE AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE TRAINING PROGRAM

The Australian Mathematical Olympiad Committee Training Program is an activity of the Australian Mathematical Olympiad Committee, a department of the Australian Mathematics Trust.

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The Mathematics/Informatics Olympiads are supported by the Australian Government through the National Innovation and Science Agenda.

The Australian Mathematical Olympiad Committee (AMOC) also acknowledges the significant financial support it has received from the Australian Government towards the training of our Olympiad candidates and the participation of our team at the International Mathematical Olympiad (IMO).

The views expressed here are those of the authors and do not necessarily represent the views of the government.

Special thanks

With special thanks to the Australian Mathematical Society, the Australian Association of Mathematics Teachers and all those schools, societies, families and friends who have contributed to the expense of sending the 2016 IMO team to Hong Kong.
ACKNOWLEDGEMENTS

The Australian Mathematical Olympiad Committee (AMOC) sincerely thanks all sponsors, teachers, mathematicians and others who have contributed in one way or another to the continued success of its activities. The editors sincerely thank those who have assisted in the compilation of this book, in particular the students who have provided solutions to the 2016 IMO. Thanks also to members of AMOC and Challenge Problems Committees, Adjunct Professor Mike Clapper, staff of the Australian Mathematics Trust and others who are acknowledged elsewhere in the book.
This year has seen some remarkable progress in the Mathematics Challenge for Young Australians (MCYA) program. We had a record number of entries for the Challenge stage, and the Enrichment stage entries were also very strong, with just over 4300 entries. The new Ramanujan book was well received as was the revised Polya book. A number of schools have made a commitment to use Challenge and/or Enrichment with whole year levels and are reporting back very positively on the effect of this on problem-solving capacity in their students. The rise in Australian Intermediate Mathematics Olympiad (AIMO) entries continues, with 1829 entries this year, including over 400 from Vietnam. We have more than doubled our numbers in this competition over the last three years. Fifteen students obtained perfect scores, including five Australian students.

In the Olympiad program, the 2016 Australian Mathematical Olympiad (AMO) proved fairly demanding, with just three perfect scores. The International Mathematical Olympiad (IMO) also proved quite challenging for Australia, after three very strong years in a team led by Alex Gunning. Commendably, this year’s team all obtained medals (two Silver and four Bronze) but we could not quite maintain our position of the last few years, finishing 25th out of 109 teams. Two of the team members became dual Olympians, while Seyoon Ragavan notched up his 4th IMO. All of the team were Year 12 students, so there will be a challenge ahead to build a new team for 2017. In the Mathematics Ashes we performed above expectations, but still lost in a close contest, to a very strong and experienced British team.

The start for selection of the 2017 team begins with the AMOC Senior contest, held in August, and it was encouraging to see five perfect scores in this competition, including one from Hadyn Tang, a year 7 student from Trinity Grammar School in Melbourne. Director of Training and IMO Team Leader Angelo Di Pasquale, along with Deputy Team Leader Andrew Elvey Price and a dedicated team of tutors, continue to innovate and have made some changes to the structure of the December School of Excellence.

I would particularly wish to thank all the dedicated volunteers without whom this program would not exist. These include the Director of Training and the ex-Olympians who train the students at camps, the various state directors, the Challenge Director, Dr Kevin McAvaney and the various members of his Problems Committee which develop such original problems each year and Dr Norm Do, and his senior problems committee (who do likewise).

Our support from the Australian Government for the AMOC program continues, though from June this year, this is provided through the Department of Science and Industry, rather than the Department of Education. We are most grateful for this support.

Once again, the Australian Scene is produced in electronic form only. Whilst the whole book can be downloaded as a pdf, it is available on our website in two sections, one containing the MCYA reports and papers and the other containing the Olympiad reports and papers.

Mike Clapper
December 2016
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<td>Australian Mathematical Olympiad Results</td>
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<td>The Mathematics Ashes</td>
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<td>The Mathematics Ashes Results</td>
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BACKGROUND NOTES ON THE IMO AND AMOC

The Australian Mathematical Olympiad Committee

In 1980, a group of distinguished mathematicians formed the Australian Mathematical Olympiad Committee (AMOC) to coordinate an Australian entry in the International Mathematical Olympiad (IMO).

Since then, AMOC has developed a comprehensive program to enable all students (not only the few who aspire to national selection) to enrich and extend their knowledge of mathematics. The activities in this program are not designed to accelerate students. Rather, the aim is to enable students to broaden their mathematical experience and knowledge.

The largest of these activities is the MCYA Challenge, a problem-solving event held in second term, in which thousands of young Australians explore carefully developed mathematical problems. Students who wish to continue to extend their mathematical experience can then participate in the MCYA Enrichment Stage and pursue further activities leading to the Australian Mathematical Olympiad and international events.

Originally AMOC was a subcommittee of the Australian Academy of Science. In 1992 it collaborated with the Australian Mathematics Foundation (which organises the Australian Mathematics Competition) to form the Australian Mathematics Trust. The Trust, a not-for-profit organisation under the trusteeship of the University of Canberra, is governed by a Board which includes representatives from the Australian Academy of Science, Australian Association of Mathematics Teachers and the Australian Mathematical Society.

The aims of AMOC include:

1. giving leadership in developing sound mathematics programs in Australian schools
2. identifying, challenging and motivating highly gifted young Australian school students in mathematics
3. training and sending Australian teams to future International Mathematical Olympiads.

AMOC schedule from August until July for potential IMO team members

Each year hundreds of gifted young Australian school students are identified using the results from the Australian Mathematics Competition sponsored by the Commonwealth Bank, the Mathematics Challenge for Young Australians program and other smaller mathematics competitions. A network of dedicated mathematicians and teachers has been organised to give these students support during the year either by correspondence sets of problems and their solutions or by special teaching sessions.

It is these students who sit the Australian Intermediate Mathematics Olympiad, or who are invited to sit the AMOC Senior Contest each August. Most states run extension or correspondence programs for talented students who are invited to participate in the relevant programs. The 25 outstanding students in recent AMOC programs and other mathematical competitions are identified and invited to attend the residential AMOC School of Excellence held in December.

In February approximately 100 students are invited to attempt the Australian Mathematical Olympiad. The best 20 or so of these students are then invited to represent Australia in the correspondence Asian Pacific Mathematics Olympiad in March. About 12 students are selected for the AMOC Selection School in April and about 13 younger students are also invited to this residential school. Here, the Australian team of six students plus one reserve for the International Mathematical Olympiad, held in July each year, is selected. A personalised support system for the Australian team operates during May and June.

It should be appreciated that the AMOC program is not meant to develop only future mathematicians. Experience has shown that many talented students of mathematics choose careers in engineering, computing, and the physical and life sciences, while others will study law or go into the business world. It is hoped that the AMOC Mathematics Problem-Solving Program will help the students to think logically, creatively, deeply and with dedication and perseverance; that it will prepare these talented students to be future leaders of Australia.

The International Mathematical Olympiad

The IMO is the pinnacle of excellence and achievement for school students of mathematics throughout the world. The concept of national mathematics competitions started with the Eötvos Competition in Hungary during 1894. This idea
was later extended to an international mathematics competition in 1959 when the first IMO was held in Romania. The aims of the IMO include:

(1) discovering, encouraging and challenging mathematically gifted school students
(2) fostering friendly international relations between students and their teachers
(3) sharing information on educational syllabi and practice throughout the world.

It was not until the mid-sixties that countries from the western world competed at the IMO. The United States of America first entered in 1975. Australia has entered teams since 1981.

Students must be under 20 years of age at the time of the IMO and have not enrolled at a tertiary institution. The Olympiad contest consists of two four-and-a-half hour papers, each with three questions.

Australia has achieved varying successes as the following summary of results indicate. HM (Honorable Mention) is awarded for obtaining full marks in at least one question.

The IMO will be held in Rio de Janeiro, Brazil in 2017.
<table>
<thead>
<tr>
<th>Year</th>
<th>City</th>
<th>Gold</th>
<th>Silver</th>
<th>Bronze</th>
<th>HM</th>
<th>Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>1981</td>
<td>Washington</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td>23 out of 27 teams</td>
</tr>
<tr>
<td>1982</td>
<td>Budapest</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td>21 out of 30 teams</td>
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<tr>
<td>1983</td>
<td>Paris</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td>19 out of 32 teams</td>
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<tr>
<td>1984</td>
<td>Prague</td>
<td>1</td>
<td>2</td>
<td></td>
<td></td>
<td>15 out of 34 teams</td>
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<tr>
<td>1985</td>
<td>Helsinki</td>
<td>1</td>
<td>1</td>
<td>2</td>
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<td>11 out of 38 teams</td>
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<tr>
<td>1986</td>
<td>Warsaw</td>
<td>5</td>
<td>3</td>
<td></td>
<td></td>
<td>15 out of 37 teams</td>
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<tr>
<td>1987</td>
<td>Havana</td>
<td>3</td>
<td>2</td>
<td></td>
<td></td>
<td>15 out of 42 teams</td>
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<tr>
<td>1988</td>
<td>Canberra</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td>17 out of 49 teams</td>
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<tr>
<td>1989</td>
<td>Braunschweig</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
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<tr>
<td>1990</td>
<td>Beijing</td>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
<td>15 out of 54 teams</td>
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<tr>
<td>1991</td>
<td>Sigtuna</td>
<td>3</td>
<td>2</td>
<td></td>
<td></td>
<td>20 out of 56 teams</td>
</tr>
<tr>
<td>1992</td>
<td>Moscow</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>19 out of 56 teams</td>
</tr>
<tr>
<td>1993</td>
<td>Istanbul</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td>13 out of 73 teams</td>
</tr>
<tr>
<td>1994</td>
<td>Hong Kong</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td></td>
<td>12 out of 69 teams</td>
</tr>
<tr>
<td>1995</td>
<td>Toronto</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td></td>
<td>21 out of 73 teams</td>
</tr>
<tr>
<td>1996</td>
<td>Mumbai</td>
<td>2</td>
<td>3</td>
<td></td>
<td></td>
<td>23 out of 75 teams</td>
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<tr>
<td>1997</td>
<td>Mar del Plata</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td></td>
<td>9 out of 82 teams</td>
</tr>
<tr>
<td>1998</td>
<td>Taipei</td>
<td>4</td>
<td>2</td>
<td></td>
<td></td>
<td>13 out of 76 teams</td>
</tr>
<tr>
<td>1999</td>
<td>Bucharest</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>15 out of 81 teams</td>
</tr>
<tr>
<td>2000</td>
<td>Taejon</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td></td>
<td>16 out of 82 teams</td>
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<td>2001</td>
<td>Washington D.C.</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td></td>
<td>25 out of 83 teams</td>
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<tr>
<td>2002</td>
<td>Glasgow</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>26 out of 84 teams</td>
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<tr>
<td>2003</td>
<td>Tokyo</td>
<td>2</td>
<td>2</td>
<td></td>
<td>2</td>
<td>26 out of 82 teams</td>
</tr>
<tr>
<td>2004</td>
<td>Athens</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>27 out of 85 teams</td>
</tr>
<tr>
<td>2005</td>
<td>Merida</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td>25 out of 91 teams</td>
</tr>
<tr>
<td>2006</td>
<td>Ljubljana</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td></td>
<td>26 out of 90 teams</td>
</tr>
<tr>
<td>2007</td>
<td>Hanoi</td>
<td>1</td>
<td>4</td>
<td>1</td>
<td></td>
<td>22 out of 93 teams</td>
</tr>
<tr>
<td>2008</td>
<td>Madrid</td>
<td>5</td>
<td>1</td>
<td></td>
<td></td>
<td>19 out of 97 teams</td>
</tr>
<tr>
<td>2009</td>
<td>Bremen</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>23 out of 104 teams</td>
</tr>
<tr>
<td>2010</td>
<td>Astana</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>15 out of 96 teams</td>
</tr>
<tr>
<td>2011</td>
<td>Astana</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>25 out of 91 teams</td>
</tr>
<tr>
<td>2012</td>
<td>Mar del Plata</td>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
<td>27 out of 100 teams</td>
</tr>
<tr>
<td>2013</td>
<td>Santa Marta</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td>15 out of 97 teams</td>
</tr>
<tr>
<td>2014</td>
<td>Cape Town</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td></td>
<td>11 out of 101 teams</td>
</tr>
<tr>
<td>2015</td>
<td>Chiang Mai</td>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
<td>6 out of 104 teams</td>
</tr>
<tr>
<td>2016</td>
<td>Hong Kong</td>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
<td>25 out of 109 teams</td>
</tr>
</tbody>
</table>
The Mathematics Challenge for Young Australians (MCYA) started on a national scale in 1992. It was set up to cater for the needs of the top 10 percent of secondary students in Years 7–10, especially in country schools and schools where the number of students may be quite small. Teachers with a handful of talented students spread over a number of classes and working in isolation can find it very difficult to cater for the needs of these students. The MCYA provides materials and an organised structure designed to enable teachers to help talented students reach their potential. At the same time, teachers in larger schools, where there are more of these students, are able to use the materials to better assist the students in their care.

The aims of the Mathematics Challenge for Young Australians include:

- encouraging and fostering
- a greater interest in and awareness of the power of mathematics
- a desire to succeed in solving interesting mathematical problems
- the discovery of the joy of solving problems in mathematics
- identifying talented young Australians, recognising their achievements nationally and providing support that will enable them to reach their own levels of excellence
- providing teachers with
  - interesting and accessible problems and solutions as well as detailed and motivating teaching discussion and extension materials
  - comprehensive Australia-wide statistics of students' achievements in the Challenge.

There are three independent stages in the Mathematics Challenge for Young Australians:

- **Challenge (three weeks during the period March–June)**
- **Enrichment (April–September)**
- **Australian Intermediate Mathematics Olympiad (September).**

**Challenge**

Challenge consists of four levels. Middle Primary (Years 3–4) and Upper Primary (Years 5–6) present students with four problems each to be attempted over three weeks, students are allowed to work on the problems in groups of up to three participants, but each must write their solutions individually. The Junior (Years 7–8) and Intermediate (Years 9–10) levels present students with six problems to be attempted over three weeks, students are allowed to work on the problems with a partner but each must write their solutions individually.

There were 13461 submissions (1472 Middle Primary, 3472 Upper Primary, 5640 Junior, 2877 Intermediate) for the Challenge in 2016. The 2016 problems and solutions for the Challenge, together with some statistics, appear later in this book.

**Enrichment**

This is a six-month program running from April to September, which consists of seven different parallel stages of comprehensive student and teacher support notes. Each student participates in only one of these stages.

The materials for all stages are designed to be a systematic structured course over a flexible 12–14 week period between April and September. This enables schools to timetable the program at convenient times during their school year.

Enrichment is completely independent of the earlier Challenge; however, they have the common feature of providing challenging mathematics problems for students, as well as accessible support materials for teachers.

- **Ramanujan (years 4–5)** includes estimation, special numbers, counting techniques, fractions, clock arithmetic, ratio, colouring problems, and some problem-solving techniques. There were 186 entries in 2016.

- **Newton (years 5–6)** includes polyominoes, fast arithmetic, polyhedra, pre-algebra concepts, patterns, divisibility and specific problem-solving techniques. There were 593 entries in 2016.

- **Dirichlet (years 6–7)** includes mathematics concerned with tessellations, arithmetic in other bases, time/distance/speed,
patterns, recurring decimals and specific problem-solving techniques. There were 784 entries in 2016.

**Euler** (years 7–8) includes primes and composites, least common multiples, highest common factors, arithmetic sequences, figurate numbers, congruence, properties of angles and pigeonhole principle. There were 1350 entries in 2016.

**Gauss** (years 8–9) includes parallels, similarity, Pythagoras’ Theorem, using spreadsheets, Diophantine equations, counting techniques and congruence. Gauss builds on the Euler program. There were 702 entries in 2016.

**Noether** (top 10% years 9–10) includes expansion and factorisation, inequalities, sequences and series, number bases, methods of proof, congruence, circles and tangents. There were 565 entries in 2016.

**Polya** (top 10% year 10) Topics include angle chasing, combinatorics, number theory, graph theory and symmetric polynomials. There were 200 entries in 2016.

**Australian Intermediate Mathematics Olympiad**

This four-hour competition for students up to Year 10 offers a range of challenging and interesting questions. It is suitable for students who have performed well in the AMC (Distinction and above), and is designed as an endpoint for students who have completed the Gauss or Noether stage. There were 1829 entries for 2016 and 15 perfect scores.
MEMBERSHIP OF MCYA COMMITTEES

Mathematics Challenge for Young Australians Committee

**Director**
Dr K McAvaney, Victoria

**Challenge Committee**
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Mr A Edwards, Queensland Studies Authority
Mr B Henry, Victoria
Ms J McIntosh, AMSI, VIC
Mrs L Mottershead, New South Wales
Ms A Nakos, Temple Christian College, SA
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Mr A Canning, Queensland
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Mr B Darcy, Rose Park Primary School, South Australia
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Ms R Jorgenson, Australian Capital Territory
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Mr J Lawson, St Pius X School, NSW
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Mr J Oliver, Northern Territory
Mr G Pointer, Marratville High School, SA
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Ms C Stanley, Queensland Studies Authority
Mr P Swain, Victoria
Dr P Swedosh, The King David School, VIC
Mrs A Thomas, New South Wales
Ms K Trudgian, Queensland
Dr D Wells, USA
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Dr K McAvaney, Victoria (Chair)
Adj Prof M Clapper, Australian Mathematics Trust, ACT
Mr J Dowsey, University of Melbourne, VIC
Dr M Evans, International Centre of Excellence for Education in Mathematics, VIC
Mr B Henry, Victoria

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Mr K Hamann, South Australia
Mr B Henry, Victoria
Dr K McAvaney, Victoria
Dr A M Storozhev, Attorney General's Department, ACT
Emeritus Prof P Taylor, Australian Capital Territory
Dr O Yevdokimov, University of Southern Queensland
MEMBERSHIP OF AMOC COMMITTEES

Australian Mathematical Olympiad Committee 2016

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Prof C Praeger, University of Western Australia
Deputy Chair
Assoc Prof D Hunt, University of New South Wales

Executive Director
Adj Prof Mike Clapper, Australian Mathematics Trust, ACT

Treasurer
Dr P Swedosh, The King David School, VIC

Chair, Senior Problems Committee
Dr N Do, Monash University, VIC

Chair, Challenge
Dr K McAvaney, Deakin University, VIC

Director of Training and IMO Team Leader
Dr A Di Pasquale, University of Melbourne, VIC

IMO Deputy Team Leader
Mr A Elvey Price, University of Melbourne, VIC

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Dr P Swedosh, The King David School, VIC
Dr Chris Wetherell, Radford College, ACT

Representatives
Ms A Nakos, Challenge Committee
Prof M Newman, Challenge Committee
Mr H Reeves, Challenge Committee
AMOC TIMETABLE FOR SELECTION OF THE TEAM TO THE 2016 IMO

August 2016—July 2017

Hundreds of students are involved in the AMOC programs which begin on a state basis. The students are given problem-solving experience and notes on various IMO topics not normally taught in schools.

The students proceed through various programs with the top 25 students, including potential team members and other identified students, participating in a 10-day residential school in December.

The selection program culminates with the April Selection School during which the team is selected.

Team members then receive individual coaching by mentors prior to assembling for last minute training before the IMO.

<table>
<thead>
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<th>Month</th>
<th>Activity</th>
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<tbody>
<tr>
<td>August</td>
<td>Outstanding students are identified from AMC results, MCYA, other competitions and recommendations; and eligible students from previous training programs mainly ASC and AIMO AMOC state organisers invite students to participate in AMOC programs Various state-based programs AMOC Senior Contest</td>
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<tr>
<td>September</td>
<td>Australian Intermediate Mathematics Olympiad</td>
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<td>AMOC School of Excellence</td>
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<td>Personal Tutor Scheme for IMO team members</td>
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<td>July</td>
<td>Short mathematics school for IMO team members 2017 IMO in Rio de Janeiro, Brazil.</td>
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ACTIVITIES OF AMOC SENIOR PROBLEMS COMMITTEE

This committee has been in existence for many years and carries out a number of roles. A central role is the collection and moderation of problems for senior and exceptionally gifted intermediate and junior secondary school students. Each year the Problems Committee provides examination papers for the AMOC Senior Contest and the Australian Mathematical Olympiad. In addition, problems are submitted for consideration to the Problem Selection Committees of the annual Asian Pacific Mathematics Olympiad and the International Mathematical Olympiad.

AMOC Senior Problems Committee October 2015–September 2016
Dr A Di Pasquale, University of Melbourne, VIC
Dr N Do, Monash University, VIC, (Chair)
Dr M Evans, Australian Mathematical Sciences Institute, VIC
Dr I Guo, University of Sydney, NSW
Assoc Prof D Hunt, University of NSW
Dr J Kupka, Monash University, VIC
Assoc Prof H Lausch, Monash University, VIC
Dr K McAvaney, Deakin University, VIC
Dr D Mathews, Monash University, VIC
Dr A Offer, Queensland
Dr C Rao, NEC Australia, VIC
Dr B B Saad, Monash University, VIC
Assoc Prof J Simpson, Curtin University of Technology, WA
Emeritus Professor P J Taylor, Australian Capital Territory
Dr I Wanless, Monash University, VIC

1. 2016 Australian Mathematical Olympiad
The Australian Mathematical Olympiad (AMO) consists of two papers of four questions each and was sat on 9 and 10 February. There were 97 participants including 12 from New Zealand, nine less participants than 2015. Three students, Matthew Cheah, Seyoon Ragavan and Wilson Zhao, achieved perfect scores and seven other students were awarded Gold certificates, 15 students were awarded Silver certificates and 30 students were awarded Bronze certificates.

2. 2016 Asian Pacific Mathematics Olympiad
On Tuesday 8 March students from nations around the Asia-Pacific region were invited to write the Asian Pacific Mathematics Olympiad (APMO). Of the top ten Australian students who participated, there were 1 Gold, 1 Silver and 5 Bronze certificates awarded. Australia finished in 15th place overall.

3. 2016 International Mathematical Olympiad, Hong Kong.
The IMO consists of two papers of three questions worth seven points each. They were attempted by teams of six students from 109 countries on 11 and 12 July in Hong Kong, with Australia being placed 25th. The medals for Australia were two Silver and four Bronze.

4. 2016 AMOC Senior Contest
Held on Tuesday 9 August, the Senior Contest was sat by 79 students (compared to 84 in 2015). This year, the award system has been changed to reflect the same system for the other AMOC olympiads. High Distinction has been changed to Gold certificate, Distinction has been changed to Silver certificate and Credit has been changed to Bronze certificate. Honourable Mention is awarded to students who obtain a perfect score in at least one question but whose total score is below the Bronze certificate cut off. There were five students who obtained Gold certificates with perfect scores and four other students who also obtained Gold certificates. Thirteen students obtained Silver certificates and 14 students obtained Bronze certificates.
Students may work on each of these four problems in groups of up to three, but must write their solutions individually.

**MP1 Hexos**

If two identical regular hexagons are joined side-to-side, that is, a side of one meets a side of the other exactly, then only one shape can be made:

Counting the number of sides of this shape, we find that its perimeter is 10.

A shape made by joining four identical regular hexagons side-to-side is called a *hexo*. There are seven different hexos:

Two shapes are not different if one fits exactly over the other (after turning or flipping if necessary). For example, these two hexos are the same:

Notice that this hexo has perimeter 18.

a Only five of the seven different hexos have perimeter 18. On the worksheet provided, draw the other two hexos and state the perimeter of each.

b On the worksheet provided, draw two different hexos that are joined side-to-side to make a shape of perimeter 26.

c Here is a shape made by joining three different hexos side-to-side. Colour this shape with three colours to show clearly three different hexos.
d  What is the maximum perimeter for a shape that is formed by joining two different hexos side-to-side? Explain how to make such a shape and why no larger perimeter is possible.

**MP2  Cupcakes**

Bruno’s Bakery is famous for its cupcakes.

![Cupcakes](image)

a  On Monday, Bruno baked 3 dozen cupcakes. Maria, the first customer, bought \( \frac{1}{4} \) of them. The next customer, Niko, bought \( \frac{1}{3} \) of the cupcakes that were left. How many cupcakes did Bruno have left after Niko bought his?

b  On Tuesday Eleanor bought 12 cupcakes, which were \( \frac{1}{4} \) of the cupcakes baked that day. How many cupcakes were baked on Tuesday?

c  On Wednesday, the first customer bought \( \frac{1}{2} \) of the cupcakes that Bruno baked that day. The second customer bought \( \frac{1}{4} \) of what remained. Then the third customer bought \( \frac{1}{4} \) of what remained. Finally the fourth customer bought \( \frac{1}{2} \) of what remained. That left just one cupcake, which Bruno ate himself. How many cupcakes did Bruno bake on Wednesday?

d  On Thursday, Bruno baked 18 cupcakes. Maryam bought some of these and shared them equally among her 3 children. Thierry bought the rest of the cupcakes and shared them equally between his 2 children. Each child received a whole number of cupcakes. Find all possible fractions of the 18 cupcakes that Maryam could have bought.

**MP3  Kimmi Dolls**

Suriya decides to give her collection of 33 Kimmi dolls to her four little sisters.

a  Show how Suriya could give out the dolls so that the girl who receives the most has only one more than anyone else.

Suriya decides to give out the dolls so that the difference between the largest number of dolls any girl receives and the smallest number of dolls any girl receives is at most 4.

b  Show how this could be done if one girl receives 11 dolls.

c  Explain why no girl can receive more than 11 dolls.

d  What is the smallest number of dolls any girl can receive? Explain why a smaller number is not possible.
Label the vertices on the top face of a cube as $ABCD$ clockwise, and the corresponding vertices on the bottom face of the cube $EFGH$. Note that $A$ and $G$ are diagonally opposite in the cube.

We want to consider the possible routes along the edges of the cube from one vertex to another. A trail is a route that does not repeat any edge, but it may repeat vertices. For example, $ABFE$ and $ABCDAE$ are trails, but $ABCB$ is not a trail.

The number of edges in a trail is called its length.

a Which vertices can be reached from $A$ with a trail of length 2?

b List all the different trails of length 3 from $A$ to $G$.

c Find a trail of length 7 from $A$ to $G$.

d Which vertices can be reached from $A$ with a trail of length 4?
Students may work on each of these four problems in groups of up to three, but must write their solutions individually.

**UP1 Knightlines**

A *knightline* is drawn on 1 cm square dot paper. It is a line segment that starts at a dot and ends at a dot that is 3 cm away in one direction (horizontal or vertical) and 2 cm away in the other direction. Here are some knightlines:

April used knightlines to draw polygons. Here are two that she drew.

Two shapes are identical if one can be cut out and fitted exactly over the other one, flipping if necessary. If two shapes are not identical, they are different.

- **a** Draw two different quadrilaterals, each using 4 knightlines and each different from April’s quadrilateral above.
- **b** Draw three different quadrilaterals, each using 6 knightlines.
- **c** Draw six different simple hexagons, each using 6 knightlines. (A hexagon is simple if no two sides cross.)
UP2 Grandma’s Eye Drops

Grandma’s eye drops come in a small bottle. The label says there are 2.7 millilitres (2.7 mL) of solution in the bottle and 15 milligrams per millilitre (15 mg/mL) of active ingredient in the solution.

a How many milligrams of active ingredient are there in the bottle?
b The manufacturer produces the eye drops in batches of 10 litres. How many grams of active ingredient do they put in each batch?

The instructions on the bottle say to put one drop in each eye at night. One 2.7 mL bottle lasts 30 days.
c How many millilitres of solution are there in each drop?
d A different bottle contains 63 mg of active ingredient in 4.5 mL of solution. Is this solution stronger or weaker than the solution in the 2.7 mL bottle? Explain why.

UP3 Cube Trails

Label the vertices on the top face of a cube as $ABCD$ clockwise, and the corresponding vertices on the bottom face of the cube $EFGH$. Note that $A$ and $G$ are diagonally opposite on the cube.

We want to consider the possible routes along the edges of the cube from one vertex to another.

A trail is a route that does not repeat any edge, but it may repeat vertices. For example, $ABFE$ and $ABCDAE$ are trails, but $ABCB$ is not a trail.

The number of edges in a trail is called its length.

a List all the different trails of length 3 from $A$ to $G$.
b Find a trail of length 9 that starts at $A$ and finishes at $D$.
c List the vertices that can be reached from $A$ with:
   (1) a trail of length 2
   (2) a trail of length 3
   (3) a trail of length 4.
d Explain why there is no trail of even length from $A$ to $E$. 
UP4 Magic Staircase

The steps on the staircase that leads to the Mathemagician’s Castle are numbered from 1 at the bottom to 100 at the top. Step 11 is a landing on which every climber is stopped by a guard.

When you arrive at the landing, the guard rolls a 20-faced die to give you a number from 1 to 20. From then on you have to take that number of steps at a time to get as close as possible to the castle courtyard, which is on step 100. Only on your final step up to the courtyard may you take less than your given number of steps. For example, if your given number is 15, then from step 11 you move to steps 26, 41, 56, 71, 86, and then 100. If you make a mistake the courtyard guard at the top will send you back to the landing and make you do it again (instead of giving you a cup of hot chocolate, which is how you are welcomed if you climb the steps correctly)!

a If you take 5 steps at a time, what is the number of the last step you would land on before reaching the courtyard?

b Once when I climbed the stairs, the last step I landed on before the courtyard was 89. How many steps was I taking at a time from the landing? Explain your answer.

c Make a list of all the last steps you could have come from to land on step 95.

d What is the number of the lowest step that you could land on immediately before landing on the courtyard? Explain your answer.
**CHALLENGE PROBLEMS – JUNIOR**

Students may work on each of these six problems with a partner but each must write their solutions individually.

**J1 Cube Trails**

Label the vertices on the top face of a cube as $ABCD$ clockwise, and the corresponding vertices on the bottom face of the cube $EFGH$. Note that $A$ and $G$ are diagonally opposite on the cube.

We want to consider the possible routes along the edges of the cube from one vertex to another. A trail is a route that does not repeat any edge, but it may repeat vertices. For example, $ABFE$ and $ABCDAE$ are trails, but $ABCB$ is not a trail. The number of edges in a trail is called its length.

**a** List all the different trails of length 3 from $A$ to $G$.

**b** List the vertices that can be reached from $A$ with:

1. a trail of length 2
2. a trail of length 3
3. a trail of length 4.

**c** Explain why there is no trail of even length from $A$ to $E$.

**d** Determine how many trails of length 3 there are in the cube. Here a trail between two vertices is regarded as the same in either direction.

**J2 Overlaps**

Cutting along the lines on 1 cm square grid paper, Tia produced many squares. The sides of her squares were all longer than 1 cm. She then partially overlapped various pairs of squares and worked out the perimeter of the final (combined) shape. For example, here is a $7 \times 7$ square overlapped with a $4 \times 4$ square with the overlap shaded.

```
7
7
7
7

7
7
7
7

Perimeter = 7 + 7 + 7 + 1 + 1 + 4 + 1 + 2 = 30 cm
```

Note that one square must not be wholly on top of another and that the grid lines of both squares must coincide. Thus the following three shapes are not allowed.
Show how to overlap a $6 \times 6$ square and a $7 \times 7$ square so that the perimeter of the final shape is 48 cm.

Show how to overlap a $6 \times 6$ square and a $7 \times 7$ square so that the perimeter of the final shape is 30 cm.

The overlap of two squares has area $1 \text{ cm}^2$. The perimeter of the final shape is 32 cm. Find all possible sizes of the two squares.

The overlap of two squares has area $12 \text{ cm}^2$. The perimeter of the final shape is 30 cm. Find all possible sizes of the two squares.

**J3 Stocking Farms**

The number of stock that can be grazed on a paddock is given in terms of DSEs (Dry Sheep Equivalents). So if a particular area of land has a stocking capacity of 12 DSEs per hectare, this means that the maximum number of dry sheep (that is, sheep that eat dry food, not milk) that can be grazed on each hectare of land is 12. So a 5 hectare farm with a stocking capacity of 12 DSEs per hectare could be stocked with a maximum of 60 dry sheep. We call this a fully stocked farm.

For other animals, a conversion needs to be made. For dairy cattle the DSEs are as follows.

- One adult (milking) cow: 15 DSEs
- One yearling (a cow not yet milking but eating dry food): 6 DSEs
- One calf (6 to 12 months old): 4 DSEs

Farmers Green, White, Black, Grey and Brown all have farms in the Adelaide Hills, where the maximum stocking capacity of a farm is 12 DSEs per hectare.

Farmer Green has a 100 hectare farm on which there are 60 adult cows, 40 yearlings and 15 calves. Show that the farm is fully stocked.

Farmer White has an 80 hectare farm on which there are 50 adult cows. There are also some yearlings and some calves. Give two possible combinations of yearlings and calves that would fully stock the property.

Grey’s farm has 12 adult cows, 18 yearlings, and 4 calves. Brown’s farm has 10 adult cows, 4 yearlings, and 6 calves. Both farms are fully stocked. To specialise they agree to exchange stock so that Grey has only adult cows and Brown has only yearlings and calves. Any stock that can’t be accommodated will be sold at the market. What stock has to be sold?

Farmer Black has a fully stocked farm. She sells some of her adult cows and restocks with 12 yearlings and some calves, so that her farm is again fully stocked. What is the smallest number of adult cows she could have sold?
J4 Cross Number

Each number in the following cross number puzzle is a 2-, 3- or 4-digit factor of 2016.
No number starts with 0.
No number is a 2-digit square.
No number is repeated.

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<td>18</td>
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a Explain why 16 down is 168.
b Which factor is 7 down? Explain why.
c Fill in the remaining 3-digit factors.
d Complete the puzzle.

J5 Tipping Points

Two empty buckets are placed on a balance beam, one at each end. Balls of the same weight are placed in the buckets one at a time. If the number of balls is the same in each bucket, the beam stays horizontal. If there is a difference of only one ball between the buckets, the beam moves a little but the buckets and balls remain in place. However, if the difference between the number of balls is two or more, the beam tips all the way, the buckets fall off, and all the balls fall out.

There are several bowls, each containing some of the balls and each labelled L or R. If a ball is taken from a bowl labelled L, the ball is placed in the left bucket on the beam. If a ball is taken from a bowl labelled R, the ball is placed in the right bucket.

a Julie arranges six labelled bowls in a row. She takes a ball from each bowl in turn from left to right, and places it in the appropriate bucket. List all sequences of six bowls which do not result in the beam tipping.
b Julie starts again with both buckets empty and with six bowls in a row. As before, she takes a ball from each bowl in turn, places it in the appropriate bucket, and the beam does not tip. She then empties both buckets and takes a ball from the 2nd, 4th, and 6th bowl in turn and places it in the appropriate bucket. Again the beam does not tip. Once more she empties both buckets but this time takes a ball from the 3rd and 6th bowl in turn and places it in the appropriate bucket. Yet again the beam does not tip. List all possible orders in which the six bowls could have been arranged.

For a large number of bowls, a ball could be taken from every bowl, or every second bowl, or every third bowl, and so on. If a ball is taken from bowl $m$, followed by bowl $2m$, then bowl $3m$, and so on (every $m$th bowl), we say an $m$-selection was used. For example, in Part b, Julie used a 1-selection, then a 2-selection, and finally a 3-selection.

c Find all sequences of 11 bowls for which the beam does not tip no matter what $m$-selection is used.

d Show that it is impossible to have a sequence of 12 bowls so that every $m$-selection is non-tipping.

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<tr>
<th>J6 Tossing Counters</th>
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Ms Smartie told all the students in her class to write four different integers from 1 to 9 on the four faces of two counters, with one number on each face. She then asked them to toss both counters simultaneously many times and write down the sum of the numbers that appeared on the upper faces each time.

a Emily put a 1 on one face of one counter and a 7 on one face of the other counter. The only sums that Emily was able to get were 8, 9, 10 and 11. List the two possible combinations for the four numbers on her counters.

b The only sums that Jack was able to get were 7, 8, 9 and 10. List all four combinations for the four numbers on his counters.

c Jill wrote 4 and 5 on opposite sides of one counter. The only sums she was able to get were three consecutive integers. Find all possible ways the second counter could be numbered.

d Ben was only able to get sums that were four consecutive numbers. Show that either one or three of the numbers he wrote on the counters were even.
Students may work on each of these six problems with a partner but each must write their solutions individually.

I1 COP-graphs

The Roads Department in the state of Graphium uses graphs consisting of dots (called vertices) that represent towns, and lines (called edges) that represent roads between towns. To keep track of things, they label the vertices with consecutive positive integers so that if two towns are joined by a road, then the numbers on the corresponding vertices are coprime, that is, their only common factor is 1. Such a labelled graph is called a COP-graph. Note that if two towns do not have a road joining them, then their labels may or may not be coprime. Here are two examples of COP-graphs, one with labels 1 to 5, the other with labels 1 to 6.

![COP-graph A](image)

![COP-graph B](image)

a Show that the labels on COP-graphs A and B could be replaced with integers 2 to 6 and 2 to 7 respectively so they remain COP-graphs.

b Explain why COP-graph B could not remain a COP-graph if its labels were replaced with integers 5 to 10.

c Explain why COP-graph A remains a COP-graph no matter what set of five consecutive positive integers label the vertices.

d If a COP-graph has six vertices and labels 1 to 6, what is the maximum number of edges it can have? Explain.

I2 Tipping Points

See Junior Problem 5.

I3 Tossing Counters

Ms Smartie told all the students in her class to write four different integers from 1 to 9 on the four faces of two counters, with one number on each face. She then asked them to toss both counters simultaneously many times and write down the sum of the numbers that appeared on the upper faces each time.

a The only sums that Jack was able to get were 8, 9, 10, and 11. Find all five possible combinations of four numbers on the counters.

b Jill wrote 4 and 5 on opposite sides of one counter. The only sums she was able to get were three consecutive integers. Find all possible ways the second counter could be numbered.

c Ben was only able to get sums that were four consecutive numbers. Show that either one or three of the numbers he wrote on the counters were even.

d Show that it is possible to number four counters with 8 different positive integers less than 20, one number on each face, so that the sums that appear are 16 consecutive numbers.

I4 Ionofs

The Ionof (integer on number of factors) of an integer is the integer divided by the number of factors it has. For example, 18 has 6 factors so Ionof(18) = 18/6 = 3, and 27 has 4 factors so Ionof(27) = 27/4 = 6.75.

a Find Ionof(36).

b Explain why Ionof($pq$) is not an integer if $p$ and $q$ are distinct primes.

c If $p$ and $q$ are distinct primes, find all numbers of the form $pq^4$ whose Ionof is an integer.

d Show that the square of any prime number is the Ionof of some integer.
## I5 Cube Trails

Label the vertices on the top face of a cube as $ABCD$ clockwise, and the corresponding vertices on the bottom face of the cube $EFGH$. Note that $A$ and $G$ are diagonally opposite on the cube.

A robot travels along the edges of this cube always starting at $A$ and never repeating an edge. This defines a *trail* of edges. For example, $ABFE$ and $ABCDAE$ are trails, but $ABCB$ is not a trail. The number of edges in a trail is called its *length*.

At each vertex, the robot must proceed along one of the edges that has not yet been traced, if there is one. If there is a choice of untraced edges, the following probabilities for taking each of them apply.

If only one edge at a vertex has been traced and that edge is vertical, then the probability of the robot taking each horizontal edge is 1/2.

If only one edge at a vertex has been traced and that edge is horizontal, then the probability of the robot taking the vertical edge is 2/3 and the probability of the robot taking the horizontal edge is 1/3.

If no edge at a vertex has been traced, then the probability of the robot taking the vertical edge is 2/3 and the probability of the robot taking each of the horizontal edges is 1/6.

In your solutions to the following problems use *exact fractions* not decimals.

a If the robot moves from $A$ to $D$, what is the probability it will then move to $H$? If the robot moves from $A$ to $E$, what is the probability it will then move to $H$?

b What is the probability the robot takes the trail $ABCG$?

c Find two trails of length 3 from $A$ to $G$ that have probabilities of being traced by the robot that are different to each other and to the probability for the trail $ABCG$.

d What is the probability that the robot will trace a trail of length 3 from $A$ to $G$?

## I6 Coverem

A disc is placed on a grid composed of small $1 \times 1$ squares.

If the disc has diameter $2\sqrt{2}$ and its centre is at a grid point, then it completely covers four grid squares.
In all of the following questions use exact surds in your calculations, not decimal approximations.

a Find the smallest diameter for a disc that will completely cover all 8 grid squares shown here.

b Find the smallest diameter for a disc that can cover 5 grid squares if they are in some suitable configuration.

c Find the minimum and maximum number of grid squares that can be completely covered by a disc of diameter 3.

d Find the smallest radius for a disc that will completely cover all 7 grid squares shown here.
**MP1 Hexos**

**a**

![Hexo shapes with perimeters 16 and 14](image)

**b** Here are two shapes with perimeter 26. There are others.

![Hexo shapes](image)

**c** Here are two colourings showing three different hexos. There are others.

![Hexo colourings](image)

**d** The perimeter of the combined shape is the sum of the perimeters of the separate hexos minus twice the number of common sides where they meet. There must be at least one common side.

The maximum perimeter of a hexo is 18. Hence the perimeter of the combined shape can’t be more than $18 + 18 - 2 = 34$.

The following shape has perimeter 34.

![Combined shape](image)

So the maximum perimeter is 34.
**MP2 Cupcakes**

a Alternative i

The number of cupcakes baked on Monday is $3 \times 12 = 36$.
The number of cupcakes bought by Maria is $\frac{1}{4} \times 36 = 9$.
The number of cupcakes left is $36 - 9 = 27$.
The number of cupcakes bought by Niko is $\frac{1}{3} \times 27 = 9$.
So the number of cupcakes Bruno had left is $27 - 9 = 18$.

Alternative ii

The number of cupcakes baked on Monday is $3 \times 12 = 36$.
The number of cupcakes left after Maria is $\frac{3}{4} \times 36 = 27$.
The number of cupcakes left after Niko is $\frac{2}{3} \times 27 = 18$.

b Eleanor bought 12 cupcakes baked on Tuesday.
This was $\frac{1}{4}$ of all cupcakes baked on Tuesday.
So the number of cupcakes baked on Tuesday is $4 \times 12 = 48$.

c Alternative i

Work backwards.
Only 1 cupcake remained after the fourth customer. So the fourth customer bought exactly 1 cupcake.
Hence the third customer left exactly 2 cupcakes. So the third customer bought exactly 2 cupcakes.
Hence the second customer left exactly 4 cupcakes. So the second customer bought exactly 4 cupcakes.
Hence the first customer left exactly 8 cupcakes. So the first customer bought exactly 8 cupcakes.
Hence there were originally 16 cupcakes.

Alternative ii

The first customer bought $\frac{1}{2}$ of the cupcakes baked that day. So the fraction left by the first customer was $1 - \frac{1}{2} = \frac{1}{2}$.
The second customer bought $\frac{1}{2}$ of those left, that is, $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. So the fraction left by the second customer was $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.
The third customer bought $\frac{1}{2}$ of those left, that is, $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$. So the fraction left by the third customer was $\frac{1}{4} - \frac{1}{8} = \frac{1}{16}$.
The fourth customer bought $\frac{1}{2}$ of those left, that is, $\frac{1}{2} \times \frac{1}{8} = \frac{1}{16}$. So the fraction left by the fourth customer was $\frac{1}{8} - \frac{1}{16} = \frac{1}{16}$.
Since only one was left, there must have been 16 baked.

d The number of cupcakes that Maryam bought is 3, 6, 9, 12, or 15. The number of cupcakes that Thierry bought is 2, 4, 6, 8, 10, 12, 14, or 16.
The only combinations that total 18 are 6 for Maryam and 12 for Thierry, and 12 for Maryam and 6 for Thierry.
So the only fractions of the 18 cupcakes that Maryam could have bought were $\frac{6}{18} = \frac{1}{3}$ and $\frac{12}{18} = \frac{2}{3}$.

**MP3 Kimmi Dolls**

a Alternative i

Since $33 \div 4 = 8$ with a remainder of 1, three girls could get 8 dolls each and the fourth girl 9 dolls.

Alternative ii

If one girl gets 7 or fewer dolls, then each of the other three girls gets at most 8 dolls. This means the total number of dolls can’t be any more than $7 + 8 + 8 + 8 = 31$, which is not enough.
So every girl must get at least 8 dolls. Since $4 \times 8 = 32$, which is 1 less than 33, Suriya must give 8 dolls to each of 3 sisters and 9 dolls to the fourth sister.

b If one girl gets 11 dolls, each of the other three girls must get at least 7 dolls. Since $11 + 7 + 7 + 7 = 32$, which is 1 less than 33, two girls get 7 dolls each and one girl gets 8.

c Alternative i

If one girl gets 12 or more dolls, the other three must get at least 8 each. Then the total number of dolls would have to be at least $12 + 8 + 8 + 8 = 36$, which is too many. So no girl can get 12 or more dolls.

Alternative ii

If one girl gets 12 or more dolls, then there are at most 21 dolls for the other three girls. This means that one girl gets at most 7 dolls, which is 5 less than 12. So no girl can get 12 or more dolls.
In Part b we saw how it’s possible for at least one girl to get 7 dolls. So we need to see if we can give one girl less than 7 dolls.

Suppose one girl gets 6 dolls. This leaves 27 dolls for the other three girls. If each of them gets 9, all the dolls are used up and Suriya’s rule is satisfied.

If one girl gets 5 or fewer dolls, then each of the other girls gets at most 9. This means the total number of dolls can’t be any more than $5 + 9 + 9 + 9 = 32$, which is not enough. So no girl can get 5 dolls.

(Alternatively, if one girl gets 5 or fewer dolls, then there are at least 28 dolls for the other three girls. This means that one girl gets at least 10 dolls, which is 5 more than 5. So no girl can get 5 dolls.)

Therefore 6 is the least number of dolls that any girl can get.

---

**MP4 Cube Trails**

**a** Trails of length 2 from $A$ are:

\[
\begin{align*}
ABC & \quad ADC & \quad AEF \\
ABF & \quad ADH & \quad AEH
\end{align*}
\]

So the vertices that can be reached from $A$ with a trail of length 2 are $C$, $F$, $H$.

**b** Trails of length 3 from $A$ to $G$ are the trails of length 2 in Part a followed by $G$:

\[
ABCG, ABFG, ADCG, ADHG, AEFG, AEHG.
\]

**c** Here is one trail of length 7: $ABFEHDCG$. There are others.

**d** Alternative i

We list, in alphabetical order, all trails of length 4 from $A$:

\[
\begin{align*}
ABCDA & \quad ADCBA & \quad AEFBA \\
ABCDH & \quad ADCBF & \quad AEFBC \\
ABCGF & \quad ADCGF & \quad AEFGC \\
ABCGH & \quad ADCGH & \quad AEFGH \\
ABFEA & \quad ADHEA & \quad AEHDA \\
ABFEH & \quad ADHEF & \quad AEHDC \\
ABFGC & \quad ADHGC & \quad AEHGC \\
ABFGH & \quad ADHGF & \quad AEHGF
\end{align*}
\]

Hence trails of length 4 from $A$ reach only vertices $A$, $C$, $F$, $H$.

**Alternative ii**

We can reach vertices $A$, $C$, $F$, $H$ from $A$ with trails of length 4. For example: $ABCDA$, $AEHGC$, $ADHEF$, $ABCGH$.

We now show that no other vertices can be reached from $A$ with a trail of length 4.

Any trail of length 4 can be divided into 2 trails of length 2.

From Part a, any trail of length 2 that starts at $A$ must finish at one of $C$, $F$, $H$. Notice that $C$, $F$, $H$ are the diagonal opposites of $A$ on the 3 cube faces that have $A$ as a vertex. From the symmetry of the cube, any trail of length 2 that starts at one of $A$, $C$, $F$, $H$ must finish at one of the other three vertices.

So each of the trails of length 2 starts at one of $A$, $C$, $F$, $H$ and finishes at one of $A$, $C$, $F$, $H$. Hence any trail of length 4 that starts at $A$ must finish at one of $A$, $C$, $F$, $H$.

**Alternative iii**

We can reach vertices $A$, $C$, $F$, $H$ from $A$ with trails of length 4. For example: $ABCDA$, $AEHGC$, $ADHEF$, $ABCGH$.

We now show that no other vertices can be reached from $A$ with a trail of length 4.

Notice that the vertices $A$, $C$, $F$, $H$ are black and the vertices $B$, $D$, $E$, $G$ are white. Also each edge in the cube joins a black vertex to a white vertex. So the vertices in any trail of length 4 that starts at $A$ must alternate black and white, starting with black and finishing with black. Hence any trail of length 4 that starts at $A$ must finish at one of $A$, $C$, $F$, $H$. 

---

33  Mathematics Contests The Australian Scene 2016
UP1 Knightlines

a

b
UP2  Grandma’s Eye Drops

a  The amount of active ingredient in the bottle is $15 \times 2.7 = 40.5\, \text{mg}$.

b  Since $1000\, \text{mL} = 1\, \text{L}$ and each mL of solution has $15\, \text{mg}$ of active ingredient, $1\, \text{L}$ of solution has $15000\, \text{mg}$ of active ingredient. Since $1000\, \text{mg} = 1\, \text{g}$, $1\, \text{L}$ of solution has $15\, \text{g}$ of active ingredient. Hence the amount of active ingredient in each batch of $10\, \text{L}$ is $150\, \text{g}$.

c  The bottle lasts 30 days and each day 2 drops are used. Hence there are 60 drops in each $2.7\, \text{mL}$ bottle. So the amount of solution in each drop is

$$\frac{2.7}{60} = \frac{27}{600} = \frac{9}{200} = \frac{4.5}{100} = 0.045\, \text{mL}.$$  

d  The number of milligrams of active ingredient per millilitre of solution in the $4.5\, \text{mL}$ bottle is

$$\frac{63}{4.5} = \frac{630}{45} = \frac{126}{9} = 14.$$  

This is less than $15\, \text{mg/mL}$. So the solution in the larger bottle is weaker.
UP3 Cube Trails

a $ABCG, ABFG, ADCG, ADHG, AEFG, AEHG$.

b Here is one trail: $ADHEABFGCD$. There are others.

c Alternative i

(1) The trails of length 2 from $A$ are:

$$
\begin{array}{ccc}
ABC & ADC & AEF \\
ABF & ADH & AEH \\
\end{array}
$$

So trails of length 2 from $A$ reach only vertices $C, F, H$.

(2) The trails of length 3 from $A$ are:

$$
\begin{array}{ccc}
ABCD & ADCB & AEFB \\
ABCG & ADCG & AEFG \\
ABFE & ADHE & AEHD \\
ABFG & ADHG & AEHG \\
\end{array}
$$

So trails of length 3 from $A$ reach only vertices $B, D, E, G$.

(3) The trails of length 4 from $A$ are:

$$
\begin{array}{ccc}
ABCD & ADCB & AEFB \\
ABCG & ADCG & AEFG \\
ABFE & ADHE & AEHD \\
ABFG & ADHG & AEHG \\
\end{array}
$$

So trails of length 4 from $A$ reach only vertices $A, C, F, H$.

Alternative ii

(1) We can reach vertices $C, F, H$ from $A$ with trails of length 2. For example: $ABC, ABF, ADH$.

(2) We can reach vertices $B, D, E, G$ from $A$ with trails of length 3. For example: $ABCD, ABCG, ABFE, ADCB$.

(3) We can reach vertices $A, C, F, H$ from $A$ with trails of length 4. For example: $ABCD, ABCDH, ABCGF, ABFGC$.

We now show that, in each case, no other vertices can be reached from $A$. Notice that the vertices $A, C, F, H$ are black and the vertices $B, D, E, G$ are white. Also each edge in the cube joins a black vertex to a white vertex. So the vertices in any trail that starts at $A$ must alternate black and white, starting with black.

Hence, for any trail that starts at $A$, (1) if it has length 2 then it must finish at one of the black vertices $C, F, H$ (A is excluded because it would repeat an edge), (2) if it has length 3 then it must finish at one of the white vertices $B, D, E, G$, (3) if it has length 4 then it must finish at one of the black vertices $A, C, F, H$.

d Alternative i

Any trail of even length can be divided into trails of length 2.

From Part c, any trail of length 2 that starts at $A$ must finish at one of $C, F, H$. Notice that $C, F, H$ are the diagonal opposites of $A$ on the 3 cube faces that have $A$ as a vertex. From the symmetry of the cube, any trail of length 2 that starts at one of $A, C, F, H$ must finish at one of the other three vertices.

So each of the trails of length 2 starts at one of $A, C, F, H$ and finishes at one of $A, C, F, H$. Hence any trail of even length that starts at $A$ must finish at one of $A, C, F, H$. This excludes $E$.

Alternative ii

Notice that vertices $A, C, F, H$ are black and vertices $B, D, E, G$ are white. Also each edge in the cube joins a black vertex to a white vertex.

So the vertices in any trail from $A$ to $E$ must alternate black and white, starting with black and finishing with white. Therefore any trail from $A$ to $E$ must have odd length.
You would start from the landing like this: 11, 16, 21, 26, \ldots. Each step you land on before the courtyard has a number ending in 1 or 6. So the number of the last step you land on before the courtyard is 96.

Since $100 - 89 = 11$, the number of steps you were taking at a time was at least 11. The number of steps from the landing at step 11 to step 89 is $89 - 11 = 78$. So the number of steps you were taking at a time was also a factor of 78. The factors of 78 are 1, 2, 3, 6, 13, 26, 39 and 78. The only one of these factors that is greater than or equal to 11 and less than or equal to 20 is 13.

Since $95 - 11 = 84$, the number of steps you were taking at a time was a factor of 84. The factors of 84 are 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 84. If you were taking 1 step at a time you came from 94. If 2, 93. If 3, 92. If 4, 91. If 6, 89. If 7, 88. If 12, 83. If 14, 81. The other factors of 84 are greater than 20 so could not have been rolled with the die.

d Alternative i

Since the number of steps you take at a time is less than or equal to 20, your last step cannot be lower than 80. If your last step is 80, then you must take 20 steps at a time to reach the courtyard. Since $80 - 11 = 69$, the number of steps you take at a time must also be a factor of 69. This is impossible. So the last step was not 80.

Suppose your last step is 81. Since $81 - 11 = 70$, the number of steps you take at a time must be a factor of 70 that is less than or equal to 20. These factors are 1, 2, 5, 7, 10, 14. In each case at least one more step could be reached before 100. So the last step was not 81.

Suppose your last step is 82. Since $82 - 11 = 71$, the number of steps you take at a time must be a factor of 71 that is less than or equal to 20. The only such factor is 1. Since 82 + 1 is less than 100, the last step was not 82.

Suppose your last step is 83. Since $83 - 11 = 72$, the number of steps you take at a time must be a factor of 72 less than or equal to 20. Since $100 - 83 = 17$, the factor must be greater than 17. Since 18 is such a factor, the last step could be 83.

Thus the number of the lowest step that you could land on before stepping onto the courtyard is 83.

Alternative ii

The table shows, for each number $n$ from 1 to 20, the largest number $m$ less than 100 that is 11 plus a multiple of $n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m$</th>
<th>$n$</th>
<th>$m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$99 = 11 + 88 \times 1$</td>
<td>11</td>
<td>$99 = 11 + 8 \times 11$</td>
</tr>
<tr>
<td>2</td>
<td>$99 = 11 + 44 \times 2$</td>
<td>12</td>
<td>$95 = 11 + 7 \times 12$</td>
</tr>
<tr>
<td>3</td>
<td>$98 = 11 + 29 \times 3$</td>
<td>13</td>
<td>$89 = 11 + 6 \times 13$</td>
</tr>
<tr>
<td>4</td>
<td>$99 = 11 + 22 \times 4$</td>
<td>14</td>
<td>$95 = 11 + 6 \times 14$</td>
</tr>
<tr>
<td>5</td>
<td>$96 = 11 + 17 \times 5$</td>
<td>15</td>
<td>$86 = 11 + 5 \times 15$</td>
</tr>
<tr>
<td>6</td>
<td>$95 = 11 + 14 \times 6$</td>
<td>16</td>
<td>$91 = 11 + 5 \times 16$</td>
</tr>
<tr>
<td>7</td>
<td>$95 = 11 + 12 \times 7$</td>
<td>17</td>
<td>$96 = 11 + 5 \times 17$</td>
</tr>
<tr>
<td>8</td>
<td>$99 = 11 + 11 \times 8$</td>
<td>18</td>
<td>$83 = 11 + 4 \times 18$</td>
</tr>
<tr>
<td>9</td>
<td>$92 = 11 + 9 \times 9$</td>
<td>19</td>
<td>$87 = 11 + 4 \times 19$</td>
</tr>
<tr>
<td>10</td>
<td>$91 = 11 + 8 \times 10$</td>
<td>20</td>
<td>$91 = 11 + 4 \times 20$</td>
</tr>
</tbody>
</table>

Thus the number of the lowest step that you could land on before stepping onto the courtyard is 83.

Alternative iii

If your step size is 20, then the last step number before the courtyard will be $11 + 4 \times 20 = 91$.
If your step size is 19, then the last step number before the courtyard will be $11 + 4 \times 19 = 87$.
If your step size is 18, then the last step number before the courtyard will be $11 + 4 \times 18 = 83$.
If your step size is 17 or smaller, then the last step number before the courtyard will be at least $100 - 17 = 83$.

Thus the number of the lowest step that you could land on before stepping onto the courtyard is 83.
J1 Cube Trails

a \( ABCG, ABFG, ADCG, ADHG, AEFG, AEHG \).

b Alternative i

(1) The trails of length 2 from \( A \) are:

\[
\begin{array}{ccc}
ABC & ADC & AEF \\
ABF & ADH & AEH \\
\end{array}
\]

So trails of length 2 from \( A \) can reach vertices \( C, F, H \).

(2) The trails of length 3 from \( A \) are:

\[
\begin{array}{ccc}
ABCD & ADCB & AEFB \\
ABCG & ADCG & AEFG \\
ABFE & ADHE & AEHD \\
ABFG & ADHG & AEHG \\
\end{array}
\]

So trails of length 3 from \( A \) can reach vertices \( B, D, E, G \).

(3) The trails of length 4 from \( A \) are:

\[
\begin{array}{ccc}
ABCDA & ADCBA & AEFBA \\
ABCDH & ADCBF & AEFBC \\
ABCGF & ADCGF & AEFGC \\
ABCGH & ADCGH & AEFGH \\
ABFEA & ADHEA & AEHDA \\
ABFEH & ADHEF & AEHDC \\
ABFGC & ADHGC & AEHGC \\
ABFGH & ADHGF & AEHGF \\
\end{array}
\]

So trails of length 4 from \( A \) can reach vertices \( A, C, F, H \).

Alternative ii

(1) We can reach vertices \( C, F, H \) from \( A \) with trails of length 2. For example: \( ABC, ABF, ADH \).

(2) We can reach vertices \( B, D, E, G \) from \( A \) with trails of length 3. For example: \( ABCD, ABCG, ABFE, ADCB \).

(3) We can reach vertices \( A, C, F, H \) from \( A \) with trails of length 4. For example: \( ABCDA, ABCDH, ABCGF, ABFGC \).

We now show that, in each case, no other vertices can be reached from \( A \). Notice that the vertices \( A, C, F, H \) are black and the vertices \( B, D, E, G \) are white. Also each edge in the cube joins a black vertex to a white vertex. So the vertices in any trail that starts at \( A \) must alternate black and white, starting with black.

Hence, for any trail that starts at \( A \), (1) if it has length 2 then it must finish at one of the black vertices \( C, F, H \) (\( A \) is excluded because it would repeat an edge), (2) if it has length 3 then it must finish at one of the white vertices \( B, D, E, G \), (3) if it has length 4 then it must finish at one of the black vertices \( A, C, F, H \).

c Alternative i

Any trail of even length can be divided into trails of length 2.

From Part b, any trail of length 2 that starts at \( A \) must finish at one of \( C, F, H \). Notice that \( C, F, H \) are the diagonal opposites of \( A \) on the 3 cube faces that have \( A \) as a vertex. From the symmetry of the cube, any trail of length 2 that starts at one of \( A, C, F, H \) must finish at one of the other three vertices.

So each of the trails of length 2 starts at one of \( A, C, F, H \) and finishes at one of \( A, C, F, H \). Hence any trail of even length that starts at \( A \) must finish at one of \( A, C, F, H \). This excludes \( E \).
Alternative ii

Notice that vertices $A$, $C$, $F$, $H$ are black and vertices $B$, $D$, $E$, $G$ are white. Also each edge in the cube joins a black vertex to a white vertex.

So the vertices in any trail from $A$ to $E$ must alternate black and white, starting with black and finishing with white. Therefore any trail from $A$ to $E$ must have odd length.

d From the first solution to Part b, there are 12 trails of length 3 starting at $A$. From the symmetry of the cube there are 12 trails of length 3 starting at each of the 8 vertices. Listing these gives 96 trails.

No trail of length 3 starts and finishes at the same vertex. So in our list of 96 trails, each trail appears exactly twice: once in one direction and once in the opposite direction. Hence the number of trails of length 3 in the cube is $96/2 = 48$.

**J2 Overlaps**

a

![Diagram of overlapping shapes]

Perimeter is $(2 \times 7) + (2 \times 6) + (2 \times 6) + (2 \times 5) = 14 + 12 + 12 + 10 = 48$ cm, or perimeter is $(4 \times 7) + (4 \times 6) - 4 = 28 + 24 - 4 = 48$ cm.

b

![Diagram of overlapping shapes]

Perimeter is $(2 \times 7) + 8 + 6 + 1 + 1 = 14 + 16 = 30$ cm, or perimeter is $(4 \times 7) + (4 \times 6) - (2 \times 6) - (2 \times 5) = 28 + 24 - 12 - 10 = 30$ cm.
c Since the overlap has area $1 \text{ cm}^2$, it must be a grid square in the corner of each overlapping square.

\[ 1 \]

Since the perimeter of the final shape is 32 cm and the perimeter of the overlapping square is 4 cm, the sum of the perimeters of the original two squares is 36 cm.

**Alternative i**

Since the sides of a square are at least 2 cm, its perimeter is at least 8 cm. So the perimeter of the other square is at most 28 cm. Since the perimeter of a square is a multiple of 4, one square has perimeter 28, 24, 20 and the other has perimeter 8, 12, 16 respectively. So the squares are $7 \times 7$ and $2 \times 2$, or $6 \times 6$ and $3 \times 3$, or $5 \times 5$ and $4 \times 4$.

**Alternative ii**

Since the perimeter of a square is 4 times its side length, the sum of the perimeters of the two overlapping squares is 4 times the sum of the lengths of one side from each square. So the sum of the lengths of one side from each square is $36/4 = 9$ cm. Hence the only possible side lengths for the two overlapping squares are 2 and 7, 3 and 6, and 4 and 5 cm.

**Alternative iii**

Let the squares be $a \times a$ and $b \times b$ with $a \leq b$.

As shown in the next diagram, a shape formed from the two overlapping squares has the same perimeter as the smallest square that encloses the shape.

\[ a \]

\[ b \]

\[ 1 \]

Hence $4(a + b - 1) = 32$. So $a + b - 1 = 8$ and $a + b = 9$. Thus $a = 2$ and $b = 7$, or $a = 3$ and $b = 6$, or $a = 4$ and $b = 5$.

d The overlap is a rectangle which is wholly inside each of the overlapping squares and along at least one side of each. Disregarding rotations and reflections, there are four cases as indicated.
Note that the perimeter of the final shape is the sum of the perimeters of the two overlapping squares minus the perimeter of the overlap.

Since the area of the overlap is 12 cm², the overlap in each of the diagrams above is one of the rectangles 12 × 1, 6 × 2, or 4 × 3.

**Alternative i**

If the overlap is a 12 × 1 rectangle, then the side length of each overlapping square is at least 12 cm. Then the perimeter of the final shape is at least \(8 \times 12 - 2 \times (12 + 1) = 96 - 26 = 70\) cm. Since 70 > 30, the overlap rectangle is not 12 × 1.

If the overlap is a 6 × 2 rectangle, then the side length of each overlapping square is at least 6 cm. Then the perimeter of the final shape is at least \(8 \times 6 - 2 \times (6 + 2) = 48 - 16 = 32\) cm. Since 32 > 30, the overlap rectangle is not 6 × 2.

If the overlap is a 4 × 3 rectangle, the sum of the perimeters of the overlapping squares is \(30 + 2 \times (4 + 3) = 30 + 14 = 44\) cm. Hence the sum of the lengths of one side from each square is \(44 / 4 = 11\) cm. Also the side length of each overlapping square is at least 4 cm. So the only possibilities are a 4 × 4 square overlapping a 7 × 7 square, and a 5 × 5 square overlapping a 6 × 6 square.

**Alternative ii**

Let the squares be \(a \times a\) and \(b \times b\) with \(a \leq b\).

If the overlap is a 12 × 1 rectangle, the perimeter of the final shape is \(4a + 4b - 26 = 30\) cm. So \(4a + 4b = 56\) and \(a + b = 14\). This is impossible since each of \(a\) and \(b\) must be at least 12 cm.

If the overlap is a 6 × 2 rectangle, the perimeter of the final shape is \(4a + 4b - 16 = 30\) cm. So \(4a + 4b = 46\), which is impossible because 4 does not divide 46.

If the overlap is a 4 × 3 rectangle, the perimeter of the final shape is \(4a + 4b - 14 = 30\) cm. So \(4a + 4b = 44\) and \(a + b = 11\). Since \(a\) must be at least 4, we have \(a = 4\) and \(b = 7\) or \(a = 5\) and \(b = 6\).

So the two squares must be 4 × 4 and 7 × 7, or 5 × 5 and 6 × 6.

### J3 Stocking Farms

**a** The capacity of Farmer Green’s 100 hectare farm is \(100 \times 12 = 1200\) DSEs. It is stocked with 60 adult cows or \(60 \times 15 = 900\) DSEs, 40 yearling or \(40 \times 6 = 240\) DSEs, and 15 calves or \(15 \times 4 = 60\) DSEs. Since \(900 + 240 + 60 = 1200\), Green’s farm is fully stocked.

**b** The capacity of Farmer White’s 80 hectare farm is \(80 \times 12 = 960\) DSEs. The farm is carrying 50 adult cows which is \(50 \times 15 = 750\) DSEs. So, to fully stock the farm, the yearlings and calves must total \(960 - 750 = 210\) DSEs.

If the number of yearlings is \(y\) and the number of calves is \(c\), then \(6y + 4c = 210\) or \(3y + 2c = 105\). So \(y\) must be odd. There are many possible combinations:

<table>
<thead>
<tr>
<th>Yearlings</th>
<th>Calves</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 3 5 7 9 11 13 15 17</td>
<td>51 48 45 42 39 36 33 30 27</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Yearlings</th>
<th>Calves</th>
</tr>
</thead>
<tbody>
<tr>
<td>19 21 23 25 27 29 31 33</td>
<td>24 21 18 15 12 9 6 3</td>
</tr>
</tbody>
</table>

**c** The table shows the number of DSEs for the various combinations of animals.

<table>
<thead>
<tr>
<th>Farm</th>
<th>Adult cows</th>
<th>Yearlings and calves</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grey’s</td>
<td>12 × 15 = 180</td>
<td>18 × 6 + 4 × 4 = 124</td>
</tr>
<tr>
<td>Brown’s</td>
<td>10 × 15 = 150</td>
<td>4 × 6 + 6 × 4 = 48</td>
</tr>
</tbody>
</table>

If Brown took away all his adult cows (150 DSEs), then Grey could move all her yearlings and calves (124 DSEs) to his farm. Since 124 = \((8 \times 15) + 4\), Grey’s farm can then accommodate only 8 of Brown’s adult cows. Hence Brown must sell 2 adult cows at the market.
d If Black’s farm is to remain fully stocked the number of adult cows must have the same DSE as the total DSE for the yearlings and calves. So, if the number of adult cows she sold is \( x \) and the number of calves she bought is \( y \), we have \( 15x = 12 \times 6 + 4y = 72 + 4y \). Thus \( x \) is even and as small as possible. If \( x = 2 \) or \( 4 \), then \( y \) is negative. If \( x = 6 \), then \( y \) is not an integer. If \( x = 8 \), then \( y = 12 \). So the minimum number of adult cows Farmer Black could have sold is 8.

**J4 Cross Number**

The only available numbers to fill the puzzle are:

12, 14, 18, 21, 24, 28, 32, 42, 48, 56, 63, 72, 84, 96, 112, 126, 144, 168, 224, 252, 288, 336, 504, 672, 1008, 2016.

In the following, A means across and D means down.

a Number 15A must be 2016 or 1008. Since 16D cannot start with 0, 15A is 2016. So 18A is 1008. Then 168 is the only possible factor for 16D.

b From Part a, we have:

Now 12D is 112, 252 or 672. If it is 112, then 11A is also 112, which is not allowed. If 12D is 672, then 11A is 168, which is already used. So 12D must be 252. Hence 11A is 126 or 224. Since 7D cannot end in 1, 11A must be 224. Then 7D is 112 or 672. If 7D is 672, then 7A is 63, leaving no factor for 5D. So 7D is 112 and we have:
c The remaining 3-digit factors are: 126, 144, 288, 336, 504, 672.

Now 14D is 56 or 96. Since no 3-digit factor starts with 9, 14D is 56. Hence 14A is 504, the only 3-digit factor starting with 5. Then 13D is 144, the only 3-digit factor with middle digit 4.

The remaining 3-digit factors are now: 126, 288, 336, 672.

Since 8A and 4D end in the same digit, they must be 126 and 336 in some order. Hence 9A is 288 or 672. If 9A is 672, then 9D is 64 which is not a factor of 2016. So 9A is 288. Since 8A is 126 or 336 and 38 is not a factor of 2016, 8D is 18. So 8A is 126, 4D is 336, and we have:

d The remaining 2-digit factors are:

12, 14, 21, 28, 32, 42, 48, 63, 72, 84, 96.

Then 17D is 21 (the only 2-digit factor ending in 1), 6A is 32 (the only 2-digit factor starting with 3), 3A is 63 (the only 2-digit factor ending in 3), and 1D is 96 (the only 2-digit factor ending in 6).

The remaining 2-digit factors are now: 12, 14, 28, 42, 48, 72, 84.

Since 5A and 5D start with the same digit, that digit is 1 or 4. Since 7A starts with 1, 5A and 5D are 42 and 48 in some order. Since 7A is 12 or 14, 5D is 42. So 7A is 12 and 5A is 48. Hence 2D is 28, 13A is 14, and 10D is 84. So we have:
J5 Tipping Points

a Alternative i

Moving through the sequence of bowls from the first to the last, the beam will tip if and only if the difference in the number of Ls and Rs is at any stage greater than 1. The following tree diagrams show the possible sequences, from left to right, of 6 bowls that avoid the beam tipping.

So the only sequences of bowls for which the beam does not tip are:
LRLRLR, LRLRRL, LRRLLR, LRLRLR,
RLLRLR, RLLRRL, RLRLLR, RLRLRL.

Alternative ii

To avoid tipping the beam, the first two bowls in the sequence must be LR or RL. Either way, the beam remains perfectly balanced. So the next two bowls in the sequence must be LR or RL. Again, either way, the beam remains perfectly balanced. So the last two bowls in the sequence must be LR or RL. Hence the only sequences of bowls for which the beam does not tip are:
LRLRLR, LRLRRL, LRRLLR, LRLRLR,
RLLRLR, RLLRRL, RLRLLR, RLRLRL.
b Since Julie uses all bowls and does not tip the beam, the bowls must be in one of the eight sequences found in Part a:

LRLRLR, LRLRLR, LRRLLR, LRRRLR, RLLRLR, RLLRLR, RLRLLR, RLRRLR.

Julie uses bowls 2, 4, 6 without tipping the beam. This eliminates the sequences LRLRLR, LRLRLR, RLRLLR, RLRRLR. So she is left with the sequences LRRLLR, LRRRLR, RLLRLR, RLRRLR.

Julie uses bowls 3 and 6 without tipping the beam. This eliminates the sequences LRRLLR and RLLRRL.

So the only sequences that work for all three procedures are:

LRRLLR and RLLRLR.

c Alternative i

The beam will not tip for any \( m \)-selection with \( m \geq 6 \) since, in those cases, a ball is drawn from only one bowl. For each \( m \)-selection with \( m \leq 5 \), the first 2 bowls must be RL or LR. For \( m = 1 \), let the first 2 bowls be LR.

Then, for \( m = 2 \), bowl 4 must be L. Hence bowl 3 is R.

For \( m = 3 \), bowl 6 must be L. Hence bowl 5 is R.

For \( m = 2 \), bowl 8 must be R. Hence bowl 7 is L.

For \( m = 5 \), bowl 10 must be L. Hence bowl 9 is R.

Bowl 11 can be L or R. Thus we have two sequences starting with LR such that no \( m \)-selection causes the beam to tip.

Similarly, there are two sequences starting with RL such that no \( m \)-selection causes the beam to tip.

So there are four sequences of 11 bowls such that no \( m \)-selection causes the beam to tip:

LRRRLRLRLL, LRRRLRRLR, RLLRLRRLL, RLLRLRRLLRL.

Alternative ii

As in the second solution to Part a, since the first two letters in any non-tipping sequence must be different, the next two letters in the sequence must be different, then the next two and so on.

If a sequence of 11 bowls is non-tipping for every \( m \)-selection, then for \( m = 1 \), the 5th and 6th bowls are different and the 7th and 8th bowls are different. So these four bowls are LRLR, LRRL, RLLR, or RLRL. For \( m = 2 \), the 6th and 8th bowls must be different. This eliminates LRLR and RLRL, and leaves us with:

For \( m = 3 \), the 3rd and 6th letters must be different. For \( m = 4 \), the 4th and 8th letters must be different. So we have:

For \( m = 2 \), the 2nd and 4th letters must be different. For \( m = 1 \), the 1st and 2nd letters must be different. So we have:
For \( m = 5 \), the 5th and 10th letters must be different. For \( m = 1 \), the 9th and 10th letters must be different. So we have:

<table>
<thead>
<tr>
<th>Bowl</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Seq. 1</td>
<td>R</td>
<td>L</td>
<td>L</td>
<td>R</td>
<td>R</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>L</td>
</tr>
<tr>
<td>Seq. 2</td>
<td>L</td>
<td>R</td>
<td>R</td>
<td>L</td>
<td>R</td>
<td>L</td>
<td>L</td>
<td>R</td>
<td>L</td>
<td>L</td>
<td>L</td>
</tr>
</tbody>
</table>

Finally, the 11th bowl could be either L or R. So there are four sequences of 11 bowls for which every \( m \)-selection is non-tipping:

- RLLRLRRLLR, RLLRLRRLLR,
- LRRLLLRLR, LRRLLLRLR.

d Alternative i

Suppose we have a sequence of 12 bowls for which every \( m \)-selection is non-tipping. Then, as in Part c, up to bowl 10 we have only two possible sequences:

<table>
<thead>
<tr>
<th>Bowl</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Letter</td>
<td>L</td>
<td>R</td>
<td>R</td>
<td>L</td>
<td>R</td>
<td>L</td>
<td>R</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>L</td>
<td>L</td>
</tr>
<tr>
<td>Bowl</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>Letter</td>
<td>R</td>
<td>L</td>
<td>L</td>
<td>R</td>
<td>L</td>
<td>R</td>
<td>L</td>
<td>L</td>
<td>R</td>
<td>L</td>
<td>L</td>
<td>L</td>
</tr>
</tbody>
</table>

For \( m = 6 \), bowl 12 must be R and L respectively. However, for \( m = 3 \), bowl 12 must be L and R respectively. So there is no sequence of 12 bowls for which every \( m \)-selection is non-tipping.

Alternative ii

As in the second solution to Part a, since the first two letters in any non-tipping sequence must be different, the next two letters in the sequence must be different, then the next two and so on.

Suppose we have a sequence of 12 bowls for which every \( m \)-selection is non-tipping. Then for \( m = 1 \), the 9th and 10th bowls are different and the 11th and 12th bowls are different. So the last 4 letters in the sequence are LRLR, LRLR, or RLLR.

For \( m = 2 \), the 10th and 12th letters must be different. This eliminates LRLR and RLRL. For \( m = 3 \), the 9th and 12th letters must be different. This eliminates LRRL and RLLR. So there is no sequence of 12 bowls for which every \( m \)-selection is non-tipping.

J6 Tossing Counters

a Alternative i

The smallest sum, 8, is the sum of the smaller numbers on the two counters. Since \( 1 + 7 = 8 \), the smaller numbers on the two counters are 1 and 7.

Since the sum of the larger numbers on the two counters is 11, the sum of 7 and the larger number on the first counter is 9 or 10. So the larger number on the first counter is 2 or 3.

If the first counter is 1/2, then the second counter must be 7/9. If the first counter is 1/3, then the second counter must be 7/8.

Alternative ii

The smallest sum, 8, is the sum of the smaller numbers on the two counters. The largest sum, 11, is the sum of the larger numbers on the two counters. So we have either of the following addition tables for the four numbers on the counters.

\[
\begin{array}{ccc}
+ & 1 & ? \\
7 & 8 & 9 \\
? & 10 & 11 \\
\end{array}
\quad
\begin{array}{ccc}
+ & 1 & ? \\
7 & 8 & 10 \\
? & 9 & 11 \\
\end{array}
\]

There is only one way to complete each table:

\[
\begin{array}{ccc}
+ & 1 & 2 \\
7 & 8 & 9 \\
9 & 10 & 11 \\
\end{array}
\quad
\begin{array}{ccc}
+ & 1 & 3 \\
7 & 8 & 10 \\
8 & 9 & 11 \\
\end{array}
\]

Thus the two counters are 1/2 and 7/9 or 1/3 and 7/8.
**b Alternative i**

The smallest sum, 7, is the sum of the smaller numbers on the two counters. The largest sum, 10, is the sum of the larger numbers on the two counters. Since $7 = 1 + 6 = 2 + 5 = 3 + 4$, the two smaller numbers on the counters are 1 and 6 or 2 and 5 or 3 and 4. Since $10 = 1 + 9 = 2 + 8 = 3 + 7 = 4 + 6$, the two larger numbers on the counters are 1 and 9 or 2 and 8 or 3 and 7 or 4 and 6. The four numbers on the two counters are all different. The table shows the only combinations we need to consider.

<table>
<thead>
<tr>
<th>Smaller numbers</th>
<th>Larger numbers</th>
<th>Give sums 7, 8, 9, 10?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 and 6</td>
<td>2 and 8</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>3 and 7</td>
<td>yes</td>
</tr>
<tr>
<td>2 and 5</td>
<td>1 and 9</td>
<td>no</td>
</tr>
<tr>
<td></td>
<td>3 and 7</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>4 and 6</td>
<td>yes</td>
</tr>
<tr>
<td>3 and 4</td>
<td>1 and 9</td>
<td>no</td>
</tr>
<tr>
<td></td>
<td>2 and 8</td>
<td>no</td>
</tr>
</tbody>
</table>

So the two counters are: 1/2 and 6/8, 1/3 and 6/7, 2/3 and 5/7, or 2/4 and 5/6.

**Alternative ii**

Let the numbers on one counter be $a$ and $a + r$.

Let the numbers on the other counter be $b$ and $b + s$.

Then $a + b = 7$ and $a + r + b + s = 10$. So $r + s = 3$.

Hence $(a, b) = (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), or (6, 1), and (r, s) = (1, 2) or (2, 1)$.

From symmetry we may assume $r = 1$ and $s = 2$.

So the two counters are: 1/2 and 6/8, 2/3 and 5/7, 3/4 and 4/6 (disallowed), 4/5 and 3/5 (disallowed), 5/6 and 2/4, or 6/7 and 1/3.

**Alternative iii**

The smallest sum, 7, is the sum of the smaller numbers on the two counters. So these numbers are 1 and 6, 2 and 5, or 3 and 4. The largest sum, 10, is the sum of the larger numbers on the two counters. So we have the following addition tables for the four numbers on the counters.

<table>
<thead>
<tr>
<th>+ 1 ?</th>
<th>+ 1 ?</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 7 8</td>
<td>6 7 9</td>
</tr>
<tr>
<td>? 9 10</td>
<td>? 8 10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>+ 2 ?</th>
<th>+ 2 ?</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 7 8</td>
<td>5 7 9</td>
</tr>
<tr>
<td>? 9 10</td>
<td>? 8 10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>+ 3 ?</th>
<th>+ 3 ?</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 7 8</td>
<td>4 7 9</td>
</tr>
<tr>
<td>? 9 10</td>
<td>? 8 10</td>
</tr>
</tbody>
</table>

There is only one way to complete each table:
We must exclude the last two tables because the four numbers on the counters must be different. So the two counters are:
1/2 and 6/8, 1/3 and 6/7, 2/3 and 5/7, or 2/4 and 5/6.

c Alternative i
Suppose the numbers on the second counter are $c$ and $d$ with $c < d$. The minimum sum is $4 + c$ and the maximum sum is $5 + d$. The other two sums, $4 + d$ and $5 + c$, are between these two. If these four sums form three consecutive integers, then $4 + d = 5 + c$ and $d - c = 1$. Since $d$ is less than 10 and the numbers 4 and 5 already appear on the first counter, the second counter is 1/2, 2/3, 6/7, 7/8, or 8/9.

Alternative ii
Suppose the three sums are $s$, $s + 1$, $s + 2$. The smallest sum, $s$, is the sum of the smaller numbers on the two counters. The largest sum, $s + 2$, is the sum of the larger numbers on the two counters. Let the smaller number on the second counter be $x$. Then the addition table for the four numbers on the counters is:

<table>
<thead>
<tr>
<th></th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$s$</td>
<td>$s + 1$</td>
</tr>
<tr>
<td>?</td>
<td>?</td>
<td>$s + 2$</td>
</tr>
</tbody>
</table>

So the larger number on the second counter is $x + 1$. Since each of $x$ and $x + 1$ is less than 10 and is neither 4 nor 5, $x$ is one of the numbers 1, 2, 6, 7, 8. So the second counter is 1/2, 2/3, 6/7, 7/8, or 8/9.

d Alternative i
Suppose the numbers on the first counter are $a$ and $b$ with $a < b$, and on the second counter $c$ and $d$ with $c < d$. The minimum sum is $a + c$ and the maximum sum is $b + d$. The other two sums, $a + d$ and $b + c$, are between these two.

Suppose $a + d < b + c$. Since the four sums are consecutive, we have $a + d = a + c + 1$ and $b + c = a + d + 1$. Hence $d = c + 1$ and $b = a + 2$. So only one of $c$ and $d$ is even and $a$ and $b$ are either both odd or both even.

Similarly, if $b + c < a + d$, then only one of $a$ and $b$ is even and $c$ and $d$ are either both odd or both even.

Thus either one or three of $a$, $b$, $c$, $d$ are even.

Alternative ii
Suppose the numbers on the first counter are $a$ and $b$ with $a < b$, and on the second counter $c$ and $d$ with $c < d$. The minimum sum is $a + c$ and the maximum sum is $b + d$. Since the four sums are consecutive, we have $b + d = a + c + 3$.

If $a + c$ is odd, then $b + d$ is even. So one of $a$ and $c$ is even and neither or both of $b$ and $d$ are even.

If $a + c$ is even, then $b + d$ is odd. So one of $b$ and $d$ is even and neither or both of $a$ and $c$ are even.

In both cases one or three of $a$, $b$, $c$, $d$ are even.

Alternative iii
Suppose the four sums are $s$, $s + 1$, $s + 2$, $s + 3$. The smallest sum, $s$, is the sum of the smaller numbers on the two counters. The largest sum, $s + 3$, is the sum of the larger numbers on the two counters. So we can arrange the addition table for the four numbers on the counters as follows:

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>7</td>
<td>9</td>
<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>6</td>
<td>8</td>
<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>9</td>
<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>10</td>
</tr>
</tbody>
</table>
Since $s$ is either even or odd, we have:

\[
\begin{array}{ccc}
+ & ? & ? \\
? & s & s+1 \\
? & s+2 & s+3 \\
\end{array}
\]

Since $s$ is either even or odd, we have:

\[
\begin{array}{ccc}
+ & x & ? \\
? & \text{even} & \text{odd} \\
? & \text{even} & \text{odd} \\
\end{array}
\quad
\begin{array}{ccc}
+ & x & ? \\
? & \text{odd} & \text{even} \\
? & \text{odd} & \text{even} \\
\end{array}
\]

Since $x$ is either even or odd, we can complete each of these tables in two ways:

\[
\begin{array}{ccc}
+ & \text{even} & \text{odd} \\
\text{even} & \text{even} & \text{odd} \\
\text{even} & \text{even} & \text{odd} \\
\end{array}
\quad
\begin{array}{ccc}
+ & \text{even} & \text{odd} \\
\text{odd} & \text{odd} & \text{even} \\
\text{odd} & \text{odd} & \text{even} \\
\end{array}
\]

\[
\begin{array}{ccc}
+ & \text{odd} & \text{even} \\
\text{odd} & \text{even} & \text{odd} \\
\text{odd} & \text{even} & \text{odd} \\
\end{array}
\quad
\begin{array}{ccc}
+ & \text{odd} & \text{even} \\
\text{even} & \text{odd} & \text{even} \\
\text{even} & \text{odd} & \text{even} \\
\end{array}
\]

In each case either one or three of the four numbers on the counters are even.

**Alternative iv**

If all four numbers on the counters were even or all four were odd, then all sums would be even and therefore not consecutive.

Suppose two numbers on the counters were even and the other two odd.

If the two even numbers were on the same counter, then all sums would be odd and therefore not consecutive. So each counter must have an even and an odd number.

If the two smaller numbers on the counters were odd, then the two larger numbers would be even. Then the lowest and highest sums would both be even and the four sums would not be consecutive. Similarly, the two smaller numbers on the counters cannot be even.

If one of the smaller numbers on the counters was odd and the other smaller number was even, then one of the larger numbers would be even and the other odd. Then the lowest and highest sums would both be odd and the four sums would not be consecutive.

So either one or three of the four numbers on the counters are even.
II COP-graphs

a Here are two COP-graphs with the required labels. There are other ways to place the labels.

\[ \text{\begin{tabular}{cccc}
3 & 2 & 5 & 4 \\
\end{tabular}} \quad \text{\begin{tabular}{cccc}
2 & 4 & 7 & 6 \\
\end{tabular}} \]

b Alternative i

Amongst the integers 5, 6, 7, 8, 9, 10, only 5 and 7 are coprime to 6. So the vertex with label 6 must be joined to at most two other vertices. Hence, only vertex b or f can have label 6.

\[ \text{\begin{tabular}{cccccc}
c & b & a & \text{c} & \text{d} & \text{e} & \text{f} \\
\end{tabular}} \]

If vertex f has label 6, then vertices a and c have labels 5 and 7. So vertex b has label 10. Then neither of the remaining vertices d and e can have label 8.

If vertex b has label 6, then vertices d and e have labels 5 and 7. So vertex f has label 10. Then neither of the remaining vertices a and c can have label 8.

Hence COP-graph B cannot remain a COP-graph if its labels are replaced with integers 5 to 10.

Alternative ii

Amongst the integers 5, 6, 7, 8, 9, 10, there are three even numbers and three odd numbers. No two vertices with even labels can be joined by an edge. So 6, 8, 10 must be assigned in some order to vertices a, b, c or vertices d, e, f.

\[ \text{\begin{tabular}{cccccc}
c & b & a & \text{c} & \text{d} & \text{e} & \text{f} \\
\end{tabular}} \]

If 6, 8, 10 are assigned to vertices a, b, c, then either 5 or 9 is assigned to one of d and e. Since 5 and 10 are not coprime and 9 and 6 are not coprime, the labelling does not give a COP-graph.

If 6, 8, 10 are assigned to vertices d, e, f, then either 5 or 9 is assigned to one of a and c. Again, the labelling does not give a COP-graph.

Hence COP-graph B cannot remain a COP-graph if its labels are replaced with integers 5 to 10.

c Let $a, b, c, d, e$ be consecutive positive integers. Any two consecutive integers are coprime and any two consecutive odd integers are coprime.

If $a$ is odd, we have the following table which has each integer in the top row and the integers coprime to it in the bottom row.

\[
\begin{array}{cccccc}
a \text{ (odd)} & b \text{ (even)} & c \text{ (odd)} & d \text{ (even)} & e \text{ (odd)} \\
b, e & a, c & a, b, d, e & c, e & c, d \\
\end{array}
\]

So we have the following COP-graph.

\[ \text{\begin{tabular}{cccc}
e & d & c & a \\
\end{tabular}} \]
If \( a \) is even, we have the following table which has each integer in the top row and the integers coprime to it in the bottom row.

<table>
<thead>
<tr>
<th>( a ) (even)</th>
<th>( b ) (odd)</th>
<th>( c ) (even)</th>
<th>( d ) (odd)</th>
<th>( e ) (even)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a, c, d )</td>
<td>( b, d )</td>
<td>( b, c, e )</td>
<td>( d )</td>
<td>( e )</td>
</tr>
</tbody>
</table>

So we have the following COP-graph.

So we have the following COP-graph.

\[ a \] \( \bullet \) \( b \) \( c \) \( d \) \( e \)

\( d \) Alternative i

By joining each vertex to every other vertex, we can see that a graph with 6 vertices has at most 15 edges. We cannot have edges \{2,4\}, \{2,6\}, \{4,6\} and \{3,6\}. That eliminates 4 edges. Here is a COP-graph with 6 vertices, 11 edges, and labels 1 to 6.

\[ 1 \] \( \bullet \) \( 2 \) \( 3 \) \( 4 \) \( 5 \) \( 6 \)

\( (\text{This graph can be drawn in many ways, for example, without edges crossing by placing edge } \{3,4\} \text{ inside triangle } \{1,2,5\}.\) So the maximum number of edges in a COP-graph that has 6 vertices and labels 1 to 6 is 11.

Alternative ii

The following table lists all the integers from 1 to 6 that are coprime to each integer from 1 to 6.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2, 3, 4, 5, 6</td>
<td>1, 3, 5</td>
<td>1, 2, 4, 5</td>
<td>1, 3, 5</td>
<td>1, 2, 3, 4, 6</td>
<td>1, 5</td>
</tr>
</tbody>
</table>

Thus, amongst the integers 1 to 6 there are exactly 11 pairs that are coprime: \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}, \{2,3\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}, \{5,6\}. The COP-graph in Alternative i displays these edges. So 11 is the maximum number of edges in a COP-graph that has 6 vertices and labels 1 to 6.

I2 Tipping Points

See Junior Problem 5.

I3 Tossing Counters

a Alternative i

The smallest sum, 8, is the sum of the smaller numbers on the two counters. The largest sum, 11, is the sum of the larger numbers on the two counters. Since \( 8 = 1 + 7 = 2 + 6 = 3 + 5 \), the two smaller numbers on the counters are 1 and 7 or 2 and 6 or 3 and 5. Since \( 11 = 2 + 9 = 3 + 8 = 4 + 7 = 5 + 6 \), the two larger numbers on the counters are 2 and 9 or 3 and 8 or 4 and 7 or 5 and 6. The four numbers on the two counters are all different. The table shows the only combinations we need to consider.

<table>
<thead>
<tr>
<th>Smaller numbers</th>
<th>Larger numbers</th>
<th>Give sums 8, 9, 10, 11?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 and 7</td>
<td>2 and 9</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>3 and 8</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>5 and 6</td>
<td>no</td>
</tr>
<tr>
<td>2 and 6</td>
<td>3 and 8</td>
<td>yes</td>
</tr>
<tr>
<td></td>
<td>4 and 7</td>
<td>yes</td>
</tr>
<tr>
<td>3 and 5</td>
<td>2 and 9</td>
<td>no</td>
</tr>
<tr>
<td></td>
<td>4 and 7</td>
<td>yes</td>
</tr>
</tbody>
</table>
So the two counters are:
1/2 and 7/9, 1/3 and 7/8, 2/3 and 6/8, 2/4 and 6/7, or 3/4 and 5/7.

Alternative ii
Let the numbers on one counter be $a$ and $a + r$.
Let the numbers on the other counter be $b$ and $b + s$.
Then $a + b = 8$ and $a + r + b + s = 11$. So $r + s = 3$.
Hence $(a,b) = (1,7), (2,6), (3,5), (5,3), (6,2), \text{ or } (7,1),$
and $(r,s) = (1,2) \text{ or } (2,1)$.

From symmetry we may assume $r = 1 \text{ and } s = 2$.

So the two counters are:
1/2 and 7/9, 2/3 and 6/8, 3/4 and 5/7, 5/6 and 3/5 (disallowed), 6/7 and 2/4, or 7/8 and 1/3.

Alternative iii
The smallest sum, 8, is the sum of the smaller numbers on the two counters. So these numbers are 1 and 7, 2 and 6, or 3 and 5. The largest sum, 11, is the sum of the larger numbers on the two counters. So we have the following addition tables for the four numbers on the counters.

\[
\begin{array}{ccc}
+ & 1 & ? \\
7 & 8 & 9 \\
? & 10 & 11 \\
\end{array}
\quad
\begin{array}{ccc}
+ & 1 & ? \\
7 & 8 & 10 \\
? & 9 & 11 \\
\end{array}
\]

\[
\begin{array}{ccc}
+ & 2 & ? \\
6 & 8 & 9 \\
? & 10 & 11 \\
\end{array}
\quad
\begin{array}{ccc}
+ & 2 & ? \\
6 & 8 & 10 \\
? & 9 & 11 \\
\end{array}
\]

\[
\begin{array}{ccc}
+ & 3 & ? \\
5 & 8 & 9 \\
? & 10 & 11 \\
\end{array}
\quad
\begin{array}{ccc}
+ & 3 & ? \\
5 & 8 & 10 \\
? & 9 & 11 \\
\end{array}
\]

There is only one way to complete each table:

\[
\begin{array}{ccc}
+ & 1 & 2 \\
7 & 8 & 9 \\
9 & 10 & 11 \\
\end{array}
\quad
\begin{array}{ccc}
+ & 1 & 3 \\
7 & 8 & 10 \\
8 & 9 & 11 \\
\end{array}
\]

\[
\begin{array}{ccc}
+ & 2 & 3 \\
6 & 8 & 9 \\
8 & 10 & 11 \\
\end{array}
\quad
\begin{array}{ccc}
+ & 2 & 4 \\
6 & 8 & 10 \\
7 & 9 & 11 \\
\end{array}
\]

\[
\begin{array}{ccc}
+ & 3 & 4 \\
5 & 8 & 9 \\
7 & 10 & 11 \\
\end{array}
\quad
\begin{array}{ccc}
+ & 3 & 5 \\
5 & 8 & 10 \\
6 & 9 & 11 \\
\end{array}
\]
We must exclude the last table because the four numbers on the counters must be different. So the two counters are 1/2 and 7/9, 1/3 and 7/8, 2/3 and 6/8, 2/4 and 6/7, or 3/4 and 5/7.

b Alternative i

Suppose the numbers on the second counter are \( c \) and \( d \) with \( c < d \). The minimum sum is \( 4 + c \) and the maximum sum is \( 5 + d \). The other two sums, \( 4 + d \) and \( 5 + c \), are between these two. If these four sums form three consecutive integers, then \( 4 + d = 5 + c \) and \( d - c = 1 \). Since \( d \) is less than 10 and the numbers 4 and 5 already appear on the first counter, the second counter is 1/2, 2/3, 6/7, 7/8, or 8/9.

Alternative ii

Suppose the three sums are \( s \), \( s + 1 \), \( s + 2 \). The smallest sum, \( s \), is the sum of the smaller numbers on the two counters. The largest sum, \( s + 2 \), is the sum of the larger numbers on the two counters. Let the smaller number on the second counter be \( x \). Then the addition table for the four numbers on the counters is:

<table>
<thead>
<tr>
<th></th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( s )</td>
<td>( s + 1 )</td>
</tr>
<tr>
<td>?</td>
<td>?</td>
<td>( s + 2 )</td>
</tr>
</tbody>
</table>

So the larger number on the second counter is \( x + 1 \). Since each of \( x \) and \( x + 1 \) is less than 10 and is neither 4 nor 5, \( x \) is one of the numbers 1, 2, 6, 7, 8. So the second counter is 1/2, 2/3, 6/7, 7/8, or 8/9.

c Alternative i

Suppose the numbers on the first counter are \( a \) and \( b \) with \( a < b \), and on the second counter \( c \) and \( d \) with \( c < d \). The minimum sum is \( a + c \) and the maximum sum is \( b + d \). The other two sums, \( a + d \) and \( b + c \), are between these two.

Suppose \( a + d < b + c \). Since the four sums are consecutive, we have \( a + d = a + c + 1 \) and \( b + c = a + d + 1 \). Hence \( d = c + 1 \) and \( b = a + 2 \). So only one of \( c \) and \( d \) is even and both of \( a \) and \( b \) are either both odd or both even.

Similarly, if \( b + c < a + d \), then only one of \( a \) and \( b \) is even and both of \( c \) and \( d \) are either both odd or both even.

Thus either one or three of \( a \), \( b \), \( c \), \( d \) are even.

Alternative ii

Suppose the numbers on the first counter are \( a \) and \( b \) with \( a < b \), and on the second counter \( c \) and \( d \) with \( c < d \). The minimum sum is \( a + c \) and the maximum sum is \( b + d \). Since the four sums are consecutive, we have \( b + d = a + c + 3 \).

If \( a + c \) is odd, then \( b + d \) is even. So one of \( a \) and \( c \) is even and neither of \( b \) and \( d \) are even.

If \( a + c \) is even, then \( b + d \) is odd. So one of \( b \) and \( d \) is even and neither of \( a \) and \( c \) are even.

In both cases one or three of \( a \), \( b \), \( c \), \( d \) are even.

Alternative iii

Suppose the four sums are \( s \), \( s + 1 \), \( s + 2 \), \( s + 3 \). The smallest sum, \( s \), is the sum of the smaller numbers on the two counters. The largest sum, \( s + 3 \), is the sum of the larger numbers on the two counters. So we can arrange the addition table for the four numbers on the counters as follows:

<table>
<thead>
<tr>
<th></th>
<th>?</th>
<th>?</th>
</tr>
</thead>
<tbody>
<tr>
<td>?</td>
<td>( s )</td>
<td>( s + 1 )</td>
</tr>
<tr>
<td>?</td>
<td>( s + 2 )</td>
<td>( s + 3 )</td>
</tr>
</tbody>
</table>

Since \( s \) is either even or odd, we have:

<table>
<thead>
<tr>
<th></th>
<th>( x )</th>
<th>?</th>
</tr>
</thead>
<tbody>
<tr>
<td>?</td>
<td>even</td>
<td>odd</td>
</tr>
</tbody>
</table>

Since \( s \) is either even or odd, we have:

<table>
<thead>
<tr>
<th></th>
<th>( x )</th>
<th>?</th>
</tr>
</thead>
<tbody>
<tr>
<td>?</td>
<td>odd</td>
<td>even</td>
</tr>
</tbody>
</table>
Since $x$ is either even or odd, we can complete each of these tables in two ways:

<table>
<thead>
<tr>
<th>+</th>
<th>even</th>
<th>odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>even</td>
<td>even</td>
<td>odd</td>
</tr>
<tr>
<td>even</td>
<td>even</td>
<td>odd</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>+</th>
<th>odd</th>
<th>even</th>
</tr>
</thead>
<tbody>
<tr>
<td>odd</td>
<td>even</td>
<td>odd</td>
</tr>
<tr>
<td>odd</td>
<td>even</td>
<td>odd</td>
</tr>
</tbody>
</table>

In each case either one or three of the four numbers on the counters are even.

**Alternative iv**

If all four numbers on the counters were even or all four were odd, then all sums would be even and therefore not consecutive.

Suppose two numbers on the counters were even and the other two odd.

If the two even numbers were on the same counter, then all sums would be odd and therefore not consecutive. So each counter must have an even and an odd number.

If the two smaller numbers on the counters were odd, then the two larger numbers would be even. Then the lowest and highest sums would both be even and the four sums would not be consecutive. Similarly, the two smaller numbers on the counters cannot be even.

If one of the smaller numbers on the counters was odd and the other smaller number was even, then one of the larger numbers would be even and the other odd. Then the lowest and highest sums would both be odd and the four sums would not be consecutive.

So either one or three of the four numbers on the counters are even.

d In Part a we found that to get the specified four consecutive sums from two counters, the difference of the numbers on one counter was 1 and the difference for the other counter was 2.

If the counters are so numbered, then they will always produce four consecutive sums. To see why, suppose counter 1 is numbered $a$ and $a + 1$ and counter 2 is numbered $b$ and $b + 2$. Then the sums will be $a + b$, $a + b + 1$, $a + b + 2$, $a + b + 3$.

If a third counter is introduced and its two numbers differ by 4, say $c$ and $c + 4$, then the three counters will produce 8 consecutive sums: $a + b + c$ to $a + b + c + 7$.

If a fourth counter is introduced and its two numbers differ by 8, say $d$ and $d + 8$, then the four counters will produce 16 consecutive sums: $a + b + c + d$ to $a + b + c + d + 15$.

For example, number counter 1 with 1/2, counter 2 with 3/5, counter 3 with 4/8, and counter 4 with 6/14.

<table>
<thead>
<tr>
<th>I4</th>
<th>Ionofs</th>
</tr>
</thead>
</table>
| a  | The factors of 36 are: 1, 2, 3, 4, 6, 9, 12, 18, 36.  
Hence Ionof(36) = 36/9 = 4. |
| b  | The factors of $pq$ are: 1, $p$, $q$, $pq$.  
Hence Ionof($pq$) = $pq$/4.  
The only even prime is 2. Hence at most one of $p$ and $q$ is 2 and the other is odd. Therefore 4 does not divide $pq$ and Ionof($pq$) is not an integer. |
| c  | There are 10 factors of $pq^4$: 1, $p$, $q$, $q^2$, $q^3$, $q^4$, $pq$, $pq^2$, $pq^3$, $pq^4$.  
So Ionof($pq^4$)=$pq^4$/10.  
Since the only prime factors of 10 are 2 and 5 and we want 10 to divide $pq^4$, we must have $p = 2$ and $q = 5$ or $p = 5$ and $q = 2$.  
So $pq^4 = 2 \times 5^4 = 1250$ or $pq^4 = 5 \times 2^4 = 80$. |
d Suppose \( p \) is prime, and \( p^2 = \text{Ionof}(m) \) for some integer \( m \). Let \( k \) be the number of factors of \( m \). Then \( m = kp^2 \). Since \( 1, p, p^2 \) are factors of \( m \), we know \( k \geq 3 \).

In the following table, we check integers of the form \( kp^2 \) to see if \( \text{Ionof}(kp^2) = p^2 \). To simplify calculations, for each value of \( k \) we choose a prime \( p \) that is not a factor of \( k \).

<table>
<thead>
<tr>
<th>( m )</th>
<th>factors of ( m )</th>
<th>( \text{Ionof}(m) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3p^2 )</td>
<td>( 1, 3, p, 3p, p^2 )</td>
<td>( 3p^2 / 6 = p^2 / 2 )</td>
</tr>
<tr>
<td>( 4p^2 )</td>
<td>( 1, 2, 4, p, 2p, 4p, 2p^2, 4p^2 )</td>
<td>( 3p^2 / 9 = p^2 / 3 )</td>
</tr>
<tr>
<td>( 5p^2 )</td>
<td>( 1, 5, p, 5p, p^2, 5p^2 )</td>
<td>( 5p^2 / 6 )</td>
</tr>
<tr>
<td>( 6p^2 )</td>
<td>( 1, 2, 3, 6, p, 2p, 3p, 6p, ) ( p^2, 2p^2, 3p^2, 6p^2 )</td>
<td>( 6p^2 / 12 = p^2 / 2 )</td>
</tr>
<tr>
<td>( 7p^2 )</td>
<td>( 1, 7, p, 7p, p^2, 7p^2 )</td>
<td>( 7p^2 / 6 )</td>
</tr>
<tr>
<td>( 8p^2 )</td>
<td>( 1, 2, 4, 8, p, 2p, 4p, 8p, ) ( p^2, 2p^2, 4p^2, 8p^2 )</td>
<td>( 8p^2 / 12 = 2p^2 / 3 )</td>
</tr>
<tr>
<td>( 9p^2 )</td>
<td>( 1, 3, 9, p, 3p, 9p, p^2, 3p^2, 9p^2 )</td>
<td>( 9p^2 / 9 = p^2 )</td>
</tr>
</tbody>
</table>

Thus, if \( p \) is a prime and not 3, then \( \text{Ionof}(9p^2) = p^2 \).

We now want \( m = k3^2 \) with \( k \) equal to the number of factors of \( m \). Again, \( k \geq 3 \). The next table is a check list for possible \( k \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>prime factorisation of ( m = k9 )</th>
<th>( f = \text{number of factors of } m )</th>
<th>( f = k? )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>( 3^3 )</td>
<td>4</td>
<td>no</td>
</tr>
<tr>
<td>4</td>
<td>( 2^2 \times 3^2 )</td>
<td>9</td>
<td>no</td>
</tr>
<tr>
<td>5</td>
<td>( 5 \times 3^2 )</td>
<td>6</td>
<td>no</td>
</tr>
<tr>
<td>6</td>
<td>( 2 \times 3^3 )</td>
<td>8</td>
<td>no</td>
</tr>
<tr>
<td>7</td>
<td>( 7 \times 3^2 )</td>
<td>6</td>
<td>no</td>
</tr>
<tr>
<td>8</td>
<td>( 2^3 \times 3^2 )</td>
<td>12</td>
<td>no</td>
</tr>
<tr>
<td>9</td>
<td>( 3^4 )</td>
<td>5</td>
<td>no</td>
</tr>
<tr>
<td>10</td>
<td>( 2 \times 5 \times 3^2 )</td>
<td>12</td>
<td>no</td>
</tr>
<tr>
<td>11</td>
<td>( 11 \times 3^2 )</td>
<td>6</td>
<td>no</td>
</tr>
<tr>
<td>12</td>
<td>( 2^2 \times 3^3 )</td>
<td>12</td>
<td>yes</td>
</tr>
</tbody>
</table>

So \( \text{Ionof}(12 \times 3^2) = 3^2 \).

Thus the square of any prime number is the Ionof of some integer.

## 15 Cube Trails

**a** At \( D \) the choices are the horizontal edge \( DC \) and the vertical edge \( DH \). So the probability of choosing \( DH \) is \( \frac{2}{3} \).

At \( E \) the choices are the horizontal edge \( EF \) and the horizontal edge \( EH \). So the probability of choosing \( EH \) is \( \frac{1}{2} \).

**b** Starting at \( A \) there is a choice of vertical edge \( AE \) and horizontal edges \( AB \) and \( AD \). So the probability of choosing \( AB \) is \( \frac{1}{2} \).

Then at \( B \) the choices are vertical edge \( BF \) and horizontal edge \( BC \). So the probability of choosing \( BC \) is \( \frac{1}{2} \).

Then at \( C \) the choices are vertical edge \( CG \) and horizontal edge \( CD \). So the probability of choosing \( CG \) is \( \frac{2}{3} \).

So the probability the robot traces the trail \( ABCG \) is \( \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{1}{27} \).

**c** There are six trails of length 3 from \( A \) to \( G \):

- \( ABCG \) and \( ADCG \), which both have the edge sequence horizontal, horizontal, vertical;
- \( ABFG \) and \( ADHG \), which both have the edge sequence horizontal, vertical, horizontal;
- \( AEFG \) and \( AEHG \), which both have the edge sequence vertical, horizontal, horizontal.

The probability that the robot traces trail \( ABFG \) is \( \frac{1}{3} \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{18} \).

The probability that the robot traces trail \( AEFG \) is \( \frac{2}{3} \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{9} \).

**d** From the solutions to Parts b and c, the probability of the robot tracing a path of length 3 to \( G \) is

\[
2 \times \left( \frac{1}{27} + \frac{1}{18} + \frac{1}{3} \right) = 2 \times \left( \frac{2}{27} + \frac{1}{18} + \frac{1}{18} \right) = 2 \times \frac{11}{54} = \frac{11}{27}.
\]
I6 Coverem

a The disc must cover the line segment CD.

From Pythagoras’ theorem, \( CD^2 = 1^2 + 4^2 = 17 \). So the diameter of the disc must be at least \( \sqrt{17} \).

Now we ask: is there a disc with diameter \( \sqrt{17} \) that will cover all 8 grid squares?

A disc with its centre at the midpoint \( M \) of \( CD \) and diameter \( \sqrt{17} \) will cover all 8 grid squares, as this diagram shows.

So the smallest diameter for a disc that will completely cover all 8 grid squares is \( \sqrt{17} \).

b All five grid squares in the following group can be covered by a disc of diameter \( \sqrt{1+3^2} = \sqrt{10} \).

Now we ask: is there some other group of five grid squares that can be covered by a disc with diameter smaller than \( \sqrt{10} \)?

Any set of 5 grid squares can be enclosed in a grid rectangle of sufficient size. Reducing the rectangle if necessary, we may assume that the first and last grid rows and columns of the rectangle each contain at least one of the grid squares. The rectangle must have at least 3 columns or 3 rows. Rotating the grid if necessary, we may assume that the rectangle has at least 3 columns. This is shown in the next diagram, where \( A \) and \( B \) are two of the grid squares that have to be covered by the disc.

Since both dots must be covered by the disc, the disc diameter is at least \( \sqrt{1+3^2} = \sqrt{10} \).

So the smallest diameter for a disc that can cover a group of 5 grid squares is \( \sqrt{10} \).
c If the centre of a disc of diameter 3 is at the centre of a grid square, then it is the only grid square completely covered by the disc as this diagram shows.

Now we ask: is it possible to locate a disc of diameter 3 so that no grid square is completely covered by the disc. The centre of the disc lies somewhere on a grid square. The longest line in a square is its diagonal, which has length $\sqrt{1^2 + 1^2} = \sqrt{2}$. So the distance from the disc centre to any point on the perimeter of the square is at most $\sqrt{2} < 1.5$. Hence the disc will cover that square. So a disc of diameter 3 always covers at least one grid square.

Therefore the minimum number of grid squares that can be covered by a disc of diameter 3 is 1.

Suppose the centre of the disc is at a grid point. The diagonal of a grid square is $\sqrt{1^2 + 1^2} = \sqrt{2} < 1.5$. Hence the disc covers 4 grid squares as this diagram shows.

From Part b, the diameter of a disc that covers five grid squares is at least $\sqrt{10}$. Since $\sqrt{10} > 3$, five grid squares cannot be covered by a disc of diameter 3.

Hence 4 is the maximum number of grid squares that can be covered by a disc of diameter 3.

d Alternative i

Consider the smallest disc that covers the given figure and has its centre on $XY$, the vertical line of symmetry of the figure. Moving the disc if necessary while keeping its centre on $XY$, we may assume that the rim of the disc covers vertices $A$ and $A'$.

Then reducing the disc radius if necessary while ensuring its rim continues to cover $A$ and $A'$, we may assume the rim of the disc also covers $B$ and $B'$ or $C$ and $C'$. If the rim of the disc covers $B$ and $B'$, then its centre $O$ is located as shown.

Then its radius is $\sqrt{1^2 + (3/2)^2} = \sqrt{3.25}$ and $OC = \sqrt{(1/2)^2 + 2^2} = \sqrt{4.25}$, which places $C$ outside the disc. So the rim of the disc covers $A$, $A'$, $C$ and $C'$. Let the distance from $O$ to $AA'$ be $h$. 

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Since $OA = OC$ we have
\[ h^2 + (3/2)^2 = (3 - h)^2 + (1/2)^2 = h^2 - 6h + 9 + 1/4. \]
So $6h = 9 - 2 = 7$, and $h = 7/6$. The radius of that disc is
\[ OA = \sqrt{(7/6)^2 + (3/2)^2} = \sqrt{(49/36) + (9/4)} = \sqrt{130/36} = \sqrt{130}/6. \]

Now we ask: is there a disc whose radius is less than $\sqrt{130}/6$ that covers all seven grid squares.

If there is, then the disc has its centre off the symmetry line $XY$, say to the left. If $A$ lies under this disc, then its centre must be below the horizontal line through $O$. If $C$ lies under this disc, then its centre must be above the horizontal line through $O$. This is a contradiction. Hence there is no disc whose radius is less than $\sqrt{130}/6$ that covers all seven grid squares.

So the smallest disc that covers the given figure has its centre on $XY$, its rim covers $A$, $A'$, $C$, $C'$, and it has radius $\sqrt{130}/6$.

**Alternative ii**

Let $PQ$ be the vertical line of symmetry of the figure. Let $O$ be a point on $PQ$ and $PO = h$.

If $OA = OC$, then
\[ h^2 + (3/2)^2 = (3 - h)^2 + (1/2)^2 = h^2 - 6h + 9 + 1/4. \]
So $6h = 9 - 2 = 7$, and $h = 7/6$, and
\[ OA = \sqrt{(7/6)^2 + (3/2)^2} = \sqrt{(49/36) + (9/4)} = \sqrt{130/36} = \sqrt{130}/6. \]

Since $OB < OA$, the disc with centre $O$ and radius $\sqrt{130}/6$ covers the seven grid squares.

Now we ask: is there a disc with radius less than $\sqrt{130}/6$ that covers all seven grid squares?

Draw cartesian axes on the figure with the origin at $O$.

Suppose there is a disc that covers the seven grid squares and has centre $O'$ different from $O$.

If $O'$ is on the horizontal axis, then either $O'A' > OA'$ or $O'A > OA$.

If $O'$ is on the vertical axis, then either $O'A > OA$ or $O'C > OC$. 

If $O'$ is in the first (top-right) quadrant, then $O'A' > OA'$.
If $O'$ is in the second (top-left) quadrant, then $O'A > OA$.
If $O'$ is in the third (bottom-left) quadrant, then $O'C > OC$.
If $O'$ is in the fourth (bottom-right) quadrant, then $O'C' > OC'$.

In each case the radius of the disc must be greater than $\sqrt{\frac{130}{6}}$.

So the smallest radius for a disc that covers the seven grid squares is $\sqrt{\frac{130}{6}}$. 
### Mean Score/School Year/Problem

<table>
<thead>
<tr>
<th>Year</th>
<th>Number of Students</th>
<th>Number of Students</th>
<th>Mean</th>
<th>Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Overall</td>
<td>Problem</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>537</td>
<td>8.8</td>
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<tr>
<td>4</td>
<td>927</td>
<td>10.9</td>
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</tr>
<tr>
<td><em>ALL YEARS</em></td>
<td>1472</td>
<td>10.1</td>
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<td>2.8</td>
</tr>
</tbody>
</table>

Please note:* This total includes students who did not provide their school year.

### Score Distribution %/Problem

<table>
<thead>
<tr>
<th>Score</th>
<th>1 Hexos</th>
<th>2 Cupcakes</th>
<th>3 Kimmi Dolls</th>
<th>4 Cube Trails</th>
</tr>
</thead>
<tbody>
<tr>
<td>Did not attempt</td>
<td>2%</td>
<td>1%</td>
<td>4%</td>
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</tr>
<tr>
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<td>14%</td>
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<td>11%</td>
</tr>
<tr>
<td>2</td>
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<td>20%</td>
<td>24%</td>
<td>18%</td>
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<td>34%</td>
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<td>26%</td>
</tr>
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<td>Mean</td>
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<td>2.8</td>
<td>2.4</td>
<td>2.5</td>
</tr>
<tr>
<td>Discrimination Factor</td>
<td>0.6</td>
<td>0.6</td>
<td>0.7</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Please note:
The discrimination factor for a particular problem is calculated as follows:

1. The students are ranked in regard to their overall scores.
2. The mean score for the top 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the ‘mean top score’.
3. The mean score for the bottom 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the ‘mean bottom score’.
4. The discrimination factor \( = \frac{\text{mean top score} - \text{mean bottom score}}{4} \)

Thus the discrimination factor ranges from 1 to –1. A problem with a discrimination factor of 0.4 or higher is considered to be a good discriminator.
### CHALLENGE STATISTICS – UPPER PRIMARY

#### Mean Score/School Year/Problem

<table>
<thead>
<tr>
<th>Year</th>
<th>Number of Students</th>
<th>Mean Overall</th>
<th>1 Knightlines</th>
<th>2 Grandma’s Eye Drops</th>
<th>3 Cube Trails</th>
<th>4 Magic Staircase</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1401</td>
<td>9.0</td>
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<td>2.3</td>
<td>2.6</td>
<td>2.3</td>
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<tr>
<td>6</td>
<td>1943</td>
<td>10.2</td>
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<td>2.6</td>
<td>2.8</td>
<td>2.6</td>
</tr>
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<td>7</td>
<td>109</td>
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<td>3.2</td>
<td>3.2</td>
<td>2.9</td>
</tr>
<tr>
<td><em>ALL YEARS</em></td>
<td>3472</td>
<td>9.8</td>
<td>2.2</td>
<td>2.5</td>
<td>2.8</td>
<td>2.5</td>
</tr>
</tbody>
</table>

Please note:* This total includes students who did not provide their school year.

#### Score Distribution %/Problem

<table>
<thead>
<tr>
<th>Score</th>
<th>1 Knightlines</th>
<th>2 Grandma’s Eye Drops</th>
<th>3 Cube Trails</th>
<th>4 Magic Staircase</th>
</tr>
</thead>
<tbody>
<tr>
<td>Did not attempt</td>
<td>2%</td>
<td>1%</td>
<td>2%</td>
<td>3%</td>
</tr>
<tr>
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<td>20%</td>
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<td>5%</td>
<td>6%</td>
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<tr>
<td>1</td>
<td>17%</td>
<td>15%</td>
<td>10%</td>
<td>14%</td>
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<td>28%</td>
</tr>
<tr>
<td>4</td>
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<td>28%</td>
<td>31%</td>
<td>21%</td>
</tr>
<tr>
<td>Mean</td>
<td>2.2</td>
<td>2.5</td>
<td>2.8</td>
<td>2.5</td>
</tr>
</tbody>
</table>

Please note: The discrimination factor for a particular problem is calculated as follows:

1. The students are ranked in regard to their overall scores.
2. The mean score for the top 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the ‘mean top score’.
3. The mean score for the bottom 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the ‘mean bottom score’.
4. The discrimination factor = \( \frac{\text{mean top score} - \text{mean bottom score}}{4} \)

Thus the discrimination factor ranges from 1 to –1. A problem with a discrimination factor of 0.4 or higher is considered to be a good discriminator.
### CHALLENGE STATISTICS – JUNIOR

#### Mean Score/School Year/Problem

<table>
<thead>
<tr>
<th>Year</th>
<th>Number of Students</th>
<th>Overall Mean</th>
<th>Challenge Problem Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>2978</td>
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<tr>
<td>8</td>
<td>2638</td>
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<tr>
<td><em>ALL YEARS</em></td>
<td>5640</td>
<td>13.5</td>
<td>2.5</td>
</tr>
</tbody>
</table>

Please note: * This total includes students who did not provide their school year.

#### Score Distribution %/Problem

<table>
<thead>
<tr>
<th>Score</th>
<th>Challenge Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 Cube Trails</td>
</tr>
<tr>
<td><strong>Did not attempt</strong></td>
<td>2%</td>
</tr>
<tr>
<td>0</td>
<td>4%</td>
</tr>
<tr>
<td>1</td>
<td>15%</td>
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<tr>
<td>2</td>
<td>28%</td>
</tr>
<tr>
<td>3</td>
<td>32%</td>
</tr>
<tr>
<td>4</td>
<td>20%</td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td>2.5</td>
</tr>
<tr>
<td><strong>Discrimination Factor</strong></td>
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</tr>
</tbody>
</table>

Please note:

The discrimination factor for a particular problem is calculated as follows:

(1) The students are ranked in regard to their overall scores.

(2) The mean score for the top 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the ‘mean top score’.

(3) The mean score for the bottom 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the ‘mean bottom score’.

(4) The discrimination factor = \( \frac{\text{mean top score} - \text{mean bottom score}}{4} \)

Thus the discrimination factor ranges from 1 to –1. A problem with a discrimination factor of 0.4 or higher is considered to be a good discriminator.
## Challenge Statistics – Intermediate

### Mean Score/School Year/Problem

<table>
<thead>
<tr>
<th>Year</th>
<th>Number of Students</th>
<th>Year</th>
<th>Overall Mean</th>
<th>Problem Mean</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<tbody>
<tr>
<td>9</td>
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<td><em>ALL YEARS</em></td>
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<td>2.3</td>
<td>2.3</td>
<td>2.7</td>
<td>1.8</td>
<td></td>
</tr>
</tbody>
</table>

*Please note:* This total includes students who did not provide their school year.

### Score Distribution %/Problem

<table>
<thead>
<tr>
<th>Score</th>
<th>1 Indim Integers</th>
<th>2 Digital Sums</th>
<th>3 Coin Flips</th>
<th>4 Jogging</th>
<th>5 Folding Fractions</th>
<th>6 Crumbling Cubes</th>
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</thead>
<tbody>
<tr>
<td>Did not attempt</td>
<td>2%</td>
<td>5%</td>
<td>6%</td>
<td>5%</td>
<td>10%</td>
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<tr>
<td>0</td>
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<td>13%</td>
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<td>9%</td>
<td>23%</td>
</tr>
<tr>
<td>1</td>
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<td>13%</td>
<td>22%</td>
<td>11%</td>
<td>13%</td>
</tr>
<tr>
<td>2</td>
<td>18%</td>
<td>20%</td>
<td>22%</td>
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<td>11%</td>
<td>18%</td>
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<tr>
<td>3</td>
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<td>23%</td>
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<tr>
<td>4</td>
<td>37%</td>
<td>37%</td>
<td>21%</td>
<td>17%</td>
<td>34%</td>
<td>10%</td>
</tr>
<tr>
<td>Mean Discrimination Factor</td>
<td>2.9</td>
<td>2.7</td>
<td>2.3</td>
<td>2.3</td>
<td>2.7</td>
<td>1.8</td>
</tr>
</tbody>
</table>

*Please note:*

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2. The mean score for the top 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the ‘mean top score’.
3. The mean score for the bottom 25% of these overall ranked students is calculated for that particular problem including no attempts. Call this mean score the ‘mean bottom score’.
4. The discrimination factor = \( \frac{\text{mean top score}}{4} – \frac{\text{mean bottom score}}{4} \)

Thus the discrimination factor ranges from 1 to −1. A problem with a discrimination factor of 0.4 or higher is considered to be a good discriminator.
AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD

Time allowed: 4 hours. NO calculators are to be used.

Questions 1 to 8 only require their numerical answers all of which are non-negative integers less than 1000. Questions 9 and 10 require written solutions which may include proofs. The bonus marks for the Investigation in Question 10 may be used to determine prize winners.

1. Find the smallest positive integer $x$ such that $12x = 25y^2$, where $y$ is a positive integer. [2 marks]

2. A 3-digit number in base 7 is also a 3-digit number when written in base 6, but each digit has increased by 1. What is the largest value which this number can have when written in base 10? [2 marks]

3. A ring of alternating regular pentagons and squares is constructed by continuing this pattern.

![Diagram of a ring of regular pentagons and squares]

How many pentagons will there be in the completed ring? [3 marks]

4. A sequence is formed by the following rules: $s_1 = 1$, $s_2 = 2$ and $s_{n+2} = s_n^2 + s_{n+1}^2$ for all $n \geq 1$. What is the last digit of the term $s_{200}$? [3 marks]

5. Sebastien starts with an $11 \times 38$ grid of white squares and colours some of them black. In each white square, Sebastien writes down the number of black squares that share an edge with it. Determine the maximum sum of the numbers that Sebastien could write down. [3 marks]

6. A circle has centre $O$. A line $PQ$ is tangent to the circle at $A$ with $A$ between $P$ and $Q$. The line $PO$ is extended to meet the circle at $B$ so that $O$ is between $P$ and $B$. $\angle APB = x^\circ$ where $x$ is a positive integer. $\angle BAQ = kx^\circ$ where $k$ is a positive integer. What is the maximum value of $k$? [4 marks]

7. Let $n$ be the largest positive integer such that $n^2 + 2016n$ is a perfect square. Determine the remainder when $n$ is divided by 1000. [4 marks]

8. Ann and Bob have a large number of sweets which they agree to share according to the following rules. Ann will take one sweet, then Bob will take two sweets and then, taking turns, each person takes one more sweet than what the other person just took. When the number of sweets remaining is less than the number that would be taken on that turn, the last person takes all that are left. To their amazement, when they finish, they each have the same number of sweets.

They decide to do the sharing again, but this time, they first divide the sweets into two equal piles and then they repeat the process above with each pile, Ann going first both times. They still finish with the same number of sweets each.

What is the maximum number of sweets less than 1000 they could have started with? [4 marks]
9. All triangles in the spiral below are right-angled. The spiral is continued anticlockwise.

\[
\begin{align*}
X_4 & \quad 1 \\
X_3 & \quad 1 \\
X_2 & \quad 1 \\
X_1 & \quad 1 \\
O & \quad X_0
\end{align*}
\]

Prove that \(X_0^2 + X_1^2 + X_2^2 + \cdots + X_n^2 = X_0^2 \times X_1^2 \times X_2^2 \times \cdots \times X_n^2.\) \([5 \text{ marks}]\)

10. For \(n \geq 3,\) consider \(2n\) points spaced regularly on a circle with alternate points black and white and a point placed at the centre of the circle.

The points are labelled \(-n, -n+1, \ldots, n-1, n\) so that:
(a) the sum of the labels on each diameter through three of the points is a constant \(s,\) and
(b) the sum of the labels on each black-white-black triple of consecutive points on the circle is also \(s.\)

Show that the label on the central point is 0 and \(s = 0.\) \([5 \text{ marks}]\)

Investigation
Show that such a labelling exists if and only if \(n\) is even. \([3 \text{ bonus marks}]\)
1. **Method 1**

We have $2^2 \times 3x = 5^2 y^2$ where $x$ and $y$ are integers. So 3 divides $y^2$.

Since 3 is prime, 3 divides $y$.

Hence 3 divides $x$. Also 25 divides $x$. So the smallest value of $x$ is $3 \times 25 = 75$.

**Method 2**

The smallest value of $x$ will occur with the smallest value of $y$.

Since 12 and 25 are relatively prime, 12 divides $y^2$.

The smallest value of $y$ for which this is possible is $y = 6$.

So the smallest value of $x$ is $(25 \times 36) / 12 = 75$.

2. $abc_7 = (a + 1)(b + 1)(c + 1)_6$.

This gives $9a + 7b + c = 36(a + 1) + 6(b + 1) + c + 1$. Simplifying, we get $13a + b = 43$. Remembering that $a + 1$ and $b + 1$ are less than 6, and therefore $a$ and $b$ are less than 5, the only solution of this equation is $a = 3$, $b = 4$.

Hence the number is $34c_7$ or $45(c + 1)_6$. But $c + 1 \leq 5$ so, for the largest such number, $c = 4$.

Hence the number is $344_7 = 179$.

3. **Method 1**

The interior angle of a regular pentagon is $108^\circ$. So the angle inside the ring between a square and a pentagon is $360^\circ - 108^\circ - 90^\circ = 162^\circ$.

Thus on the inside of the completed ring we have a regular polygon with $n$ sides whose interior angle is $162^\circ$.

The interior angle of a regular polygon with $n$ sides is $180^\circ (n - 2)/n$.

So $162n = 180(n - 2)/n = 180n - 360$. Then $18n = 360$ and $n = 20$.

Since half of these sides are from pentagons, the number of pentagons in the completed ring is $10$.

**Method 2**

The interior angle of a regular pentagon is $108^\circ$. So the angle inside the ring between a square and a pentagon is $360^\circ - 108^\circ - 90^\circ = 162^\circ$.

Thus on the inside of the completed ring we have a regular polygon with $n$ sides whose exterior angle is $180^\circ - 162^\circ = 18^\circ$. Hence $18n = 360$ and $n = 20$.

Since half of these sides are from pentagons, the number of pentagons in the completed ring is $10$.

**Method 3**

The interior angle of a regular pentagon is $108^\circ$. So the angle inside the ring between a square and a pentagon is $360^\circ - 108^\circ - 90^\circ = 162^\circ$.

Thus on the inside of the completed ring we have a regular polygon whose interior angle is $162^\circ$.

The bisectors of these interior angles form congruent isosceles triangles on the sides of this polygon. So all these bisectors meet at a point, $O$ say.

The angle at $O$ in each of these triangles is $180^\circ - 162^\circ = 18^\circ$. If $n$ is the number of pentagons in the ring, then $18n = 360/2 = 180$. So $n = 10$.

4. Working modulo 10, we can make a sequence of last digits as follows:

$$1, 2, 5, 9, 6, 7, 5, 4, 1, 7, 0, 9, 1, 2, \ldots$$

Thus the last digits repeat after every 12 terms. Now $200 = 16 \times 12 + 8$. Hence the 200th last digit will be the same as the 8th last digit.

So the last digit of $s_{200}$ is 4.
5. For each white square, colour in red the edges that are adjacent to black squares. Observe that the sum of the numbers that Sebastien writes down is the number of red edges.

The number of red edges is bounded above by the number of edges in the $11 \times 38$ grid that do not lie on the boundary of the grid. The number of such horizontal edges is $11 \times 37$, while the number of such vertical edges is $10 \times 38$. Therefore, the sum of the numbers that Sebastien writes down is bounded above by $11 \times 37 + 10 \times 38 = 787$.

Now note that this upper bound is obtained by the usual chessboard colouring of the grid. So the maximum sum of the numbers that Sebastien writes down is 787.

6. **Method 1**

Draw $OA$.

Since $OA$ is perpendicular to $PQ$, $\angle OAB = 90^\circ - kx^\circ$.

Since $OA = OB$ (radii), $\angle OBA = 90^\circ - kx^\circ$.

Since $\angle QAB$ is an exterior angle of $\triangle PAB$, $kx^\circ = x + (90 - kx)$.

Rearranging gives $(2k - 1)x = 90$.

For maximum $k$ we want $2k - 1$ to be the largest odd factor of 90. Then $2k - 1 = 45$ and $k = 23$.

**Method 2**

Let $C$ be the other point of intersection of the line $PB$ with the circle.

By the Tangent-Chord theorem, $\angle ACB = \angle QAB = kx^\circ$. Since $BC$ is a diameter, $\angle CAB = 90^\circ$. By the Tangent-Chord theorem, $\angle PAC = \angle ABC = 180 - 90 - kx = 90 - kx$.

Since $\angle ACB$ is an exterior angle of $\triangle PAC$, $kx^\circ = x + 90 - kx$.

Rearranging gives $(2k - 1)x = 90$.

For maximum $k$ we want $2k - 1$ to be the largest odd factor of 90. Then $2k - 1 = 45$ and $k = 23$. 

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7. **Method 1**

If \( n^2 + 2016n = m^2 \), where \( n \) and \( m \) are positive integers, then \( m = n + k \) for some positive integer \( k \). Then \( n^2 + 2016n = (n + k)^2 \). So \( 2016 = 2nk + k^2 \), or \( n = k^2/(2016 - 2k) \). Since both \( n \) and \( k^2 \) are positive, we must have \( 2016 - 2k > 0 \), or \( 2k < 2016 \). Thus \( 1 \leq k \leq 1007 \).

As \( k \) increases from 1 to 1007, \( k^2 \) increases and \( 2016 - 2k \) decreases, so \( n \) increases. Conversely, as \( k \) decreases from 1007 to 1, \( k^2 \) decreases and \( 2016 - 2k \) increases, so \( n \) decreases. If we take \( k = 1007 \), then \( n = 1007^2/2 \), which is not an integer. If we take \( k = 1006 \), then \( n = 1006^2/4 = 503^2 \). So \( n \leq 503^2 \).

If \( k = 1006 \) and \( n = 503^2 \), then \( (n + k)^2 = (503^2 + 1006)^2 = (503^2 + 2 \times 503)^2 = 503^2(503 + 2)^2 = 503^2(503^2 + 4 \times 503 + 4) = 503^2(503^2 + 2016) = n^2 + 2016n \). So \( n^2 + 2016n \) is indeed a perfect square. Thus \( 503^2 \) is the largest value of \( n \) such that \( n^2 + 2016n \) is a perfect square.

Since \( 503^2 = (500 + 3)^2 = 500^2 + 2 \times 500 \times 3 + 3^2 = 250000 + 3000 + 9 = 253009 \), the remainder when \( n \) is divided by 1000 is 9.

**Method 2**

If \( n^2 + 2016n = m^2 \), where \( n \) and \( m \) are positive integers, then \( m^2 = (n + 1008)^2 - 1008^2 \).

So \( 1008^2 = (n + 1008)(n + 1008 - m) \) and both factors are even and positive. Hence \( n + 1008 + m = 1008^2/(n + 1008 - m) \leq 1008^2/2 \).

Since \( m \) increases with \( n \), maximum \( n \) occurs when \( n + 1008 + m \) is maximum. If \( n + 1008 + m = 1008^2/2 \), then \( n + 1008 - m = 2 \). Adding these two equations and dividing by 2 gives \( n + 1008 = 504^2 + 1 \) and \( m = 504^2 - 1008 + 1 = (504 - 1)^2 = 503^2 \).

If \( n = 503^2 \), then \( n^2 + 2016n = 503^2(503^2 + 2016) \). Now \( 503^2 + 2016 = (504 - 1)^2 + 2016 = 504^2 + 1008 + 1 = (504 + 1)^2 = 505^2 \). So \( n^2 + 2016n \) is indeed a perfect square. Thus \( 503^2 \) is the largest value of \( n \) such that \( n^2 + 2016n \) is a perfect square.

Since \( 503^2 = (500 + 3)^2 = 500^2 + 2 \times 500 \times 3 + 3^2 = 250000 + 3000 + 9 = 253009 \), the remainder when \( n \) is divided by 1000 is 9.

**Method 3**

If \( n^2 + 2016n = m^2 \), where \( n \) and \( m \) are positive integers, then solving the quadratic for \( n \) gives \( n = (-2016 + \sqrt{2016^2 + 4m^2})/2 = \sqrt{1008^2 + m^2} - 1008 \). So \( 1008^2 + m^2 = k^2 \) for some positive integer \( k \).

Hence \( (k - m)(k + m) = 1008^2 \) and both factors are even and positive. Hence \( k + m = 1008^2/(k - m) \leq 1008^2/2 \).

Since \( m \), \( n \), \( k \) increase together, maximum \( n \) occurs when \( m + k \) is maximum. If \( k + m = 1008^2/2 \), then \( k - m = 2 \).

Subtracting these two equations and dividing by 2 gives \( m = 504^2 - 1 \) and \( 1008^2 + m^2 = 1008^2 + (504^2 - 1)^2 = 4 \times 504^2 + 504^4 - 2 \times 504^2 + 1 + 504^2 + 2 \times 504^2 + 1 = (504^2 + 1)^2 \). So \( n = 504^2 + 1 - 2 \times 504 = (504 - 1)^2 = 503^2 \).

If \( n = 503^2 \), then \( n^2 + 2016n = 503^2(503 + 2016) \). Now \( 503^2 + 2016 = (504 - 1)^2 + 2016 = 504^2 + 1008 + 1 = (504 + 1)^2 = 505^2 \). So \( n^2 + 2016n \) is indeed a perfect square. Thus \( 503^2 \) is the largest value of \( n \) such that \( n^2 + 2016n \) is a perfect square.

Since \( 503^2 = (500 + 3)^2 = 500^2 + 2 \times 500 \times 3 + 3^2 = 250000 + 3000 + 9 = 253009 \), the remainder when \( n \) is divided by 1000 is 9.
8. Suppose Ann has the last turn. Let \( n \) be the number of turns that Bob has. Then the number of sweets that he takes is \( 2 + 4 + 6 + \cdots + 2n = 2(1 + 2 + \cdots + n) = n(n + 1) \). So the total number of sweets is \( 2n(n + 1) \).

Suppose Bob has the last turn. Let \( n \) be the number of turns that Ann has. Then the number of sweets that she takes is \( 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \). So the total number of sweets is \( 2n^2 \).

So half the number of sweets is \( n(n + 1) \) or \( n^2 \). Applying the same sharing procedure to half the sweets gives, for some integer \( m \), one of the following four cases:

1. \( n(n + 1) = 2m(m + 1) \)
2. \( n(n + 1) = 2m^2 \)
3. \( n^2 = 2m(m + 1) \)
4. \( n^2 = 2m^2 \).

In the first two cases we want \( n \) such that \( n(n + 1) < 500 \). So \( n \leq 21 \).

In the first case, since 2 divides \( m \) or \( m + 1 \), we also want 4 to divide \( n(n + 1) \). So \( n \leq 10 \).

Since \( 20 \times 21 = 420 = 2 \times 21 \times 20 \), 15 \( \times \) 16 = 240 while 16 \( \times \) 17 = 272.

In the fourth case, \( m^2 = 2m(n + 1) \). Applying this or similar triangles we have

\[
Y_{n+1}^2 = X_n^2 + X_n^2 = X_n^2 + Y_{n-1}^2 = X_n^2 + X_{n-1}^2 - Y_{n-2}^2 = X_n^2 + X_{n-1}^2 + X_{n-2}^2 + \cdots + X_1^2 + Y_1^2
\]

The area of the triangle shown is given by \( \frac{1}{2}Y_{n+1} \) and by \( \frac{1}{2}X_nY_n \). Using this or similar triangles we have

\[
Y_{n+1} = X_n \times Y_n = X_n \times X_{n-1} \times Y_{n-1}
\]

So

\[
X_0^2 + X_1^2 + X_2^2 + \cdots + X_n^2 = X_0^2 \times X_1^2 \times X_2^2 \times \cdots \times X_n^2
\]
Method 2

For each large triangle, one leg is \( X_n \). Let \( Y_{n-1} \) be the other leg and let \( Y_n \) be the hypotenuse. Note that \( Y_0 = X_0 \).

\[
\begin{array}{c}
Y_n \\
1 \\
Y_{n-1}
\end{array}
\]

From similar triangles we have \( Y_1/X_1 = X_0/1 \). So \( Y_1 = X_0 \times X_1 \).

By Pythagoras, \( Y_n^2 = X_n^2 + X_{n-1}^2 \). So \( X_0^2 + X_1^2 = Y_1^2 = X_0^2 \times X_1^2 \).

Assume for some \( k \geq 1 \)

\[
Y_k^2 = X_0^2 + X_1^2 + X_2^2 + \cdots + X_k^2 = X_0^2 \times X_1^2 \times X_2^2 \times \cdots \times X_k^2
\]

From similar triangles we have \( Y_{k+1}/X_{k+1} = Y_k/1 \). So \( Y_{k+1} = Y_k \times X_{k+1} \).

By Pythagoras, \( Y_{k+1}^2 = X_{k+1}^2 + Y_k^2 \). So \( X_{k+1}^2 + Y_k^2 = Y_{k+1}^2 = Y_k^2 \times X_{k+1}^2 \). Hence

\[
X_0^2 + X_1^2 + X_2^2 + \cdots + X_k^2 + X_{k+1}^2 = X_0^2 \times X_1^2 \times X_2^2 \times \cdots \times X_k^2 \times X_{k+1}^2
\]

By induction,

\[
X_0^2 + X_1^2 + X_2^2 + \cdots + X_n^2 = X_0^2 \times X_1^2 \times X_2^2 \times \cdots \times X_n^2
\]

for all \( n \geq 1 \).

10. Method 1

Let \( b \) and \( w \) denote the sum of the labels on all black and white vertices respectively. Let \( c \) be the label on the central vertex. Then

\[
b + w + c = 0 \quad (1)
\]

Summing the labels over all diameters gives

\[
b + w + nc = ns \quad (2)
\]

Summing the labels over all black-white-black arcs gives

\[
2b + w = ns \quad (3)
\]

From (1) and (2),

\[
(n-1)c = ns \quad (4)
\]

Hence \( n \) divides \( c \). Since \(-n \leq c \leq n, c = 0, -n, or n\).

Suppose \( c = \pm n \). From (2) and (3), \( b = nc = \pm n^2 \).

Since \(|b| \leq 1 + 2 + \cdots + n < n^2 \), we have a contradiction.

So \( c = 0 \). From (4), \( s = 0 \).

Method 2

Case 1. \( n \) is even.

For any label \( x \) not at the centre, let \( x' \) denote the label diametrically opposite \( x \). Let the centre have label \( c \). Then

\[
x + c + x' = s.
\]

If \( x, y, z \) are any three consecutive labels where \( x \) and \( z \) are on black points, then we have

\[
x + c + x' = y + c + y' = z + c + z' = s.
\]

Adding these yields

\[
x + y + z + 3c + x' + y' + z' = 3s.
\]

Since \( n \) is even, diametrically opposite points have the same colour. So

\[
x + y + z = s = x' + y' + z' \quad \text{and} \quad s = 3c.
\]

Hence \( p + p' = 2c \) for any label \( p \) on the circle. Since there are \( n \) such diametrically opposite pairs, the sum of all labels on the circle is \( 2nc \).

Since the sum of all the labels is zero, we have \( 0 = 2nc + c = c(2n + 1) \). Thus \( c = 0 \), and \( s = 3c = 0 \).
Case 2. \( n \) is odd.

We show that the required labelling is impossible. In each of the following diagrams, the arrows indicate the order in which the labels are either arbitrarily prescribed \((x, c, y)\) or dictated by the given conditions.

If \( n = 3 \), we have:

From the first, last, and fourth last labels, \( s = x + (s + x - 2c - y) + y = s + 2x - 2c \). Hence \( x = c \), which is disallowed.

If \( n = 5 \), we have:

From the first, last, and fourth last labels, \( s = x + (s + y - 3c) + (2c - x) = s + y - c \). Hence \( y = c \), which is disallowed.

If \( n \geq 7 \), we have:

The first and last labels are the same, which is disallowed.
**Investigation**

Since \( c = 0 = s \), for each diameter, the label at one end is the negative of the label at the other end.

Let \( n \) be an odd number.
Each diameter is from a black point to a white point.
If \( n = 3 \), we have:

\[
\begin{align*}
-\ a & \quad b \quad c \\
\quad b & \quad c \\
\quad a & \quad -\ a
\end{align*}
\]

Hence \( a + b - c = 0 = a - b + c \). So \( b = c \), which is disallowed.
If \( n > 3 \), we have:

\[
\begin{align*}
-\ a & \quad b \quad c \quad d \\
\quad c & \quad b \quad a \\
\quad d & \quad -\ a \quad -\ b \quad -\ c \quad -\ d
\end{align*}
\]

Hence \( b + c + d = 0 = -a - b - c = a + b + c \). So \( a = d \), which is disallowed.
So the required labelling does not exist for odd \( n \).
(Alternatively, the argument in Method 2 for odd \( n \) could be awarded a bonus mark.)
Now let $n$ be an even number.

We show that a required labelling does exist for $n = 2m \geq 4$. It is sufficient to show that $n + 1$ consecutive points on the circle from a black point to a black point can be assigned labels from $\pm 1, \pm 2, \ldots, \pm n$, so that the absolute values of the labels are distinct except for the two end labels, and the sum of the labels on each black-white-black arc is 0. We demonstrate such labellings with a zigzag pattern for clarity. Essentially, with some adjustments at the ends and in small cases, we try to place the odd labels on the black points, which are at the corners of the zigzag, and the even labels on the white points in between.

Case 1. $m$ odd.

$m = 3$

$m = 5$

General odd $m$. 

...
Case 2. $m$ even.

$m = 2$

$m = 4$

$m = 6$

General even $m$.

Thus the required labelling exists if and only if $n$ is even.

Comments

1. The special case $m = 2$ gives the classical magic square:

2. It is easy to check that, except for rotations and reflections, there is only one labelling for $m = 2$. Are the general labellings given above unique for all $m$?

3. Method 2 shows that the conclusion of the Problem 10 also holds for non-integer labels provided their sum is 0.
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## AUSTRALIAN INTERMEDIATE MATHEMATICS OLYMPIAD RESULTS

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**High Distinction**

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1. Determine all triples \((a, b, c)\) of distinct integers such that \(a, b, c\) are solutions of
\[x^3 + ax^2 + bx + c = 0.\]

2. Each unit square in a 2016 \(\times\) 2016 grid contains a positive integer. You play a game on the grid in which the following two types of moves are allowed.
- Choose a row and multiply every number in the row by 2.
- Choose a positive integer, choose a column, and subtract the positive integer from every number in the column.

You win if all of the numbers in the grid are 0. Is it always possible to win after a finite number of moves?

3. Show that in any sequence of six consecutive integers, there is at least one integer \(x\) such that
\[(x^2 + 1)(x^4 + 1)(x^6 - 1)\]
is a multiple of 2016.

4. Consider the sequence \(a_1, a_2, a_3, \ldots\) defined by \(a_1 = 1\) and
\[a_n = n - \lfloor \sqrt{a_{n-1}} \rfloor, \quad \text{for } n \geq 2.\]
Determine the value of \(a_{800}\).
(Here, \(\lfloor x \rfloor\) denotes the largest integer that is less than or equal to \(x\).)

5. Triangle \(ABC\) is right-angled at \(A\) and satisfies \(AB > AC\). The line tangent to the circumcircle of triangle \(ABC\) at \(A\) intersects the line \(BC\) at \(M\). Let \(D\) be the point such that \(M\) is the midpoint of \(BD\). The line through \(D\) that is parallel to \(AM\) intersects the circumcircle of triangle \(ACD\) again at \(E\).
Prove that \(A\) is the incentre of triangle \(EBM\).
1. Determine all triples \((a, b, c)\) of distinct integers such that \(a, b, c\) are solutions of
\[
x^3 + ax^2 + bx + c = 0.
\]

**Solution 1** (Norman Do)
Using Vieta’s formulas to relate the coefficients of the polynomial to its roots, we obtain the following equations.

\[
\begin{align*}
a + b + c &= -a \\
ab + bc + ca &= b \\
abc &= -c
\end{align*}
\]

The third equation implies that one of the following two cases must hold.

- **Case 1:** \(c = 0\)
  The equations then reduce to \(a + b = -a\) and \(ab = b\). The second of these implies that either \(b = 0\) or \(a = 1\). Since \(a, b, c\) are distinct, we cannot have \(b = 0\). So \(a = 1\) and the first equation implies that \(b = -2\), which leads to the triple \((a, b, c) = (1, -2, 0)\).

- **Case 2:** \(ab = -1\)
  Since \(a\) and \(b\) are integers, we must have \((a, b) = (1, -1)\) or \((a, b) = (-1, 1)\). Using the first equation above, this yields the triples \((a, b, c) = (1, -1, -1)\) and \((a, b, c) = (-1, 1, 1)\). Since we require \(a, b, c\) to be distinct, we do not obtain a solution in this case.

We can now check that the polynomial \(x^3 + x^2 - 2x = x(x - 1)(x + 2)\) does indeed have the roots 1, -2, 0. So the only triple that satisfies the conditions of the problem is \((a, b, c) = (1, -2, 0)\).

**Solution 2** (Angelo Di Pasquale and Alan Offer)
Substitute \(x = a, b, c\) into the polynomial to find

\[
\begin{align*}
2a^3 + ab + c &= 0 \quad (1) \\
b^3 + ab^2 + b^2 + c &= 0 \quad (2) \\
c^3 + ac^2 + bc + c &= 0. \quad (3)
\end{align*}
\]

- **Case 1:** At least one of \(a, b\) or \(c\) is equal to zero.
  In this case, the above equations imply that \(c = 0\).
Since $a, b, c$ are distinct, we have $a, b \neq 0$. Thus, equations (1) and (2) simplify to

$$2a^2 + b = 0 \quad \text{and} \quad b + a + 1 = 0.$$ 

Eliminating $b$ from these yields the equation

$$2a^2 - a - 1 = 0 \quad \Rightarrow \quad (a - 1)(2a + 1) = 0.$$ 

Since $a$ is an integer, we have $a = 1$. It follows from equation (1) that $b = -2$.

**Case 2:** All of $a, b$ and $c$ are not equal to zero.

From equation (2), we have $b \mid c$. From equation (3), we have $c^2 \mid c(b+1)$, so $c \mid b+1$.

Thus, $b \mid b + 1$, and so $b \mid 1$.

If $b = -1$, then equations (1) and (2) become

$$2a^3 - a + c = 0 \quad \text{and} \quad a + c = 0.$$ 

Eliminating $c$ from these yields $a^3 = a$. But since $a, b, c$ are distinct and non-zero, we have $a = 1$. Then equation (1) yields $c = -1$ and $b = -1$, a contradiction.

If $b = 1$, then equations (1) and (2) become

$$2a^3 + a + c = 0 \quad \text{and} \quad a + c + 2 = 0.$$ 

Eliminating $c$ from these yields $a^3 = 1$. Thus $a = 1$ and $b = 1$, a contradiction.

It is routine to verify that $(a, b, c) = (1, -2, 0)$ solves the problem. So the only triple that satisfies the conditions of the problem is $(a, b, c) = (1, -2, 0)$. 
2. Each unit square in a 2016 × 2016 grid contains a positive integer. You play a game on the grid in which the following two types of moves are allowed.

- Choose a row and multiply every number in the row by 2.
- Choose a positive integer, choose a column, and subtract the positive integer from every number in the column.

You win if all of the numbers in the grid are 0. Is it always possible to win after a finite number of moves?

Solution (Norman Do)

We will prove that it is possible to make every number in an $m \times n$ grid equal to 0 after a finite number of moves. The proof will be by induction on $n$, the number of columns in the grid.

Consider the base case, in which the number of columns is 1. Let the difference between the maximum and minimum numbers in the column be $D$. We will show that if $D > 0$, then it is possible to reduce the value of $D$ after a finite number of moves. First, we subtract a positive integer from every number in the column to make the minimum number in the column 1. Now multiply all of the entries equal to 1 by 2, which reduces $D$ by 1. Therefore, after a finite number of moves, it is possible to make all numbers in the column equal to each other. By subtracting this number from each entry of the column, we have made every number in the column equal to 0.

Now consider an $m \times n$ grid with $n \geq 2$. Suppose that we can make every number in a grid with $n-1$ columns equal to 0 after a finite number of moves. We simply apply the construction of the previous paragraph to the leftmost column in the grid. Note that the entries in the remaining columns may change, but remain positive. Therefore, after a finite number of moves, we obtain a grid whose leftmost column contains only 0. By the inductive hypothesis, we can make every number in the remaining $m \times (n-1)$ grid equal to 0 after a finite number of moves. Furthermore, observe that these moves do not change the 0 entries in the leftmost column. Therefore, it is always possible to make every number in an $m \times n$ grid equal to 0 after a finite number of moves.
3. Show that in any sequence of six consecutive integers, there is at least one integer \(x\) such that
\[(x^2 + 1)(x^4 + 1)(x^6 - 1)\]
is a multiple of 2016.

**Solution 1** (Alan Offer)
Let \(N = (x^2 + 1)(x^4 + 1)(x^6 - 1)\). Suppose that \(x\) is relatively prime to 2016, whose prime factorisation is \(2^5 \times 3^2 \times 7^1\).

- Since \(x\) is not divisible by 7, Fermat’s little theorem tells us that \(x^6 - 1\) is divisible by 7. Therefore, \(7 \mid N\).
- Since \(x\) is relatively prime to 9, Euler’s theorem tells us that \(x^{\phi(9)} - 1 = x^6 - 1\) is divisible by 9. Therefore, \(9 \mid N\).
- Since \(x\) is odd, the expressions \(x^2 + 1, x^4 + 1, x + 1\) and \(x - 1\) are all even, with one of the last two necessarily a multiple of 4. Therefore,
\[N = (x^2 + 1)(x^4 + 1)(x + 1)(x - 1)(x^4 + x^2 + 1)\]
is divisible by \(2^5\).

Thus, if \(x\) is relatively prime to 2016, then \(N\) is a multiple of 2016.

Now consider the following residues modulo 42: 1, 5, 11, 17, 23, 25, 31, 37, 1. Note that the largest difference between consecutive numbers in this sequence is 6. It follows that, among any six consecutive integers, there is an integer \(x\) that is congruent modulo 42 to one of these residues. Since these residues are relatively prime to 42, it follows that \(x\) is relatively prime to 42. And since 42 and 2016 have the same prime factors — namely 2, 3 and 7 — it follows that \(x\) is relatively prime to 2016. So, by the above reasoning, \(N\) is a multiple of 2016.

**Solution 2** (Angelo Di Pasquale, Chaitanya Rao and Jamie Simpson)
Let \(N = (x^2 + 1)(x^4 + 1)(x^6 - 1)\). Among any six consecutive integers, there must be one of the form \(6k + 1\) and one of the form \(6k - 1\) for some integer \(k\). At most one of these numbers can be divisible by 7, so let \(x\) be one that is not divisible by 7.

- Observe that
\[N = (x^2 + 1)(x^4 + 1)(x - 1)(x + 1)(x^4 + x + 1)(x^2 - x + 1)\]
One may check directly that if \(x\) is congruent to 1, 2, 3, 4, 5, or 6 modulo 7, then, the third, fourth, sixth, fourth, sixth, or fifth factor is divisible by 7, respectively. In any case, \(N\) is divisible by 7.
- If \(x = 6k + 1\), then the third and fourth factors in the expression above are divisible by 3. If \(x = 6k - 1\), then the fifth and sixth factors are divisible by 3. In any case, \(N\) is divisible by 9.
Since $x$ is odd, we have $x^2 \equiv 1 \pmod{8}$, so $x^6 \equiv 1 \pmod{8}$. Furthermore, $x^2 + 1$ and $x^4 + 1$ are even. So $N = (x^2 + 1)(x^4 + 1)(x^6 - 1)$ is divisible by $2^3 \times 2 \times 2 = 32$.

We conclude that in any sequence of six consecutive integers, there is at least one integer $x$ such that $N = (x^2 + 1)(x^4 + 1)(x^6 - 1)$ is divisible by 7, 9, and 32. Hence, $N$ is divisible by their lowest common multiple $7 \times 9 \times 32 = 2016$.

**Solution 3** (Kevin McAvaney)

Note that

$$(x^2 + 1)(x^4 + 1)(x^6 - 1) = (x^2 + 1)(x^4 + 1)(x^4 - 1)(x^4 + x^2 + 1) = (x^8 - 1)(x^4 + x^2 + 1).$$

- Let $x = 7k + r$, where $r = 0, 1, 2, 3, 4, 5$ or 6. From the binomial theorem, we know that $x^6 = (a$ multiple of $7) + r^6$. By testing each of the values for $r$, we see that $7 \mid x^6 - 1$ if $r$ is not equal to 0.
- Let $x = 3k + r$, where $r = 0, 1$ or 2. From the binomial theorem, we know that $x^6 = (a$ multiple of $9) + r^6$. By testing each of the values of $r$, we see that $9 \mid x^6 - 1$ if $r$ is not equal to 0.
- Let $x = 2k + 1$. From the binomial theorem, we know that

$$x^8 = (a$ multiple of $32) + 28(2k)^2 + 8(2k) + 1 = (a$ multiple of $32) + 16k(7k + 1) + 1.$$

Hence, $32 \mid x^8 - 1$.

In any six consecutive integers, exactly three are odd and they take the form $a, a + 2, a + 4$ for some integer $a$. Of these, exactly one is a multiple of 3 and at most one is a multiple of 7. So there is at least one of them, $x$ say, that is not divisible by 2, 3 or 7. By the above reasoning, it follows that this value of $x$ makes $(x^2 + 1)(x^4 + 1)(x^6 - 1)$ divisible by 2016.
4. Consider the sequence $a_1, a_2, a_3, \ldots$ defined by $a_1 = 1$ and

$$a_n = n - \lfloor \sqrt{a_{n-1}} \rfloor, \quad \text{for } n \geq 2.$$ 

Determine the value of $a_{800}$.

(Here, $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to $x$.)

Solution 1 (Norman Do)

The first several terms of the sequence are 1, 1, 2, 3, 4, 4, 5, 6, 7, 8, 9, 9, 10, \ldots. It appears that the following are true.

- We have $a_n - a_{n-1} = 0$ or 1 for all $n \geq 2$, so the sequence is non-decreasing and contains every positive integer.

- Every positive integer occurs exactly once, apart from perfect squares, which appear exactly twice.

First, we prove by induction that $a_n - a_{n-1} = 0$ or 1 for all $n \geq 2$. Note that this is certainly true for the first several terms listed above. Using the inductive hypothesis that $a_n - a_{n-1} = 0$ or 1, we have

$$a_{n+1} - a_n = (n + 1 - \lfloor \sqrt{a_n} \rfloor) - (n - \lfloor \sqrt{a_{n-1}} \rfloor) = 1 + \lfloor \sqrt{a_{n-1}} \rfloor - \lfloor \sqrt{a_n} \rfloor$$

$$= \begin{cases} 1, & \text{if } a_n = a_{n-1}, \\ 1 + \lfloor \sqrt{a_{n-1}} \rfloor - \lfloor \sqrt{a_n} \rfloor, & \text{if } a_n = a_{n-1} + 1. \end{cases}$$ (1)

For all positive integers $a$, we have

$$\lfloor \sqrt{a} - \sqrt{a-1} \rfloor^2 = 2a - 1 - 2\sqrt{a\sqrt{a-1}} < 2a - 1 - 2\sqrt{a-1} = 1.$$  

So we have $\sqrt{a} - \sqrt{a-1} < 1$, which implies that $0 \leq \lfloor \sqrt{a} \rfloor - \lfloor \sqrt{a-1} \rfloor \leq 1$. Combining this with equation (1), we obtain $a_{n+1} - a_n = 0$ or 1. Therefore, by induction, we have $a_n - a_{n-1} = 0$ or 1 for all $n \geq 2$, so the sequence is non-decreasing and contains every positive integer.

Next, we prove that every positive integer occurs exactly once, apart from perfect squares, which appear exactly twice. Since we can deduce from equation (1) that

$$a_n - a_{n-1} = 0 \quad \Rightarrow \quad a_{n+1} - a_n = 1,$$

every positive integer occurs either once or twice. From equation (1) again, we know that a term occurs twice when $a_{n+1} - a_n = 0$, which arises only if $\lfloor \sqrt{a_{n-1}} \rfloor - \lfloor \sqrt{a_{n-1}} \rfloor = 1$ or $n = 1$. However, this occurs only if $a_n$ is a perfect square.

Finally, we observe that the sequence can be defined from the following properties.

- We have $a_n - a_{n-1} = 0$ or 1 for all $n \geq 2$, so the sequence is non-decreasing and contains every positive integer.
Every positive integer occurs exactly once, apart from perfect squares, which appear exactly twice.

It follows that the only time that the number $27^2 - 1 = 728$ appears in the sequence is at term number $728 + 26 = 754$. Then $a_{755} = 729, a_{756} = 729, a_{757} = 730, a_{758} = 731, \ldots$ all the way up to $a_{800} = 773$. In this part of the sequence, no term appears twice, since there is no square number between 731 and 773.

**Solution 2** (Angelo Di Pasquale and Alan Offer)

The function $f(m) = m^2 + m$ is strictly increasing on the non-negative integers. Hence, there is a unique non-negative integer $m$ such that $f(m) \leq n < f(m+1)$ for each positive integer $n$. It follows that any positive integer $n$ may be written uniquely in the form

$$n = f(m) + k,$$

where $m$ is a non-negative integer and $k$ is an integer with $0 \leq k \leq 2m + 1$. We shall refer to the form in (1) as the special form of $n$.

We shall prove by induction that if $n$ has special form $n = f(m) + k$, then $a_n = m^2 + k$.

For the base case $n = 1$, we have $1 = f(0) + 1$, and $a_1 = 0^2 + 1 = 1$, as desired. Assume that $a_n = m^2 + k$, where $n$ has special form $n = f(m) + k$. We compute $a_{n+1}$.

- **Case 1:** $k \leq 2m$
  
  Since $k + 1 \leq 2m + 1$, the special form of $n + 1$ is given by $n + 1 = f(m) + k + 1$.
  
  Also observe that $[\sqrt{a_n}] = m$. Hence,

  $$a_{n+1} = n + 1 - \lfloor \sqrt{a_n} \rfloor = f(m) + k + 1 - m = m^2 + k + 1,$$

  as desired.

- **Case 2:** $k = 2m + 1$
  
  Note that $n + 1 = f(m) + k + 1 = m^2 + m + 2m + 1 + 1 = (m+1)^2 + (m+1)$. Thus the special form of $n + 1$ is given by $n + 1 = f(m+1) + 0$. Observe that $[\sqrt{a_n}] = m + 1$.
  
  Hence,

  $$a_{n+1} = n + 1 - \lfloor \sqrt{a_n} \rfloor = f(m+1) - (m+1) = (m+1)^2 + 0,$$

  as desired. This concludes the induction.

To conclude, note that $f(27) = 756$ and $f(28) = 812$. Hence, 800 has special form $800 = f(27) + 44$. It follows that $a_{800} = 27^2 + 44 = 773$.

**Solution 3** (Daniel Mathews)

Let $a_n = n - b_n$. For any $n \geq 1$, we claim that $b_n = k$ for $k(k+1) \leq n \leq (k+1)(k+2) - 1$.

We note that, for each $n \geq 1$, there is precisely one integer $k \geq 0$ such that the above inequality holds, so the above gives a well-defined formula for $b_n$. We observe that the
claimed value for \( b_n \) is true for \( n = 1 \); now assume it is true for all \( b_n \), and we prove it is true for \( b_{n+1} \).

We have \( a_{n+1} = n + 1 - \lfloor \sqrt{a_n} \rfloor \), and \( a_n = n - k \), where \( k(k + 1) \leq n \leq (k + 1)(k + 2) - 1 \) as above.

Thus \( k^2 \leq n - k \leq k^2 + 2k + 1 = (k + 1)^2 \), and hence, \( k \leq \sqrt{a_n} \leq k + 1 \). Thus \( \lfloor \sqrt{a_n} \rfloor = k \) or \( k + 1 \), and \( \lfloor \sqrt{a_n} \rfloor = k + 1 \) if and only if \( n = (k + 1)(k + 2) - 1 \). Hence, we can consider the two cases separately: \( n = (k + 1)(k + 2) - 1 \), and \( k(k + 1) \leq n \leq (k + 1)(k + 2) - 2 \).

In the first case \( \lfloor \sqrt{a_n} \rfloor = k + 1 \), and in the second case \( \lfloor \sqrt{a_n} \rfloor = k \).

So, first suppose \( n = (k + 1)(k + 2) - 1 \) and \( \lfloor \sqrt{a_n} \rfloor = k + 1 \). Then \( a_{n+1} = n + 1 - (k + 1) \), so \( b_{n+1} = k + 1 \). And indeed, since \( (k + 1)(k + 2) - 1 = n \), we have the inequality \( (k + 1)(k + 2) \leq n + 1 \leq (k + 1)(k + 2) - 1 \), so the formula for \( b_n \) holds.

Now suppose that \( k(k + 1) \leq n \leq (k + 1)(k + 2) - 2 \) and \( \lfloor \sqrt{a_n} \rfloor = k \). Then \( a_{n+1} = n + 1 - k \), and \( b_{n+1} = k \).

And since \( k(k + 1) \leq n \leq (k + 1)(k + 2) - 2 < (k + 1)(k + 2) - 1 \) the formula holds for \( b_n \) in this case also.

By induction, then our formula holds for all \( n \). When \( n = 800 \) we have \( k = 27 \), so \( b_{800} = 27 \) and \( a_{800} = 773 \).

Solution 4 (Kevin McAvaney)

The first few terms of the sequence are 1, 1, 2, 3, 4, 5, 6, 7, 8, 9, 9, 10, \ldots. We want to show this pattern continues (terms increase by 1, except each square appears exactly twice).

Suppose for some \( k \) and \( m \), \( a_{k-1} = a_k = m^2 \). Then \( a_k = k - \lfloor \sqrt{m^2} \rfloor = k - m \), so \( m^2 = k - m \).

Hence, \( a_{k+1} = (k + 1) - \lfloor \sqrt{m^2} \rfloor = (k + 1) - m = m^2 + 1 \).

For \( 1 \leq r \leq (m + 1)^2 - m^2 \), we have inductively \( a_{k+r} = (k + r) - \lfloor \sqrt{m^2 + r - 1} \rfloor = (k + r) - m = m^2 + r \). In particular, \( a_{k+(m+1)^2-m^2} = m^2 + (m + 1)^2 - m^2 = (m + 1)^2 \).

Hence \( a_{k+(m+1)^2-m^2+1} = (k + 2m + 2) - (m + 1) = k + (m + 1) = m^2 + m + (m + 1) = (m + 1)^2 \).

So the pattern continues.

Moreover, the index for the second appearance of the square \( m^2 \) is \( k = m^2 + m \). The largest \( m \) such that \( m^2 + m \leq 800 \) is 27 and \( 27^2 + 27 = 756 \). Since \( a_{756} = 27^2 = 729 \) and \( 800 - 756 = 44 \), we have \( a_{800} = 729 + 44 = 773 \).
5. Triangle $ABC$ is right-angled at $A$ and satisfies $AB > AC$. The line tangent to the circumcircle of triangle $ABC$ at $A$ intersects the line $BC$ at $M$. Let $D$ be the point such that $M$ is the midpoint of $BD$. The line through $D$ that is parallel to $AM$ intersects the circumcircle of triangle $ACD$ again at $E$.

Prove that $A$ is the incentre of triangle $EBM$.

**Solution 1** (Angelo Di Pasquale)

Let the line $AM$ intersect the circumcircle of triangle $ACD$ again at $N$.

Using the alternate segment theorem in circle $ABC$, and then the cyclic quadrilateral $ACND$, we have

$$\angle CBA = \angle CAN = \angle CDN.$$ 

Since $MB = MD$, it follows that $\triangle MAB \cong \triangle MND$ (ASA). Hence, $MA = MN$.

We also have $\triangle MND \cong \triangle MAE$, since these triangles are related by reflection in the perpendicular bisector of the parallel chords $AN$ and $ED$ in circle $ANDE$. Hence, $\triangle MAB \cong \triangle MAE$ and $MA$ is a line of symmetry for triangle $EBM$.

Now let $x = \angle AEM = \angle MBA = \angle CAM$ and $y = \angle ABE = \angle BEA$. The angle sum in $\triangle ABM$ yields $\angle AMB = 90^\circ - 2x$. By symmetry, we also have $\angle EMA = 90^\circ - 2x$.

Finally, the angle sum in $\triangle EBM$ yields $2x + 2y + 2(90^\circ - 2x) = 180^\circ$, which implies that $x = y$.

Thus, $A$ is the incentre of $\triangle EBM$ because it is the intersection of its angle bisectors.

**Solution 2** (Angelo Di Pasquale)

With $N$ defined as in the solution above, we have

$$MA \cdot MN = MC \cdot MD \quad \text{(power of $M$ with respect to circle $ACD$)}$$

$$= MC \cdot MB \quad \text{($M$ is the midpoint of $BD$)}$$

$$= MA^2. \quad \text{(power of $M$ with respect to circle $ABC$)}$$

So $MA = MN$ and since $MB = MD$, we have $\triangle MAB \cong \triangle MND$ (SAS). Hence, $MA = MN$.

The rest is as in the solution above.
## AMOC SENIOR CONTEST RESULTS

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<td>Arun Jha</td>
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## AMOC SENIOR CONTEST STATISTICS

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The 2015 AMOC School of Excellence was held 3–12 December at Newman College, University of Melbourne. The main qualifying exams for this are the AIMO and the AMOC Senior Contest.

A total of 28 students from around Australia attended the school. All states and territories were represented except for the Northern Territory. A further student from New Zealand also attended.

The students are divided into a senior group and a junior group. There were 14 junior students, 13 of whom were attending for the first time. There were 15 students making up the senior group, 7 of whom were first-time seniors.

The program covered the four major areas of number theory, geometry, combinatorics and algebra. Each day would start at 8am with lectures or an exam and go until 12 noon or 1pm. After a one-hour lunch break they would have a lecture at 2pm. At 4pm participants would usually have free time, followed by dinner at 6pm. Finally, each evening would round out with a problem session, topic review, or exam review from 7pm until 9pm.

Two highly experienced senior students were assigned to give a lecture each. Seyoon Ragavan was assigned the senior Functional Equations lecture, and Kevin Xian was assigned the senior Inspired Constructions geometry lecture. They both did an excellent job!

Many thanks to Alexander Chua, Ivan Guo, Victor Khou, and Andy Tran, who served as live-in sta. Also my thanks go to Adrian Agisilaou, Natalie Aisbett, Aaron Chong, Norman Do, Alexander Gunning, Patrick He, Daniel Mathews, Chaitanya Rao, and Jeremy Yip, who assisted in lecturing and marking.

Angelo Di Pasquale
Director of Training, AMOC
## Participants at the 2015 AMOC School of Excellence

<table>
<thead>
<tr>
<th>Name</th>
<th>School</th>
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<tr>
<td><strong>Seniors</strong></td>
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<tr>
<td>Matthew Cheah</td>
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<td>Michelle Chen</td>
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<td>Linus Cooper</td>
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<td>William Hu</td>
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<td>Ilia Kucherov</td>
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<tr>
<td>Miles Lee</td>
<td>Auckland International College NZ</td>
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<td>Isabel Longbottom</td>
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<td>Seyoon Ragavan</td>
<td>Knox Grammar School NSW</td>
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<td>Kevin Xian</td>
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<td><strong>Juniors</strong></td>
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<td>Hadyn Tang</td>
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<tr>
<td>Stanley Zhu</td>
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1. Find all positive integers $n$ such that $2^n + 7^n$ is a perfect square.

2. Let $ABC$ be a triangle. A circle intersects side $BC$ at points $U$ and $V$, side $CA$ at points $W$ and $X$, and side $AB$ at points $Y$ and $Z$. The points $U, V, W, X, Y, Z$ lie on the circle in that order. Suppose that $AY = BZ$ and $BU = CV$.

Prove that $CW = AX$.

3. For a real number $x$, define $\lfloor x \rfloor$ to be the largest integer less than or equal to $x$, and define $\{x\} = x - \lfloor x \rfloor$.

(a) Prove that there are infinitely many positive real numbers $x$ that satisfy the inequality

$$\{x^2\} - \{x\} > \frac{2015}{2016}.$$ 

(b) Prove that there is no positive real number $x$ less than 1000 that satisfies this inequality.

4. A binary sequence is a sequence in which each term is equal to 0 or 1. We call a binary sequence superb if each term is adjacent to at least one term that is equal to 1. For example, the sequence 0, 1, 1, 0, 0, 1, 1, 1 is a superb binary sequence with eight terms. Let $B_n$ denote the number of superb binary sequences with $n$ terms.

Determine the smallest integer $n \geq 2$ such that $B_n$ is divisible by 20.
5. Find all triples \((x, y, z)\) of real numbers that simultaneously satisfy the equations
\[
xy + 1 = 2z \\
yz + 1 = 2x \\
zx + 1 = 2y.
\]

6. Let \(a, b, c\) be positive integers such that \(a^3 + b^3 = 2^c\).
Prove that \(a = b\).

7. Each point in the plane is assigned one of four colours.
Prove that there exist two points at distance 1 or \(\sqrt{3}\) from each other that are assigned the same colour.

8. Three given lines in the plane pass through a point \(P\).
(a) Prove that there exists a circle that contains \(P\) in its interior and intersects the three lines at six points \(A, B, C, D, E, F\) in that order around the circle such that \(AB = CD = EF\).
(b) Suppose that a circle contains \(P\) in its interior and intersects the three lines at six points \(A, B, C, D, E, F\) in that order around the circle such that \(AB = CD = EF\).
Prove that
\[
\frac{1}{2} \text{area}(\text{hexagon } ABCDEF) \geq \text{area}(\triangle APB) + \text{area}(\triangle CPD) + \text{area}(\triangle EPF).
\]
1. Clearly \( n = 1 \) is obviously a solution. We show that there is no solution for \( n > 1 \). For reference, in each of the solutions that follow, we suppose that

\[
2^n + 7^n = m^2, \tag{1}
\]

for some positive integers \( n \) and \( m \).

**Solution 1** (Sharvil Kesarwani, year 9, Merewether High School, NSW)

**Case 1** \( n \) is odd and \( n > 1 \)

Considering equation (1) modulo 4 yields

\[
\text{LHS}(1) \equiv 0 + (-1)^n \pmod{4} \\
\equiv 3 \pmod{4}.
\]

However, \( m^2 \equiv 0 \) or 1 (mod 4) for any integer \( m \). Thus there are no solutions in this case.

**Case 2** \( n \) is even

Considering equation (1) modulo 3 yields

\[
\text{LHS}(1) \equiv (-1)^n + 1^n \pmod{3} \\
\equiv 2 \pmod{3}.
\]

However, \( m^2 \equiv 0 \) or 1 (mod 3) for any integer \( m \). Thus there are no solutions in this case either.

Having covered all possible cases, the proof is complete. \( \square \)
Solution 2 (Keiran Lewellen, year 11, Te Kura (The Correspondence School), NZ)

Case 1 $n$ is odd and $n > 1$

Let $n = 2k + 1$, where $k$ is a positive integer. Equation (1) can be rewritten as

\[ 2 \cdot 2^{2k} + 7 \cdot 7^{2k} = m^2 \]
\[ \iff 2(2^{2k} - 7^{2k}) = m^2 - 9 \cdot 7^{2k} \]
\[ = (m - 3 \cdot 7^k)(m + 3 \cdot 7^k). \]

The LHS of the above equation is even, so the RHS must be even too. This implies that $m$ is odd. But the both factors on the RHS are even, and so the RHS is a multiple of 4. However, the LHS is not a multiple of 4. So there are no solutions in this case.

Case 2 $n$ is even

Let $n = 2k$, where $k$ is a positive integer. Equation (1) can be rewritten as

\[ 2^{2k} = m^2 - 7^{2k} \]
\[ = (m - 7^k)(m + 7^k). \]

Hence there exist non-negative integers $r < s$ satisfying the following.

\[ m - 7^k = 2^r \]
\[ m + 7^k = 2^s \]

Subtracting the first equation from the second yields

\[ 2^r(2^s - r - 1) = 2 \cdot 7^k. \]

Hence $r = 1$, and so $m = 7^k + 2$. Substituting this into equation (1) yields

\[ 2^{2k} + 7^{2k} = (7^k + 2)^2 \]
\[ = 7^{2k} + 4 \cdot 7^k + 4 \]
\[ \iff 4^k = 4 \cdot 7^k + 4. \]

However, this is clearly impossible because the LHS is smaller than the RHS.

Having covered all possible cases, the proof is complete. \[\square\]

1Equivalent to year 10 in Australia.
Solution 3 (Anthony Tew, year 10, Pembroke School, SA)

Case 1  \( n \) is odd and \( n > 1 \)
This is handled as in solution 1.

Case 2  \( n \) is even

Let \( n = 2k \), where \( k \) is a positive integer so that equation (1) becomes

\[
2^{2k} + 7^{2k} = m^2.
\]

Observe that

\[
7^{2k} < 7^{2k} + 2^{2k} = 7^{2k} + 4^k < 7^{2k} + 2 \cdot 7^k + 1 = (7^k + 1)^2.
\]

It follows that \( 7^k < m < 7^k + 1 \). Hence there are no solutions in this case. \( \square \)
**Solution 4**  (Xutong Wang, year 10\(^2\), Auckland International College, NZ)

**Case 1**  \(n\) is odd and \(n > 1\)
This may be handled as in solution 1.

**Case 2**  \(n\) is even
Let \(n = 2k\), where \(k\) is a positive integer so that equation (1) becomes

\[
2^{2k} + 7^{2k} = m^2.
\]

Observe that \((2^k, 7^k, m)\) is a primitive a Pythagorean triple.\(^3\) It follows that there exist positive integers \(u > v\) such that

\[
u^2 - v^2 = 7^k \quad \text{and} \quad 2uv = 2^k.
\]

However, the second equation above ensures that \(u, v < 2^k\). Hence \(u^2 - v^2 < 4^k < 7^k\).
Hence there are no solutions in this case. \(\square\)

\(^2\)Equivalent to year 9 in Australia.

\(^3\)Recall that \((x, y, z)\) is a primitive Pythagorean triple if \(x, y,\) and \(z\) are pairwise relatively prime positive integers satisfying \(x^2 + y^2 = z^2\). All primitive Pythagorean triples take the form

\[
(x, y, z) = (u^2 - v^2, 2uv, u^2 + v^2),
\]

for some relatively prime positive integers \(u\) and \(v\), of opposite parity, and where, without loss of generality, \(x\) is odd.
Solution 5  (Zefeng Li, year 9, Camberwell Grammar School, VIC)

Case 1 $n$ is odd and $n > 1$
This is handled as in solution 1.

Case 2 $n$ is even
The following table shows the units digit of $2^n$, $7^n$, and $2^n + 7^n$, for $n = 1, 2, \ldots$.
Observe that in each case, the units digit forms a 4-cycle.

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<tr>
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<td>3</td>
<td>1</td>
<td>7</td>
<td>\ldots</td>
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<tr>
<td>$2^n + 7^n$</td>
<td>9</td>
<td>3</td>
<td>1</td>
<td>7</td>
<td>9</td>
<td>\ldots</td>
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Therefore, for $n$ even, the units digit of $2^n + 7^n$ is either 3 or 7. However, the units digit of a perfect square can only be 0, 1, 4, 5, 6, or 9. Hence there are no solutions in this case. $\square$
2. Solution 1 (Linus Cooper, year 10, James Ruse Agricultural High School, NSW)

Let $O$ be the centre of circle $UVWXYZ$. Let $P$, $Q$, and $R$ be the midpoints of $UV$, $WX$, and $YZ$, respectively. Since the perpendicular bisector of any chord of a circle passes through the centre of the circle, we have $OP \perp UV$, $OQ \perp WX$, and $OR \perp YZ$.

Since $R$ is the midpoint of $YZ$ and $AY = BZ$, we also have that $R$ is the midpoint of $AB$. Hence $OR$ is the perpendicular bisector of $AB$. Similarly $OP$ is the perpendicular bisector of $BC$. Since $OR$ and $OP$ intersect at $O$, it follows that $O$ is the circumcentre of $\triangle ABC$.

Since $O$ is the circumcentre of $\triangle ABC$, it follows that $OQ$ is the perpendicular bisector of $AC$. Thus $Q$ is the midpoint of $AC$. However, since $Q$ is also the midpoint of $WX$, it follows that $AX = CW$. □
Solution 2  (Jack Liu, year 10, Brighton Grammar School, VIC)

Let $O$ be the centre of circle $UVWXYZ$.

From $OY = OZ$, we have $\angle OYZ = \angle YZO$. Thus

$$\angle AYO = \angle OZB.$$  

We also have $AY = BZ$. Hence $\triangle AYO \cong \triangle BZO$ (SAS). Therefore $OA = OB$.

A similar argument shows that $\triangle BUO \cong \triangle CVO$, and so $OB = OC$. Hence $OA = OC$. Thus also $\angle XAO = \angle OCW$. From $OX = OW$, we also have $\angle WXO = \angle OWX$, and so $\angle OXA = \angle CWO$. Therefore $\triangle AXO \cong \triangle CWO$ (AAS). Hence $AX = CW$.  \hfill \square
Solution 3  (James Bang, year 9, Baulkham Hills High School, NSW)

Let $BU = CV = a$, $AY = BZ = c$, $CW = x$, $AX = y$, and $WX = z$.

Considering the power of point $A$ with respect to circle $UVWXYZ$, we have

$$y(y + z) = c(c + YZ).$$

Considering the power of point $B$ with respect to circle $UVWXYZ$, we have

$$c(c + YZ) = a(a + UV).$$

Considering the power of point $C$ with respect to circle $UVWXYZ$, we have

$$a(a + UV) = x(x + z).$$

From the above, it follows that

$$y(y + z) = x(x + z)$$

$$\iff x^2 - y^2 = yz - xz$$

$$\iff (x - y)(x + y) = -z(x - y)$$

$$\iff (x - y)(x + y + z) = 0.$$

Since $x + y + z = AC > 0$, we have $x = y$. Thus $CW = AX$, as desired. \qed
3. **Solution 1** (Hadyn Tang, year 7, Trinity Grammar School, VIC)

(a) The following inequality is easily demonstrated by squaring everything.

\[ 1008 < \sqrt{1008^2 + 1} < 1008 + \frac{1}{2016} \]

Observe that

\[ \lim_{x \to \sqrt{1008^2 + 1}} \{x^2\} = 1 \]

and

\[ \lim_{x \to \sqrt{1008^2 + 1}} \{x\} < \frac{1}{2016} \]

Thus there is an open interval \( I = (r, \sqrt{1008^2 + 1}) \) such that for every \( x \in I \) we have \( \{x^2\} - \{x\} > \frac{2015}{2016} \). But \( I \) contains infinitely many real numbers. \( \square \)

(b) Suppose that \( \{x^2\} - \{x\} > \frac{2015}{2016} \).

Since \( \{x^2\} < 1 \), we require \( \{x\} < \frac{1}{2016} \). If \( x < 1000 \), then we have

\[ n < x < n + \frac{1}{2016} \]

for some \( n \in \{0, 1, \ldots, 999\} \). It follows that

\[ n^2 < x^2 < \left( n + \frac{1}{2016} \right)^2 = n^2 + \frac{n}{1008} + \frac{1}{2016^2} \]

Hence

\[ \{x^2\} - \{x\} < \frac{n}{1008} + \frac{1}{2016^2} \leq \frac{999}{1008} + \frac{1}{2016^2} = \frac{1998 + \frac{1}{2016}}{2016} < \frac{2015}{2016} \]

Thus no \( x < 1000 \) can satisfy the given inequality. \( \square \)
Solution 2  (Charles Li, year 10, Camberwell Grammar School, VIC)

(a) Let $x = n + \frac{1}{m}$ where $m$ and $n$ are positive integers yet to be chosen.

We have

$$x^2 = n^2 + \frac{2n}{m} + \frac{1}{m^2}.$$ 

Thus

$$\{x\} = \frac{1}{m} \quad \text{and} \quad \{x^2\} = \left\{\frac{2n + \frac{1}{m}}{m}\right\}.$$ 

Consider any fixed integer $m > 20160$. For $n = 1, 2, \ldots$ consider the value of

$$\frac{2n + \frac{1}{m}}{m}.$$ 

At $n = 1$, this value is less than 1. And every time $n$ increases by 1, this value increases by $\frac{2}{m}$ which is less than $\frac{2}{20160}$. Hence there is a value of $n$ for which

$$1 > \frac{2n + \frac{1}{m}}{m} > \frac{20158}{20160}.$$ 

For this value of $n$, we have

$$\{x^2\} - \{x\} = \frac{2n + \frac{1}{m}}{m} - \frac{1}{m} > \frac{20158}{20160} - \frac{1}{20160} > \frac{2015}{2016}.$$ 

Since such an $n$ can be chosen for each integer $m > 20160$ we have shown the existence of infinitely many $x$ satisfying the inequality. □

(b) Suppose $x$ is a positive real number satisfying

$$\{x^2\} - \{x\} > \frac{2015}{2016}.$$ 

Let $\lfloor x \rfloor = a$ and $\{x\} = b$. Since $\{x^2\} < 1$, we have $0 < b < \frac{1}{2016}$. We compute

$$x^2 = a^2 + 2ab + b^2 \quad \Rightarrow \quad \{x^2\} = \{2ab + b^2\}.$$ 

Since $a$ and $b$ are non-negative we have

$$2ab + b^2 \geq \{2ab + b^2\} = \{x^2\} > \{x^2\} - \{x\} > \frac{2015}{2016}.$$ 

Hence

$$b(2a + b) > \frac{2015}{2016}.$$ 

Since $0 < b < \frac{1}{2016}$ we have

$$2a + b > 2015.$$ 

And since $0 < b < \frac{1}{2016}$, this implies $a > 1007$. Since $a$ is an integer, we have $a \geq 1008$. Thus $x > 1008$. □

Comment  Part (b) of this solution shows that no positive real number $x$ satisfying the inequality can have $\lfloor x \rfloor \leq 1007$. Part (a) of solution 1 shows that the inequality can be satisfied for infinitely many $x$ with $\lfloor x \rfloor = 1008$. 

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Solution 3 (Jerry Mao, year 10, Caulfield Grammar School, VIC)

(a) Consider $x = 4032n - 1 + \frac{1}{8064}$ for any positive integer $n$. We compute

$$x^2 = (4032n - 1)^2 + n - \frac{2}{8064} + \frac{1}{8064^2}.$$  

Hence

$$\{x^2\} = 1 - \frac{2}{8064} + \frac{1}{8064^2},$$

$$\Rightarrow \{x^2\} - \{x\} = 1 - \frac{2}{8064} + \frac{1}{8064^2} - \frac{1}{8064} = \frac{8061}{8064} + \frac{1}{8064^2} > \frac{8060}{8064} = \frac{2015}{2016}.$$  

Since this is true for any positive integer $n$, we have found infinitely many positive $x$ satisfying the required inequality. \hfill \Box

(b) Let $[x] = a$ and $\{x\} = b$. Note that $0 \leq a < 1000$ and $0 < b < 1$.

As in solution 2, we have $b < \frac{1}{2016}$ and $\{x^2\} = \{2ab + b^2\}$. However since $0 \leq a < 1000$ and $0 < b < \frac{1}{2016}$, we have

$$2ab + b^2 < \frac{2000}{2016} + \frac{1}{2016^2} < \frac{2015}{2016}.$$  

Hence $\{x^2\} < \frac{2015}{2016}$. It follows that $\{x^2\} - \{x\} < \frac{2015}{2016}$.

Hence no such $x$ can satisfy the required inequality. \hfill \Box
Solution 4  (Zefeng Li, year 9, Camberwell Grammar School, VIC)

(a) Let $x = \underbrace{99\ldots9}_{n\text{ digits}}.0001$, where $n \geq 4$. Then $\{x\} = .0001$ and

$$x^2 = \underbrace{99\ldots9}_{n\text{ digits}}^2 + \underbrace{99\ldots9}_{n\text{ digits}} \times 0.0002 + 0.00000001$$

$$= \underbrace{99\ldots9}_{n\text{ digits}}^2 + 1 \underbrace{99\ldots9}_{n-4} .9998 + 0.00000001$$

Hence $\{x^2\} > .9998$.

It follows that $\{x^2\} - \{x\} > .9997 > \frac{2015}{2016}$. □

(b) This is done as in solution 3.
Solution 5  (Norman Do, AMOC Senior Problems Committee)

(a) We shall show that \( x = n + \frac{1}{n+1} \) satisfies the inequality for all sufficiently large positive integers \( n \).

\[
\{x^2\} - \{x\} = \left\{ n^2 + \frac{2n}{n+1} + \frac{1}{(n+1)^2} \right\} - \left\{ n + \frac{1}{n+1} \right\} \\
= \left\{ n^2 + 2 - \frac{2}{n+1} + \frac{1}{(n+1)^2} \right\} - \frac{1}{n+1} \\
= \left( 1 - \frac{2}{n+1} + \frac{1}{(n+1)^2} \right) - \frac{1}{n+1} \\
= 1 - \frac{3}{n+1} + \frac{1}{(n+1)^2} \\
> 1 - \frac{3}{n+1}
\]

Therefore, \( x = n + \frac{1}{n+1} \) satisfies the inequality as long as \( n \) is a positive integer such that

\[
1 - \frac{3}{n+1} > \frac{2015}{2016} \iff n > 3 \times 2016 - 1. \quad \square
\]

(b) This is done as in solution 2.
Solution 6 (Wilson Zhao, year 12, Killara High School, NSW)

(a) Let $\alpha$ be any irrational number satisfying $0 < \alpha < \frac{1}{2016}$. For example we could take $\alpha = \frac{1}{2016\sqrt{2}}$. Consider $x = n + \alpha$. We have

$$x^2 = n^2 + 2n\alpha + \alpha^2 \Rightarrow \{x^2\} = \{2n\alpha + \alpha^2\} \Rightarrow \{x^2\} - \{x\} = \{2n\alpha + \alpha^2\} - \alpha$$

If we could arrange for $\{2n\alpha\} < 1 - \alpha^2$, then we would have

$$\{x^2\} - \{x\} = \{2n\alpha\} + \alpha^2 - \alpha.$$  

If we could further arrange for $\{2n\alpha\} > \frac{2015}{2016} + \alpha - \alpha^2$, then the corresponding value of $x$ would satisfy the required inequality.

In summary, we would like to ensure the existence of infinitely many positive integers $n$ that satisfy

$$\frac{2015}{2016} + \alpha - \alpha^2 < \{2n\alpha\} < 1 - \alpha^2.$$  

However, this is an immediate consequence of the *Equidistribution theorem* applied to the irrational number $\frac{\alpha}{2}$.

(b) This is done as in solution 3.

Comment The full strength of the Equidistribution theorem is not needed in the above proof. The following weaker statement is sufficient.

For any irrational number $r$, the sequence $\{r\}, \{2r\}, \{3r\}, \ldots$ takes values arbitrarily close to any real number in the interval $[0, 1)$. We deem it instructive to provide a proof of the above.

Proof For any positive integer $n$, consider the intervals

$$I_1 = \left[0, \frac{1}{n}\right], I_2 = \left[\frac{1}{n}, \frac{2}{n}\right], I_3 = \left[\frac{2}{n}, \frac{3}{n}\right], \ldots, I_n = \left[\frac{n-1}{n}, 1\right].$$

By the pigeonhole principle, one of these intervals contains two terms of the sequence $\{r\}, \{2r\}, \{3r\}, \ldots$. Hence there exist positive integers $u \neq v$ such that

$$0 < \{ur\} - \{vr\} < \frac{1}{n}$$

$$\iff 0 < ur - \lfloor ur\rfloor - vr + \lfloor vr\rfloor < \frac{1}{n}$$

$$\iff 0 < \{ur - vr\} < \frac{1}{n}.$$  

The theorem states that for any irrational number $r$, the sequence $\{r\}, \{2r\}, \{3r\}, \ldots$ is uniformly distributed on the interval $[0, 1)$. In particular the sequence takes values arbitrarily close to any real number in the interval $[0, 1)$. See [https://en.wikipedia.org/wiki/Equidistribution_theorem](https://en.wikipedia.org/wiki/Equidistribution_theorem)
It follows that
\[0 < \{d\} < \frac{1}{n},\]
where \(d = (u - v)r\).

If \(d > 0\), then \(\{d\}, \{2d\}, \{3d\}, \ldots\) is a subsequence of \(\{r\}, \{2r\}, \{3r\}, \ldots\). Moreover, it is an arithmetic sequence whose first term is \(\{d\}\) and whose common difference is \(\{d\}\) up until the point where it is just about to get bigger than 1.

If \(d < 0\), then \(\{-d\}, \{-2d\}, \{-3d\}, \ldots\) is a subsequence of \(\{r\}, \{2r\}, \{3r\}, \ldots\). Moreover, it is an arithmetic sequence whose first term is \(1 - \{d\}\) and whose common difference is \(-\{d\}\) up until the point where it is just about to get smaller than 0.

In either case we have found a subsequence of \(\{r\}, \{2r\}, \{3r\}, \ldots\) which comes within \(\{d\}\) of every real number in the interval \([0, 1)\). Since \(0 < \{d\} < \frac{1}{n}\), we have shown that the sequence contains a term within \(\frac{1}{n}\) of every real number in \([0, 1)\). Since this is true for any positive integer \(n\), the conclusion follows.
4. **Solution 1** (Matthew Cheah, year 11, Penleigh and Essendon Grammar School, VIC)

Let us say that a binary sequence of length \( n \) is *acceptable* if it does not contain two consecutive 0s. Let \( A_n \) be the number of acceptable binary sequences of length \( n \).

**Lemma** The sequence \( A_1, A_2, \ldots \) satisfies \( A_1 = 2, \ A_2 = 3 \), and

\[
A_{n+1} = A_n + A_{n-1} \quad \text{for } n = 2, 3, 4, \ldots
\]

**Proof** It is easy to verify by inspection that \( A_1 = 2 \) and \( A_2 = 3 \).

Consider an acceptable sequence \( S \) of length \( n + 1 \), where \( n \geq 2 \).

If \( S \) starts with a 1, then this 1 can be followed by any acceptable sequence of length \( n \). Hence there are \( A_n \) acceptable sequences in this case.

If \( S \) starts with a 0, then the next term must be a 1, and the remaining terms can be any acceptable sequence of length \( n - 1 \). Hence there are \( A_{n-1} \) acceptable sequences in this case.

Putting it all together yields, \( A_{n+1} = A_n + A_{n-1} \). \( \square \)

Returning to the problem at hand, observe that any superb sequence satisfies the following two properties.

(i) The second and the second last digits are both 1s.

(ii) It contains no subsequence of the form 0, \( x \), 0.

Moreover, these two properties completely characterise superb sequences. Property (ii), in particular, motivates us to look at every second term \( a \) of superb sequence.

**Case 1** \( n = 2m \) for \( m \geq 2 \)

From (i), the superb sequence is \( a_1, 1, a_3, a_4, \ldots, a_{2m-3}, a_{2m-2}, 1, a_{2m} \).

From (ii), the subsequences \( a_1a_3, a_5, \ldots, a_{2m-3} \) and \( a_4, a_6, \ldots, a_{2m} \) are both acceptable. The first subsequence has \( m - 1 \) terms as does the second. Hence from the lemma we have \( B_{2m} = A_{m-1}^2 \).

**Case 2** \( n = 2m + 1 \) for \( m \geq 3 \)

From (i), the superb sequence is \( a_1, 1, a_3, a_4, \ldots, a_{2m-2}, a_{2m-1}, 1, a_{2m+1} \).

From (ii), the subsequences \( a_1, a_3, \ldots, a_{2m+1} \) and \( a_4, a_6, \ldots, a_{2m-2} \) are both acceptable. The first subsequence has \( m + 1 \) terms and the second has \( m - 2 \) terms. Hence from the lemma we have \( B_{2m+1} = A_{m+1}A_{m-2} \).

Using the lemma it is a simple matter to compute the values of \( A_1, A_2, \ldots \) modulo 20 and enter them into the following table. Then the values of \( B_{2m} \) and \( B_{2m+1} \) are calculated modulo 20 using the two formulas above.

\[
\begin{array}{c|cccccccccccccc}
 m & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \cdots \\
 A_m & 2 & 3 & 5 & 8 & 13 & 1 & 14 & 15 & 9 & 4 & 13 & 17 & \cdots \\
 B_{2m} & (1) & 4 & 9 & 5 & 4 & 9 & 1 & 16 & 5 & 1 & 16 & 9 & \cdots \\
 B_{2m+1} & (3) & (5) & 16 & 19 & 5 & 12 & 15 & 9 & 16 & 15 & 13 & 0 & \cdots \\
\end{array}
\]

Thus from the table, \( n = 25 \) is the smallest \( n \) such that \( 20 \mid B_n \). \( \square \)

---

5Here we are forgetting about whether or not a sequence is superb for the time being.
Call a binary sequence okay if it fulfils the superb requirement everywhere except possibly on its last digit. Thus all superb sequences are okay, but there are some okay sequences that are not superb. We classify all okay sequences with two or more terms into the following four types.

Type A: sequences that end in 0,0.
Type B: sequences that end in 0,1.
Type C: sequences that end in 1,0.
Type D: sequences that end in 1,1.

Observe that type A and type B sequences are okay but are not superb, while type C and type D sequences are superb. Let $a_n$, $b_n$, $c_n$, and $d_n$ be the number of okay sequences with $n$ terms of type A, B, C, and D, respectively. Thus $B_n = c_n + d_n$.

We seek a recursion for $a_n$, $b_n$, $c_n$, and $d_n$.

A type A sequence with $n + 1$ terms has 0 as its second last term. However, it cannot end in 0,0,0 because this would violate the superbness requirement for its second last digit. Thus $a_{n+1} = c_n$.

A type B sequence with $n + 1$ terms has 0 as its second last term. Thus it is built out of a type A or a type C okay sequence with $n$ terms by appending a 1 to any such sequence. Moreover, each such built sequence is okay. Thus $b_{n+1} = a_n + c_n$.

A type C sequence with $n + 1$ terms has 1 as its second last term. However, it cannot end in 0,1,0 because this would violate the superbness requirement for its second last digit. Thus $c_{n+1} = d_n$.

A type D sequence with $n + 1$ terms has 1 as its second last term. Thus it is built out of a type B or a type D okay sequence with $n$ terms by appending a 1 to any such sequence. Moreover, each such built sequence is okay. Thus $d_{n+1} = b_n + d_n$.

Hence we have a full set of recursive relations for $a_n$, $b_n$, $c_n$, and $d_n$. We also know that $B_n = c_n + d_n$. By inspection, we have $a_2 = c_2 = 0$ and $b_2 = d_2 = 1$. Thus we can use the following table to calculate $a_n$, $b_n$, $c_n$, $d_n$, and $B_n$ modulo 20 until we observe that $20 \mid B_n$.

\[
\begin{array}{c|cccccccccccc}
 n & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
 a_n & 0 & 0 & 1 & 2 & 2 & 3 & 6 & 10 & 15 & 4 & 0 & 5 \\
 b_n & 1 & 0 & 1 & 3 & 4 & 5 & 9 & 16 & 5 & 19 & 4 & 5 \\
 c_n & 0 & 1 & 2 & 2 & 3 & 6 & 10 & 15 & 4 & 0 & 5 & 4 \\
 d_n & 1 & 2 & 2 & 3 & 6 & 10 & 15 & 4 & 0 & 5 & 4 & 8 \\
 B_n & 1 & 3 & 4 & 5 & 9 & 16 & 5 & 19 & 4 & 5 & 9 & 12 \\
\end{array}
\]

\[
\begin{array}{c|cccccccccccc}
 n & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 \\
 a_n & 4 & 8 & 13 & 2 & 14 & 15 & 10 & 6 & 15 & 0 & 16 & 17 \\
 b_n & 9 & 12 & 1 & 15 & 16 & 9 & 5 & 16 & 1 & 15 & 16 & 13 \\
 c_n & 8 & 13 & 2 & 14 & 15 & 10 & 6 & 15 & 0 & 16 & 17 & 12 \\
 d_n & 13 & 2 & 14 & 15 & 10 & 6 & 15 & 0 & 16 & 17 & 12 & 8 \\
 B_n & 1 & 15 & 16 & 9 & 5 & 16 & 1 & 15 & 16 & 13 & 9 & 0 \\
\end{array}
\]

From the table, the smallest $n$ with $20 \mid B_n$ is $n = 25$. □
Solution 3  (Based on the solution of Michelle Chen, year 12, Methodist Ladies’ College, VIC)

Since no superb sequence can have 0 for its second term or for its second last term, we have following observation.

A superb sequence has 1 for its second term and for its second last term.  \hspace{.5cm} (1)

For each integer $n \geq 2$, let $F_n$ be the set of superb sequences with $n$ terms that end in 0. From observation (1), all such sequences end in 1,0. Let $f_n = |F_n|$.

For each integer $n \geq 2$, let $G_n$ be the set of superb sequences with $n$ terms that end in 1. From observation (1), all such sequences end in 1,1. Let $g_n = |G_n|$.

Consider any superb sequence $s \in F_n$ where $n \geq 3$. Since $s$ ends in 1,0, if we remove the final 0 from $s$, we end up with a member of $G_{n-1}$. Conversely, since any member of $G_{n-1}$ ends in 1,1, we may append a 0 to any member of $G_{n-1}$ to create a member of $F_n$. It follows that $f_n = g_{n-1}$. Since $B_n = f_n + g_n$ we have

$$B_n = g_{n-1} + g_n \text{ for } n \geq 3. \hspace{.5cm} (2)$$

Consider any superb sequence $s \in G_n$, where $n \geq 5$. The following two sequences are valid possibilities for $s$.

$$s = 1, 1, \ldots, 1 \hspace{.5cm} \text{and} \hspace{.5cm} s = 0, 1, 1, \ldots, 1. \hspace{.5cm} (3)$$

If $s$ is not one of the two above sequences, we may assume that the last 0 in $s$ appears at the $k$th position from the left, where $k > 1$, as shown.

$$s = a_1, a_2, \ldots, a_{k-1}, 0, 1, 1, \ldots, 1 \hspace{.5cm} \text{where } k \neq 2 \text{ and } k \neq n-1, \text{ because of observation (1), and } k \neq n \text{ because of the definition of } G_n.$$ 

Since $a_{k-2} \neq 0$, we have that $a_1, a_2, \ldots, a_{k-1}$ is a superb sequence with $k-1$ terms. Conversely, from observation (1), any superb sequence with $k-1$ terms can be built into a superb sequence with $n$ terms by appending $0, 1, 1, \ldots, 1$ to it. Since this is valid for any $k \notin \{1, 2, n-1, n\}$, we have

$$g_n = 2 + B_2 + B_3 + \cdots + B_{n-3} \text{ for } n \geq 5. \hspace{.5cm} (4)$$

Replacing $n$ with $n-1$ in (4) yields

$$g_{n-1} = 2 + B_2 + B_3 + \cdots + B_{n-4} \text{ for } n \geq 6. \hspace{.5cm} (5)$$

Comparing (4) with (5) we see that

$$g_n = g_{n-1} + B_{n-3} \text{ for } n \geq 6.$$
By checking small cases it can be verified that the last equation also holds for $n = 4, 5$. Thus we have

\begin{align*}
B_n &= g_{n-1} + g_n \quad \text{for } n \geq 3, \\
g_n &= g_{n-1} + B_{n-3} \quad \text{for } n \geq 4.
\end{align*}

It is not hard to eliminate all the $g$-terms from (2) and (6). For example, replacing $n$ with $n + 1$ in (2) and (6) yields

\begin{align*}
B_{n+1} &= g_n + g_{n+1} \quad \text{for } n \geq 2, \\
g_{n+1} &= g_n + B_{n-2} \quad \text{for } n \geq 3.
\end{align*}

Adding (6) and (8) yields

\[ g_n + g_{n+1} = g_{n-1} + g_n + B_{n-2} + B_{n-3} \quad \text{for } n \geq 4. \]

With the help of (2) and (7), the above equation becomes

\[ B_{n+1} = B_n + B_{n-2} + B_{n-3} \quad \text{for } n \geq 4. \]

We calculate the values of $B_1, B_2, B_3,$ and $B_4$ manually by inspection. After this we use the recursion in equation (9) to calculate the values of $B_n$ modulo 20. This is done in the following table.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_n$</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
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<td>16</td>
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<td>18</td>
<td>19</td>
<td>20</td>
<td>21</td>
<td>22</td>
<td>23</td>
<td>24</td>
<td>25</td>
</tr>
</tbody>
</table>

Thus the first 0 occurs at $n = 25$, as required. $\square$

---

$^6$Observation (1) speeds this up considerably. In particular, we find $B_1 = 0, B_2 = 1, g_1 = 0, g_2 = 1, g_3 = 2, g_4 = 2,$ and $g_5 = 3.$
Solution 4  (Jerry Mao, year 10, Caulfield Grammar School, VIC)

Let $F_n$, $G_n$, $f_n$, and $g_n$ be as in solution 3. As in solution 3, we have $B_n = g_{n-1} + g_n$ for $n \geq 3$, and that all members of $G_n$ end in 1,1.

Consider a sequence $s \in G_n$ where $n \geq 6$. Either $s$ ends with 1,1,1 or $s$ ends with 0,1,1. But in the second scenario this splits into two further possibilities, namely, that $s$ ends with 1,0,1,1 or $s$ ends with 0,0,1,1.

Case 1  The sequence $s$ ends with 1,1,1.
If we remove the final 1, we end up with a member of $G_{n-1}$. Conversely, we can append 1 to any member of $G_{n-1}$ to create a member of $G_n$. So there are $g_{n-1}$ possibilities for this case.

Case 2  The sequence $s$ ends with 1,0,1,1.
We cannot have the sequence 0,1,0 occurring anywhere within a superb sequence. Hence $s$ ends with 1,1,0,1,1. If we remove the final three digits we end up with a member of $G_{n-3}$. Conversely, we can append 0,1,1 to any member of $G_{n-3}$ to create a member of $G_n$. So there are $g_{n-3}$ possibilities in this case.

Case 3  The sequence $s$ ends with 0,0,1,1.
We cannot have the sequence 0,0,0 occurring anywhere within a superb sequence. Hence $s$ ends with 1,0,0,1,1. If we remove the final four digits we end up with a member of $G_{n-4}$. Conversely, we can append 0,0,1,1 to any member of $G_{n-4}$ to create a member of $G_n$. So there are $g_{n-4}$ possibilities in this case.

It follows from cases 1, 2, and 3, that
g_n = g_{n-1} + g_{n-3} + g_{n-4}  \quad \text{for all } n \geq 6.

By inspection we may calculate that $g_2 = 1$, $g_3 = 2$, $g_4 = 2$, and $g_5 = 3$.

Since $B_n = g_{n-1} + g_n$, we may use the recurrence for $g_n$ to compute successive terms modulo 20, and stop the first time we notice that consecutive terms of the sequence add to a multiple of 20. The sequence $g_2, g_3, g_4, \ldots$ modulo 20 proceeds as follows.

\begin{align*}
1, 2, 2, 3, 6, 10, 15, 4, 0, 5, 4, 8, 13, 2, 14, 15, 10, 6, 15, 0, 16, 17, 12, 8, \ldots
\end{align*}

We stop because 12 + 8 is a multiple of 20. This corresponds to $n = 25$. \hfill \square
The following remarkable solution finds a recursion for $B_n$ directly.

**Solution 5** (Ian Wanless, AMOC Senior Problems Committee)

We note that each run of $1$s in a superb binary sequence has to have length two or more. For $n \geq 5$ we partition the sequences counted by $B_n$ into three cases.

**Case 1** The first run of $1$s has length at least three.

In this case, removing one of the $1$s in the first run leaves a superb sequence of length $n - 1$. Conversely, every superb sequence of length $n - 1$ can be extended to a superb sequence of length $n$ by inserting a $1$ into the first run of $1$s. The process is illustrated via the following example.

\[
 0, 1, 1, 1, 1, 0, 0, 1, 1, 0 \quad \leftrightarrow \quad 0, 1, 1, 1, 0, 0, 1, 1, 0
\]

So there are $B_{n-1}$ sequences in this case.

**Case 2** The first run of $1$s has length two and the first term in the sequence is $1$.

In this case, the sequence begins with $1,1,0$, and what follows is any one of the $B_{n-3}$ superb sequences of length $n - 3$.

**Case 3** The first run of $1$s has length two and the first term in the sequence is $0$.

In this case, the sequence begins $0,1,1,0$, and what follows is any one of the $B_{n-4}$ superb sequences of length $n - 4$.

From cases 1, 2, and 3, we conclude that $B_n$ satisfies the recurrence

\[
B_n = B_{n-1} + B_{n-3} + B_{n-4} \quad \text{for } n \geq 5.
\]

By checking small cases, we verify that $B_1 = 0$, $B_2 = 1$, $B_3 = 3$, and $B_4 = 4$.

From the above recurrence, we calculate the sequence modulo 20. It begins as

\[
0, 1, 3, 4, 5, 9, 16, 5, 19, 4, 5, 9, 12, 1, 15, 16, 9, 5, 16, 1, 15, 16, 13, 9, 0, \ldots
\]

from which we deduce that the answer is $n = 25$. \qed
**Comment 1** It is possible to determine all \( n \) for which \( B_n \) is a multiple of 20.

We look for repeating cycles for \( B_n \) modulo 4 and modulo 5. In light of the recurrence from solution 5, it suffices to find instances of where four consecutive terms in the sequence are equal to another four consecutive terms of the sequence.

Here is a table of values that shows the sequence modulo 4.

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| \( B_n \) | 0 | 1 | 3 | 0 | 1 | 1 | 0 | 1 | 3 | 0 |

So, modulo 4, the sequence forms a repeating cycle of length 6. It follows that \( 4 \mid B_n \) whenever \( n \equiv 1, 4 \pmod{6} \). That is, \( n \equiv 1 \pmod{3} \).

Here is a table of values that shows the sequence modulo 5.

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| \( B_n \) | 0 | 1 | 3 | 4 | 0 | 4 | 1 | 0 | 4 | 0 | 4 | 0 | 4 | 2 | 1 | 0 | 1 | 4 | 0 | 1 | 1 | 0 | 1 | 3 | 4 |

So, modulo 5, the sequence forms a repeating cycle of length 20. It follows that \( 5 \mid B_n \) whenever \( n \equiv 1, 5, 8, 11, 15, 18 \pmod{20} \). That is, \( n \equiv 1, 5, 8 \pmod{10} \).

Thus \( 20 \mid n \) if and only if \( n \equiv 1 \pmod{3} \) and \( n \equiv 1, 5, 8 \pmod{10} \). Solving these congruences shows that \( 20 \mid n \) if and only if \( n \equiv 1, 25, \) or \( 28 \pmod{30} \).

Thus the smallest integer \( n \geq 2 \) with \( 20 \mid B_n \) is \( n = 25 \). □

**Comment 2** We could also find all \( n \) for which \( B_n \) is a multiple of 20 by further analysing solution 1 as follows.

Recall we had the following pair of equations.

\[
B_{2m} = A_{m-1}^2 \quad (1)
\]

\[
B_{2m+1} = A_{m+1}A_{m-2} \quad (2)
\]

Here \( A_1, A_2, \ldots \) was defined by \( A_1 = 2, A_2 = 3, \) and \( A_{n+1} = A_n + A_{n-1} \) for \( n \geq 2 \).

It is easy to compute that the sequence \( A_1, A_2, \ldots \) cycles as

\[
0, 1, 1, 0, 1, 1, \ldots \pmod{2} \quad (3)
\]

and

\[
2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, 1, 2, 3, 0, 3, \ldots \pmod{5} \quad (4)
\]

From (3), \( A_k \) is even if and only if \( k \equiv 1 \pmod{3} \). If \( n = 2m \) is even, then we need \( 2 \mid A_{m-1} \). Thus \( m \equiv 2 \pmod{3} \), and so \( n \equiv 2m \equiv 1 \pmod{3} \). If \( n = 2m + 1 \) is odd, then since \( A_{m+1} \equiv A_{m-2} \pmod{2} \), we need \( 2 \mid A_{m+1} \). Thus \( m \equiv 0 \pmod{3} \), and so again \( n \equiv 2m + 1 \equiv 1 \pmod{3} \). Hence in both cases \( n \equiv 1 \pmod{3} \).

From (4), \( 5 \mid A_k \) if and only if \( k \equiv 3 \pmod{5} \). If \( n = 2m \) is even, then we need \( 5 \mid A_{m-1} \). Thus \( m \equiv 4 \pmod{5} \), and so \( n \equiv 2m \equiv 3 \pmod{5} \). So \( n \equiv 8 \pmod{10} \). If \( n = 2m + 1 \) is odd, then we need \( 5 \mid A_{m+1} \) or \( 5 \mid A_{m-2} \). So \( m \equiv 2 \) or \( 0 \pmod{5} \), and hence \( n \equiv 2m + 1 \equiv 0 \) or \( 1 \pmod{5} \). So \( n \equiv 5 \) or \( 1 \pmod{10} \).

Thus \( 20 \mid B_n \) if and only if \( n \equiv 1 \pmod{3} \) and \( n \equiv 1, 5, \) or \( 8 \pmod{10} \). This implies that \( 20 \mid B_n \) if and only if \( n \equiv 1, 25, \) or \( 28 \pmod{30} \).

\footnote{This is the Fibonacci sequence, except that the indices have been shifted down by 2.}
5. Solution 1 (Jongmin Lim, year 12, Killara High School, NSW)

For reference the equations are

\[ \begin{align*}
xy + 1 &= 2z \quad (1) \\
yz + 1 &= 2x \quad (2) \\
zx + 1 &= 2y. \quad (3)
\end{align*} \]

Subtracting equation (2) from equation (1) yields

\[ y(x - z) = -2(x - z) \iff (x - z)(y + 2) = 0. \]

Similarly, subtracting equation (3) from equation (2) yields

\[ z(y - x) = -2(y - x) \iff (y - x)(z + 2) = 0. \]

This yields four cases which we now analyse.

**Case 1** \( x = z \) and \( y = x \)

Thus \( x = y = z \). Putting this into equation (1) yields \( x^2 + 1 = 2z \), which is equivalent to \( (x - 1)^2 = 0 \). Hence \( x = y = z = 1 \).

**Case 2** \( x = z \) and \( z = -2 \)

Thus \( x = z = -2 \). Putting this into (3) immediately yields \( y = \frac{5}{2} \).

**Case 3** \( y = -2 \) and \( y = x \)

Thus \( x = y = -2 \). Putting this into (1) immediately yields \( z = \frac{5}{2} \).

**Case 4** \( y = -2 \) and \( z = -2 \)

Putting this into equation (2) immediately yields \( x = \frac{5}{2} \).

So the only possible solutions are \((x, y, z) = (1, 1, 1), (\frac{5}{2}, -2, -2), (-2, \frac{5}{2}, -2), \) and \((-2, -2, \frac{5}{2}) \). It is readily verified that these satisfy the given equations. \( \square \)
Solution 2 (Jodie Lee, 11, Seymour College, SA)

For reference the equations are

\[ xy + 1 = 2z \] \hspace{1cm} (1)
\[ yz + 1 = 2x \] \hspace{1cm} (2)
\[ zx + 1 = 2y. \] \hspace{1cm} (3)

Subtracting equation (2) from equation (1) yields

\[ y(x - z) = -2(x - z) \Leftrightarrow (x - z)(y + 2) = 0. \]

This yields the following two cases.

Case 1 \( y = -2 \)

Putting this into equation (3) yields

\[ zx + 1 = -4 \]
\[ \Rightarrow z = \frac{-5}{x}. \]

Substituting this into equation (1) yields

\[ -2x + 1 = -\frac{10}{x} \]
\[ \Rightarrow 2x^2 - x - 10 = 0 \]
\[ \Leftrightarrow (2x - 5)(x + 2) = 0. \]

Thus either \( x = -\frac{5}{2} \) which implies \( z = -2 \), or \( x = -2 \) which implies \( z = -\frac{5}{2} \).

Case 2 \( x = z \)

Substituting \( z = x \) into equation (3) yields

\[ xy + 1 = 2x \]
\[ \Rightarrow y = \frac{x^2 + 1}{2}. \]

Putting this into equation (1) yields

\[ \frac{x(x^2 + 1)}{2} + 1 = 2x \]
\[ \Leftrightarrow x^3 - 3x + 2 = 0 \]
\[ \Leftrightarrow (x - 1)(x^2 + x - 2) = 0 \]
\[ \Leftrightarrow (x - 1)(x - 1)(x + 2) = 0. \]

Thus either \( x = 1 \), which implies \( y = z = 1 \), or \( x = -2 \) which implies \( z = -2 \) and \( y = -\frac{5}{2} \).

From cases 1 and 2, the only possible solutions are \( (x, y, z) = (1, 1, 1), \left(\frac{5}{2}, -2, -2\right), \left(-2, \frac{5}{2}, -2\right), \) and \( \left(-2, -2, \frac{5}{2}\right) \). It is straightforward to check that these satisfy the given equations. \( \square \)
Solution 3 (Wilson Zhao, year 12, Killara High School, NSW)

For reference the equations are

\[
\begin{align*}
xy + 1 &= 2z \quad (1) \\
yz + 1 &= 2x \quad (2) \\
zx + 1 &= 2y. \quad (3)
\end{align*}
\]

Observe that the three equations are symmetric in \(x, y,\) and \(z.\) So without loss of generality we may assume that \(x \geq y \geq z.\)

**Case 1** \(x \geq y \geq z \geq 0.\)

It follows that \(xy \geq xz \geq yz.\) Thus \(xy + 1 \geq xz + 1 \geq yz + 1,\) and so from equations (1), (2), and (3), it follows that \(z \geq y \geq x.\) Comparing this with \(x \geq y \geq z\) yields \(x = y = z.\) Then equation (1) becomes \(x^2 + 1 = 2x,\) that is \((x - 1)^2 = 0.\) Therefore \(x = 1,\) and so \((x, y, z) = (1, 1, 1)\) in this case.

**Case 2** \(x \geq y \geq 0 > z\)

It follows that \(0 > 2z = xy + 1 > 0,\) a contradiction. So this case does not occur.

**Case 3** \(x \geq 0 > y \geq z\)

It follows that \(xy \geq xz.\) Thus \(xy + 1 \geq xz + 1,\) and so from equations (1) and (3), we have \(z \geq y.\) Comparing this with \(y \geq z\) yields \(y = z.\) Substituting this into (2) and rearranging yields

\[
x = \frac{z^2 + 1}{2}.
\]

Substituting this into (1) and also remembering that \(y = z\) yields

\[
\begin{align*}
\frac{(z^2 + 1)z}{2} + 1 &= 2z \\
\iff z^3 - 3z + 2 &= 0 \\
\iff (z - 1)(z^2 + z - 2) &= 0 \\
\iff (z - 1)(z - 1)(z + 2) &= 0.
\end{align*}
\]

This has the two solutions \(z = 1\) and \(z = -2.\) Since \(z < 0\) we have \(z = -2.\) It follows that \(y = -2\) and \(x = \frac{5}{2}.\) Hence \((x, y, z) = \left(\frac{5}{2}, -2, -2\right)\) in this case.

**Case 4** \(0 > x \geq y \geq z\)

It follows that \(0 > 2x = yz + 1 > yz > 0,\) a contradiction. So this case does not occur.

From the four cases thus analysed we have the possibilities \((x, y, z) = (1, 1, 1)\) or any of the three three permutations of \(\left(\frac{5}{2}, -2, -2\right).\) It is readily verified that these do indeed satisfy the original equations. \(\square\)
Solution 4 (Yong See Foo, year 12, Nossal High School, VIC)

For reference the equations are

\[ xy + 1 = 2z \]  \hspace{1cm} (1)
\[ yz + 1 = 2x \]  \hspace{1cm} (2)
\[ zx + 1 = 2y. \]  \hspace{1cm} (3)

Let \( p = xyz \). Multiplying equation (1) by \( z \), yields

\[ xyz + z = 2z^2 \]
\[ \Rightarrow 2z^2 + z + p = 0 \]

Similarly, we find

\[ 2x^2 + x + p = 0 \quad \text{and} \quad 2y^2 + y + p = 0. \]

Thus \( x, y, \) and \( z \) are all roots of the same quadratic equation. From the quadratic formula we have that each of \( x, y, \) and \( z \) is equal to

\[ \frac{1 - \sqrt{1 + 8p}}{4} \quad \text{or} \quad \frac{1 + \sqrt{1 + 8p}}{4}. \]  \hspace{1cm} (4)

**Case 1** Not all of \( x, y, \) and \( z \) are equal.

Since the given equations are symmetric in \( x, y, \) and \( z \), we can assume without loss of generality that \( x \neq y, \) and so \( p \neq 0. \) It follows that the product \( xy \) is equal to the product of the two expressions found in (4). Hence

\[ xy = \left( \frac{1 - \sqrt{1 + 8p}}{4} \right) \left( \frac{1 + \sqrt{1 + 8p}}{4} \right) = -\frac{p}{2}. \]

However, since \( p = xyz \neq 0, \) it follows that \( z = -2. \) But \( z \) is also equal to one of the expressions found in (4). The second expression in (4) is definitely positive, while \( z = -2 < 0. \) Thus we have

\[ -2 = \frac{1 - \sqrt{1 + 8p}}{4}. \]

Solving for \( p \) yields \( p = 10. \) Substituting this into (4) yields \( \{x, y\} = \left\{ \frac{5}{2}, -2 \right\}. \) Thus \( (x, y, z) \) can be any permutation of \( \left( \frac{5}{2}, -2, -2 \right). \)

**Case 2** \( x = y = z \)

Equation (1) becomes \( x^2 + 1 = 2x. \) This is the same as \( (x - 1)^2 = 0. \) Therefore \( x = 1, \) and so \( (x, y, z) = (1, 1, 1) \) in this case.

It is readily verified that \( (1, 1, 1) \) and the three permutations of \( \left( \frac{5}{2}, -2, -2 \right) \) satisfy the original equations. □
6. **Solution 1**  (Based on the solution of Shivasankaran Jayabalan, year 10, Rossmoyne Senior High School, WA)

For reference we are given

\[ a^3 + b^3 = 2^c. \]  \hspace{1cm} (1)

Assume for the sake of contradiction that there are positive integers \( a \neq b \) such that \( a^3 + b^3 \) is a power of two. Of all such solutions choose one with \( |a - b| \) minimal.

**Case 1** Both \( a \) and \( b \) are even.

Let \( x = \frac{a}{2} \) and \( y = \frac{b}{2} \). Then \( x^3 + y^3 \) is also a power of two. But \( 0 < |x - y| < |a - b| \), which contradicts the minimality of \( |a - b| \).

**Case 2** One of \( a \) and \( b \) is even while the other is odd.

It follows that \( a^3 + b^3 \) is odd. But the only odd power of two is 1. Hence \( a^3 + b^3 = 1 \). However this is impossible for positive integers \( a \) and \( b \).

**Case 3** Both of \( a \) and \( b \) are odd.

Equation (1) may be rewritten as

\[ (a + b)(a^2 - ab + b^2) = 2^c. \]

Since both \( a \) and \( b \) are odd, then so is \( a^2 - ab + b^2 \). However the only odd factor of a power of two is 1. Hence

\[ a^2 - ab + b^2 = 1 \]

\[ \Leftrightarrow \quad (a - b)^2 + ab = 1. \]

However, this is also impossible because \( (a - b)^2 \geq 1 \) and \( ab \geq 1 \). \( \square \)
Solution 2  (Yasiru Jayasooriya, year 8, James Ruse Agricultural High School, NSW)

For reference we are given
\[ a^3 + b^3 = 2^c. \]  \(1\)

Let \(2^n\) be the highest power of two dividing both \(a\) and \(b\). Thus we may write
\[ a = 2^n A \quad \text{and} \quad b = 2^n B \]  \(2\)

for some positive integers \(A\) and \(B\) such that at least once of \(A\) and \(B\) is odd.

From this we see that \(2^{3n} | 2^c\), so that \(c \geq 3n\). Let \(d = c - 3n\). Substituting (2) into (1) and tidying up yields
\[ A^3 + B^3 = 2^d. \]  \(3\)

Since \(A\) and \(B\) are positive integers, we have \(A^3 + B^3 \geq 2\), and so \(d \geq 1\). Thus \(A^3\) and \(B^3\) have the same parity. Hence \(A\) and \(B\) have the same parity. Since at most one of \(A\) and \(B\) is even, it follows that both \(A\) and \(B\) are odd.

Factoring the LHS of (3) yields
\[ (A + B)(A^2 - AB + B^2) = 2^d. \]

Since \(A\) and \(B\) are both odd it follows that \(A^2 - AB + B^2\) is odd. But the only odd factor of \(2^d\) is 1. Thus we have
\[ A^2 - AB + B^2 = 1 \quad \text{and} \quad A + B = 2^d. \]

From (3), this implies
\[ A^3 + B^3 = A + B. \]  \(4\)

However \(x^3 > x\) for any integer \(x > 1\). So if (4) is true, we must have \(A = B = 1\). It follows that \(a = b = 2^n\). □
Solution 3 (Barnard Patel, year 12\textsuperscript{8}, Wellington College, NZ)

For reference we are given

\[a^3 + b^3 = 2^c.\]  \hspace{1cm} (1)

This may be rewritten as

\[(a + b)(a^2 - ab + b^2) = 2^c.\]

It follows that both \(a + b\) and \(a^2 - ab + b^2\) are powers of two. Consequently, we have the following equations for some non-negative integers \(d\) and \(e\).

\[a + b = 2^d\]  \hspace{1cm} (2)

\[a^2 - ab + b^2 = 2^e\]  \hspace{1cm} (3)

If we square equation (2) and then subtract equation (3), we find

\[3ab = 2^{2d} - 2^e.\]  \hspace{1cm} (4)

The LHS of (4) is positive, hence so also is the RHS. Thus \(2d > e\), and so

\[3ab = 2^e(2^{2d-e} - 1).\]

Since \(\text{gcd}(3, 2^e) = 1\), we have \(2^e \mid ab\). From (3), this is \(a^2 - ab + b^2 \mid ab\). Thus

\[a^2 - ab + b^2 \leq ab\]

\[\Leftrightarrow \quad (a - b)^2 \leq 0.\]

Since squares are non-negative, it follows that \(a = b\), as desired. \hfill \Box

\textsuperscript{8}Equivalent to year 11 in Australia.
Solution 4 (Keiran Lewellen, year 11\textsuperscript{9}, Te Kura (The Correspondence School), NZ)

For reference we are given
\[ a^3 + b^3 = 2^c. \]  \hspace{1cm} (1)

Without loss of generality \( a \geq b \).

We are given that \( a \) and \( b \) are positive integers. Hence \( a^3 + b^3 = 2^c \) is a power of two that is greater than 1. Thus \( c \geq 1 \) and \( a^3 \) and \( b^3 \) have the same parity. It follows that \( a \) and \( b \) have the same parity. Hence we may let \( a + b = 2x \) and \( a - b = 2y \) for integers \( 0 \leq y < x \). It follows that

\[ a = x + y \quad \text{and} \quad b = x - y. \]

Substituting this in to (1) yields
\[ 2x^3 + 6xy^2 = 2^c \]
\[ \Rightarrow x(x^2 + 3y^2) = 2^{c-1}. \]

Consequently, we have the following for some non-negative integers \( r \) and \( s \).

\[ x = 2^r \] \hspace{1cm} (2)
\[ x^2 + 3y^2 = 2^s \] \hspace{1cm} (3)

Note that \( 2r \leq s \) because \( x^2 \leq 2^s \). Substituting (2) into (3) and rearranging yields
\[ 3y^2 = 2^{2r}(2^s - 2^r - 1). \]

Since \( \gcd(3, 2^{2r}) = 1 \), we have \( 2^{2r} \mid y^2 \), and so \( 2^r \mid y \). Thus \( x \mid y \). But \( 0 \leq y < x \). Hence \( y = 0 \), and so \( a = b \). \hfill \Box
Solution 5  (Anthony Tew, year 10, Pembroke School, SA)

For reference we are given

\[ a^3 + b^3 = 2^c. \]  \hspace{1cm} (1)

This may be rewritten as

\[(a + b)(a^2 - ab + b^2) = 2^c.\]

Suppose for the sake of contradiction that \(a \neq b\). Without loss of generality \(a < b\). Let \(b = a + k\) for some positive integer \(k\). Substituting this into the above equation and tidying up yields

\[(2a + k)(a^2 + ak + k^2) = 2^c.\]

Consequently, we have the following for some non-negative integers \(d\) and \(e\).

\[
\begin{align*}
2a + k &= 2^d \\
 a^2 + ak + k^2 &= 2^e
\end{align*}
\]

Squaring (2) yields

\[4a^2 + 4ak + k^2 = 2^{2d}.\]  \hspace{1cm} (4)

Now observe that

\[4a^2 + 4ak + k^2 < 4(a^2 + ak + k^2) < 4(4a^2 + 4ak + k^2).\]

This is the same as

\[2^{2d} < 2^{e+2} < 2^{2d+2}.\]

Hence \(e + 2 = 2d + 1\). That is, \(e = 2d - 1\). Then comparing (3) and (4) yields

\[
\begin{align*}
4a^2 + 4ak + k^2 &= 2(a^2 + ak + k^2) \\
 \Leftrightarrow \quad k^2 - 2ak - 2k^2 &= 0 \\
 \Leftrightarrow \quad (k-a)^2 &= 3k^2.
\end{align*}
\]

But \(3k^2\) can never be a perfect square for any positive integer \(k\). This contradiction concludes the proof. \qed
7. Solution 1  (Matthew Cheah, year 11, Penleigh and Essendon Grammar School VIC)

Suppose, for the sake of contradiction, that we can assign every point in the plane one of four colours in such a way that no two of points at distance 1 or $\sqrt{3}$ from each other are assigned the same colour.

Consider any rhombus formed by joining two unit equilateral triangles together as shown. Call such a figure a 60°-rhombus.

![60°-rhombus diagram]

**Lemma 1** The four vertices of a 60°-rhombus are assigned different colours.

**Proof** It is easily computed that the long diagonal of the rhombus has length $\sqrt{3}$. All other distances between pairs of vertices of the rhombus are of length 1. Thus no two vertices can be assigned the same colour. □

**Lemma 2** Any two points at distance 2 from each other are assigned the same colour.

**Proof** Consider any two points $X$ and $Y$ such that $XY = 2$. We can build a net of unit equilateral triangles as follows.

![Net diagram]

From lemma 1, points $A$, $C$, $B$, and $X$ are assigned four different colours. Again from lemma 1, points $A$, $B$, $C$, and $Y$ are assigned four different colours. But since only four colours are available, $X$ and $Y$ must be assigned the same colour. □

To finish the proof of the given problem, consider the following triangle.

![Triangle diagram]

From lemma 2, points $P$ and $Q$ are assigned the same colour. Again from lemma 2, points $P$ and $R$ are assigned the same colour. Hence $Q$ and $R$ are assigned the same colour. This is a contradiction because $QR = 1$. □
Solution 2  (Angelo Di Pasquale, Director of Training, AMOC)

Suppose that the four colours are blue, red, yellow, and green. We argue by contradiction and suppose that there do not exist two points at distance 1 or $\sqrt{3}$ from each other that have the same colour.

Consider an isosceles triangle $ABC$ with $BC = 1$ and $AB = AC = 2$. Since $B$ and $C$ must be different colours, one of them is coloured differently to $A$. Without loss of generality, we may suppose that $A$ is blue and $B$ is red.

Let us orient the plane so that $AB$ is a horizontal segment.

Let $O$ be the midpoint of $AB$. Then as $AO = BO = 1$, it follows that $O$ is not blue or red. Without loss of generality we may suppose that $O$ is green.

Let $X$ be the point above the line $AB$ such that $\triangle AOX$ is equilateral. It is easy to compute that $XB = \sqrt{3}$ and $XA = XO = 1$. Hence, $X$ is not red, blue or green. Thus $X$ must be yellow.

Finally, let $Y$ be the point above the line $AB$ such that $\triangle BOY$ is equilateral. Then it is easy to compute that $YX = YO = YB = 1$ and $YA = \sqrt{3}$. Hence $Y$ cannot be any of the four colours, giving the desired contradiction. □

Comment  (Kevin Xian, year 12, James Ruse Agricultural High School, NSW)

Forget about trying to assign colours to every point in the plane! An analysis of either of the presented solutions shows that if one tries to colour the nine points shown in the diagram below using four colours, then there will always be two points of the same colour at distance 1 or $\sqrt{3}$ from each other.
8. This was the most difficult problem of the 2016 AMO. Just six contestants managed to solve it completely.

8 (a). Solution 1 (Ilia Kucherov, year 12, Westall Secondary College, VIC)

Let $\alpha$, $\beta$, and $\gamma$ be the angles between the lines as shown in the diagram on the left below.

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (3,2) -- (5,0) -- (2,-2) -- cycle;
\draw (0,0) -- (5,0);
\node at (2,1) {$P$};
\end{tikzpicture}
\end{center}

Note that $\alpha + \beta + \gamma = 180^\circ$. This permits us to locate segments $AB$, $CD$, and $EF$, all of unit length, as shown in the diagram on the right below. We shall prove that $ABCDEF$ is a cyclic hexagon.

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (3,2) -- (5,0) -- (2,-2) -- cycle;
\draw (0,0) -- (5,0);
\node at (2,1) {$P$};
\end{tikzpicture}
\end{center}

Since $\angle BAD = \angle ADC$ and $AB = CD$, it follows that $ABCD$ is an isosceles trapezium with $AD \parallel BC$. Thus $ABCD$ is a cyclic quadrilateral.\(^{10}\)

Similarly, $CDEF$ is cyclic with $CF \parallel DE$. Thus

$$\angle BED = \angle BPC = \alpha = \angle BAD.$$  

Hence $ABDE$ is cyclic. Since quadrilaterals $ABCD$ and $ABDE$ are both cyclic, it follows that $ABCDE$ is a cyclic pentagon. But then since $ABCDE$ and $CDEF$ are both cyclic, it follows that $ABCDEF$ is a cyclic, as desired. \(\square\)

\(^{10}\)Every isosceles trapezium is cyclic.
Solution 2  (Matthew Cheah, year 11, Penleigh and Essendon Grammar School, VIC)

Orient the diagram so that one of the lines is horizontal. Choose any points $B$ and $C$, one on each of the other two lines, such that $BC$ is parallel to the horizontal line. Let $x$ be the angle as shown in the diagram.

Let $D$ be the point on the horizontal line such that $\angle DCP = x$. Let $E$ be the point on the line $BP$ such that $DE \parallel CP$. Let $F$ be the point on the line $CP$ such that $\angle PFE = x$. Let $A$ be the point on the line $DP$ such that $AF \parallel BE$. We claim that $ABCDEF$ is cyclic.

From $DE \parallel CP$, we have $\angle EDC = 180^\circ - x$. And from $AD \parallel BC$, we have $\angle CBE = x$. So $BCDE$ is cyclic because $\angle CBE + \angle EDC = 180^\circ$.

Quadrilateral $CDEF$ is cyclic because $\angle EDC + \angle CFE = 180^\circ$.

From $DE \parallel CF$, we have $\angle FED = 180^\circ - x$. And from $AF \parallel BE$, we have $\angle DAF = x$. So $DEFA$ is cyclic because $\angle FED + \angle DAF = 180^\circ$.

Since $BCDE$, $CDEF$, and $DEFA$ are all cyclic, it follows that $ABCDEF$ is cyclic, as claimed. □

11Note that $\angle APC > x$. So the point $D$ really does exist and lies on the same side of the line $BP$ as $C$ as shown in the diagram.

12Note that $\angle CPE = \angle CPD + x > x$. So the point $F$ really does exist and has the property that $P$ lies between $C$ and $F$ as shown in the diagram.
Solution 3  (Angelo Di Pasquale, Director of Training, AMOC)

As in solution 2, orient the diagram so that one of the lines is horizontal, and choose any points $B$ and $C$, one on each of the other two lines, such that $BC$ is parallel to the horizontal line.

Let $E$ be a variable point on line $BP$ so that $P$ lies between $B$ and $E$. Consider the family of circles passing through points $B$, $C$, and $E$. Let $A$, $D$, and $F$ be the intersection points of circle $BCE$ with the three given lines so that $A$, $B$, $C$, $D$, $E$, and $F$ are in that order around the circle. Then $BC \parallel AD$. Thus $ABCD$ is an isosceles trapezium with $AB = CD$.

Consider the ratio $r = \frac{EF}{AB}$ as $E$ varies on the line $BP$ so that $P$ lies between $B$ and $E$. As $E$ approaches $P$, the length $AB$ approaches $\min\{BP, CP\}$ while the length $EF$ approaches 0. Hence, $r$ approaches 0.

As $E$ diverges away from $P$, we have $\angle ADB$ approaches $0^\circ$, while $\angle ECF$ approaches $\angle BPC$. Thus, eventually $\angle ECF > \angle ADB$, and so $r > 1$.

Since $r$ varies continuously with $E$, we may apply the intermediate value theorem to deduce that there is a position for $E$ such that $r = 1$. The circle $BCE$ now has the required property because $EF = AB = CD$. \qed
8 (b). **Solution 1** (Seyoon Ragavan, year 12, Knox Grammar School, NSW)

Let $AB = CD = EF = r$ and label the angles between the lines as shown.

Note that $ABCD$ is an isosceles trapezium because it is cyclic and has $AB = CD$. Thus $AD \parallel BC$. Hence $\angle CBP = \gamma$ and $\angle PCB = \beta$. Similarly we have $BE \parallel AF$ and $CF \parallel DE$, which imply $\angle A FP = \alpha$, $\angle PAF = \gamma$, $\angle EDP = \beta$, and $\angle PED = \alpha$.

Since $ABCDEF$ is cyclic we have $\angle ADC = \angle AFC = \alpha$, $\angle DCF = \angle DAF = \gamma$, $\angle EBA = \angle EDA = \beta$, $\angle BAD = \angle BED = \alpha$, $\angle CFE = \angle CBE = \gamma$, and $\angle FEB = \angle FCB = \beta$.

Let $\triangle XYZ$ be a reference triangle that satisfies $\angle YXZ = \alpha$, $\angle ZYX = \beta$, and $\angle XZY = \gamma$. Let $YZ = a$, $ZX = b$, and $XY = c$. We have

$$\triangle ABP \sim \triangle PCB \sim \triangle DPC \sim \triangle EDP \sim \triangle PEF \sim \triangle FPA \sim \triangle XYZ.$$ 

Let $S = |XYZ|$. From $\triangle ABP \sim \triangle XYZ$, we deduce

$$\frac{|ABP|}{|XYZ|} = \left(\frac{AB}{XY}\right)^2 \Rightarrow |APB| = \frac{r^2 S}{c^2}$$

and

$$\frac{AP}{XZ} = \frac{AB}{XY} \Rightarrow AP = \frac{rb}{c}.$$ 

\(^{13}\)The notation $|XYZ|$ stands for the area of $\triangle XYZ$. 

Analogously, from \( \triangle DPC \sim \triangle XYZ \), we have

\[
\frac{|DPC|}{|XYZ|} = \frac{DC^2}{XZ^2} \quad \Rightarrow \quad |CPD| = \frac{r^2 S}{b^2}
\]

and

\[
\frac{PC}{YZ} = \frac{DC}{XZ} \quad \Rightarrow \quad CP = \frac{ra}{b}.
\]

And from \( \triangle PEF \sim \triangle XYZ \), we have

\[
\frac{|PEF|}{|XYZ|} = \left(\frac{EF}{YZ}\right)^2 \quad \Rightarrow \quad |EPF| = \frac{r^2 S}{a^2}
\]

and

\[
\frac{PE}{XY} = \frac{EF}{YZ} \quad \Rightarrow \quad EP = \frac{rc}{a}.
\]

We also have \( \triangle FPA \sim \triangle XYZ \), and so

\[
\frac{|FPA|}{|XYZ|} = \left(\frac{PA}{YZ}\right)^2 = \left(\frac{rb}{ca}\right)^2 \quad \Rightarrow \quad |FPA| = \frac{r^2 Sb^2}{c^2 a^2}.
\]

Analogously, from \( \triangle PCB \sim \triangle XYZ \), we have

\[
\frac{|PCB|}{|XYZ|} = \left(\frac{PC}{XY}\right)^2 = \left(\frac{ra}{bc}\right)^2 \quad \Rightarrow \quad |BPC| = \frac{r^2 Sa^2}{b^2 c^2}.
\]

And from \( \triangle EDP \sim \triangle XYZ \), we have

\[
\frac{|EDP|}{|XYZ|} = \left(\frac{EP}{XZ}\right)^2 = \left(\frac{rc}{ab}\right)^2 \quad \Rightarrow \quad |DPE| = \frac{r^2 Sc^2}{a^2 b^2}.
\]

Since \( |ABCDEF| = |APB| + |BPC| + |CPD| + |DPE| + |EPF| + |FPA| \), the given inequality we are required to show is equivalent to

\[
|BPC| + |DPE| + |FPA| \geq |APB| + |CPD| + |EPF|
\]

\[
\Leftrightarrow \quad \frac{r^2 Sa^2}{b^2 c^2} + \frac{r^2 Sc^2}{a^2 b^2} + \frac{r^2 Sb^2}{c^2 a^2} \geq \frac{r^2 S}{c^2} + \frac{r^2 S}{b^2} + \frac{r^2 S}{a^2}
\]

\[
\Leftrightarrow \quad a^4 + b^4 + c^4 \geq a^2 b^2 + c^2 a^2 + b^2 c^2
\]

\[
\Leftrightarrow \quad (a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 \geq 0.
\]

The last inequality is trivially true, which concludes the proof. \( \square \)
Solution 2 (Kevin Xian, year 12, James Ruse Agricultural High School, NSW)

Let $X$ be the intersection of lines $AF$ and $BC$. Let $Y$ be the intersection of lines $DE$ and $BC$. And let $Z$ be the intersection of lines $DE$ and $AF$.

Since $AB = CD$ and $ABCD$ is cyclic, it follows that $ABCD$ is an isosceles trapezium with $AD \parallel BC$. Thus $AD \parallel XY$. Similarly we have $CF \parallel YZ$ and $BE \parallel XZ$. It follows that

$\triangle BCP \sim \triangle PDE \sim \triangle APF \sim \triangle XYZ$.

Let $S_1 = |PDE|, S_2 = |PFA|, S_3 = |PBC|, S_4 = |PAB|, S_5 = |PCD|$, and $S_6 = |PEF|$.

Since $BE \parallel XZ$ and $AD \parallel XY$, it follows that $APBX$ is a parallelogram. Hence $|XAB| = |PAB| = S_4$. Similarly we have $|YCD| = S_5$ and $|ZEF| = S_6$. The inequality we are asked to prove is

$$S_1 + S_2 + S_3 + S_4 + S_5 + S_6 \geq 2(S_4 + S_5 + S_6)$$

$\iff$

$$S_1 + S_2 + S_3 \geq S_4 + S_5 + S_6$$

$\iff$

$$3(S_1 + S_2 + S_3) \geq 2(S_4 + S_5 + S_6) + S_1 + S_2 + S_3$$

$\iff$

$$3(S_1 + S_2 + S_3) \geq |XYZ|$$

$\iff$

$$\frac{S_1 + S_2 + S_3}{|XYZ|} \geq \frac{1}{3}.$$ 

For the sake of less clutter, we draw another diagram. The essential details are that $AD \parallel XY$, $BE \parallel XZ$, and $CF \parallel YZ$, and that $AD$, $BE$, and $CF$ are concurrent at $P$.\footnote{It turns out that if this is the case, then the inequality}

$$\frac{S_1 + S_2 + S_3}{|XYZ|} \geq \frac{1}{3}$$

is true in general. In particular, as shall be seen, it is true whether or not we have $AB = CD = EF$. 

\[14\]
Let $u = XB$, $v = BC$, and $w = CY$. Then since $\triangle PDE$, $\triangle APF$, and $\triangle BCP$ are all similar to $\triangle XYZ$, we have

$$\frac{S_1}{|XYZ|} = \left(\frac{PD}{XY}\right)^2 = \left(\frac{CY}{XY}\right)^2 = \frac{w^2}{(u + v + w)^2},$$

$$\frac{S_2}{|XYZ|} = \left(\frac{AP}{XY}\right)^2 = \left(\frac{XB}{XY}\right)^2 = \frac{u^2}{(u + v + w)^2},$$

and

$$\frac{S_3}{|XYZ|} = \left(\frac{BC}{XY}\right)^2 = \frac{v^2}{(u + v + w)^2}.$$

Thus the inequality is equivalent to

$$\frac{u^2 + v^2 + w^2}{(u + v + w)^2} \geq \frac{1}{3} \iff u^2 + v^2 + w^2 \geq uv + vw + wu$$

$$\iff (u - v)^2 + (v - w)^2 + (w - u)^2 \geq 0$$

The last inequality is trivially true, which concludes the proof. □
Solution 3 (Matthew Cheah, year 11, Penleigh and Essendon Grammar School, VIC)

We define $\alpha$, $\beta$, and $\gamma$, and angle chase the diagram as was done in solution 1 to part (b).

![Diagram of geometric figure]

The inequality we are required to prove equivalent to

$$|BPC| + |DPE| + |FPA| \geq |APB| + |CPD| + |EPF|.$$ 

We can scale the diagram so that $AB = CD = EF = 1$. We may compute

$$PA = \frac{\sin \beta}{\sin \gamma}, \quad PB = \frac{\sin \alpha}{\sin \gamma} \quad \text{(sine rule $\triangle APB$)}$$

$$PC = \frac{\sin \alpha}{\sin \beta}, \quad PD = \frac{\sin \gamma}{\sin \beta} \quad \text{(sine rule $\triangle CPD$)}$$

$$PE = \frac{\sin \gamma}{\sin \alpha}, \quad PF = \frac{\sin \beta}{\sin \alpha} \quad \text{(sine rule $\triangle EFP$)}.$$

Using the $\frac{1}{2}bc\sin \alpha$ formula for the area of a triangle we compute

$$|APB| = \frac{\sin \alpha \sin \beta}{2 \sin \gamma}, \quad |CPD| = \frac{\sin \alpha \sin \gamma}{2 \sin \beta}, \quad |EPF| = \frac{\sin \beta \sin \gamma}{2 \sin \alpha}$$

$$|BPC| = \frac{\sin^3 \alpha}{2 \sin \beta \sin \gamma}, \quad |DPE| = \frac{\sin^3 \gamma}{2 \sin \alpha \sin \beta}, \quad |FPA| = \frac{\sin^3 \beta}{2 \sin \alpha \sin \gamma}.$$ 

Thus the inequality is equivalent to

$$\frac{\sin^3 \alpha}{2 \sin \beta \sin \gamma} + \frac{\sin^3 \beta}{2 \sin \alpha \sin \gamma} + \frac{\sin^3 \gamma}{2 \sin \alpha \sin \beta} \geq \frac{\sin \beta \sin \gamma}{2 \sin \alpha} + \frac{\sin \alpha \sin \gamma}{2 \sin \beta} + \frac{\sin \alpha \sin \beta}{2 \sin \gamma}.$$ 

After clearing denominators, this last inequality becomes

$$\sin^4 \alpha + \sin^4 \beta + \sin^4 \gamma \geq \sin^2 \alpha \sin^2 \beta + \sin^2 \beta \sin^2 \gamma + \sin^2 \alpha \sin^2 \gamma.$$ 

But this can be rewritten as

$$(\sin^2 \alpha - \sin^2 \beta)^2 + (\sin^2 \beta - \sin^2 \gamma)^2 + (\sin^2 \gamma - \sin^2 \alpha)^2 \geq 0,$$

which is obviously true. $\square$
## Score Distribution/Problem

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**AUSTRALIAN MATHEMATICAL OLYMPIAD STATISTICS**
# Australian Mathematical Olympiad Results

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* indicates New Zealand school year
Problem 1. We say that a triangle $ABC$ is great if the following holds: for any point $D$ on the side $BC$, if $P$ and $Q$ are the feet of the perpendiculars from $D$ to the lines $AB$ and $AC$, respectively, then the reflection of $D$ in the line $PQ$ lies on the circumcircle of the triangle $ABC$.

Prove that triangle $ABC$ is great if and only if $\angle A = 90^\circ$ and $AB = AC$.

Problem 2. A positive integer is called fancy if it can be expressed in the form

$$2^{a_1} + 2^{a_2} + \cdots + 2^{a_{100}},$$

where $a_1, a_2, \ldots, a_{100}$ are non-negative integers that are not necessarily distinct.

Find the smallest positive integer $n$ such that no multiple of $n$ is a fancy number.

Problem 3. Let $AB$ and $AC$ be two distinct rays not lying on the same line, and let $\omega$ be a circle with center $O$ that is tangent to ray $AC$ at $E$ and ray $AB$ at $F$. Let $R$ be a point on segment $EF$. The line through $O$ parallel to $EF$ intersects line $AB$ at $P$. Let $N$ be the intersection of lines $PR$ and $AC$, and let $M$ be the intersection of line $AB$ and the line through $R$ parallel to $AC$. Prove that line $MN$ is tangent to $\omega$.

Problem 4. The country Dreamland consists of 2016 cities. The airline Starways wants to establish some one-way flights between pairs of cities in such a way that each city has exactly one flight out of it. Find the smallest positive integer $k$ such that no matter how Starways establishes its flights, the cities can always be partitioned into $k$ groups so that from any city it is not possible to reach another city in the same group by using at most 28 flights.

Problem 5. Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$(z + 1)f(x + y) = f(xf(z) + y) + f(yf(z) + x),$$

for all positive real numbers $x, y, z$.  

---

Time allowed: 4 hours Each problem is worth 7 points

The contest problems are to be kept confidential until they are posted on the official APMO website http://apmo.ommenlinea.org. Please do not disclose nor discuss the problems over online until that date. The use of calculators is not allowed.
1. In all solutions presented for this problem, the point $E$ denotes the reflection of $D$ in the line $PQ$, and $\Gamma$ denotes the circumcircle of triangle $ABC$. Furthermore, all solutions presented split neatly into the following three steps.

**Step 1** Prove that if $\triangle ABC$ is great, then $\angle BAC = 90^\circ$.

**Step 2** Prove that if $\triangle ABC$ is great, then $AB = AC$.

**Step 3** Prove that if $\angle BAC = 90^\circ$ and $AB = AC$, then $\triangle ABC$ is great.

**Solution 1** (Based on the solution of William Hu, year 10, Christ Church Grammar School, WA)

**Step 1** Let $D \in BC$ be such that $AD$ bisects $\angle BAC$. Then $\triangle ADP \equiv \triangle ADQ$ (AAS), and so $AP = AQ$ and $DP = DQ$. Hence the line $AD$ is the perpendicular bisector of $PQ$. It follows that $E$ lies on the line $AD$.

Since $\angle DPA = \angle AQD = 90^\circ$, the points $D$, $P$, $A$, and $Q$ all lie on the circle with diameter $AD$. Furthermore, since $P$ and $Q$ lie on opposite sides of line $AD$, quadrilateral $DPAQ$ is cyclic in that order. Hence $A$ and $D$ lie on opposite sides of the line $PQ$. Therefore, $E$ and $A$ lie on the same side of $PQ$.

Note that $DPAQ$ is a cyclic kite having $AD$ as an axis of symmetry.

**Case 1** $\angle BAC < 90^\circ$.

We have $\angle QDP > 90^\circ$ due to $DPAQ$ being cyclic. Thus

$$\angle DPQ = \angle PQD < 45^\circ \quad \text{and} \quad \angle AQP = \angle QPA > 45^\circ.$$ 

It follows that

$$\angle QPE = \angle DPQ < 45^\circ < \angle QPA,$$

and similarly $\angle EQP < \angle AQP$. Thus point $E$ lies strictly inside $\triangle APQ$ and hence strictly inside $\Gamma$. Hence $\triangle ABC$ is not great.
Case 2  $\angle BAC > 90^\circ$.

Define the region $\mathcal{R}$ as follows. Let $B_1$ be any point on the extension of the ray $BA$ beyond $A$, and let $C_1$ be any point on the extension of the ray $CA$ beyond $A$. Then $\mathcal{R}$ is the (infinite) region that lies strictly between rays $AB_1$ and $AC_1$.

Let us return to our consideration of case 2. Since $DPAQ$ is cyclic, it follows that $\angle QDP = 180^\circ - \angle BAC < 90^\circ$. Thus

$$\angle DPQ = \angle PQD > 45^\circ \quad \text{and} \quad \angle AQP = \angle QPA < 45^\circ.$$  

It follows that

$$\angle QPE = \angle DPQ > 45^\circ > \angle QPA,$$

and similarly $\angle EQP > \angle AQP$. Then since $E$ lies on the same side of $PQ$ as $A$, it follows that $E$ lies strictly inside $\mathcal{R}$.

Observe that $\Gamma$ already intersects each of the lines $AB$ and $AC$ twice. Hence $\Gamma$ has no further intersection point with either of these lines. This implies that $\mathcal{R}$, and hence also the point $E$, lie strictly outside of $\Gamma$. Hence $\triangle ABC$ is not great.

Since we have ruled out cases 1 and 2, it follows that $\angle BAC = 90^\circ$.

Step 2  Let $D$ be the midpoint of $BC$. From step 1, we know that $\angle BAC = 90^\circ$. Hence $D$ is the centre of $\Gamma$. Recall that the projection of the centre of a circle onto any chord of the circle is the midpoint of the chord. So $P$ is the midpoint of $AB$ and $Q$ is the midpoint of $AC$. Thus $PQ$ is the midline of $\triangle ABC$ that is parallel to $BC$. Hence $d(A, PQ) = d(D, PQ)$. From the reflection we have $d(E, PQ) = d(D, PQ)$. It follows that points $A$ and $E$ lie at the same height above the line $PQ$, and therefore lie at the same height above the line $BC$.

For any points $X, Y, Z$, the expression $d(X, YZ)$ denotes the distance from the point $X$ to the line $YZ$.\[1\]
Observe that since $E \in \Gamma$ and $E$ lies directly above $D$, we have that the point $E$ is the unique point on $\Gamma$ that lies at maximal height above the line $BC$. Since $A \in \Gamma$ and $A$ and $E$ have the same height above $BC$, it follows that $A = E$. Since $A$ is now known to be the midpoint of arc $BC$, it follows that $AB = AC$.

**Step 3** Without loss of generality we may assume that $BD \leq DC$, so that our diagram looks like the one shown below.

Observe that $DPAQ$ is a rectangle. We have $\angle PEQ = \angle QDP = 90^\circ$ due to the reflection. It follows that $E$ lies on circle $DPQ$, so that $DPEAQ$ is cyclic in that order. However, $AD$ is a diameter of circle $DPAQ$. Consequently $\angle DEA = 90^\circ$.

Furthermore, we have $\triangle PBD \sim \triangle ABC$ because $DP \parallel QA$. Therefore $PB = PD$ and $\angle BPD = 90^\circ$. From the reflection we know $PE = PD$. Hence $P$ is the circumcentre of $\triangle DBE$. We may now compute as follows.

\[
\angle BEA = \angle BED + 90^\circ \\
= \frac{1}{2} \angle BPD + 90^\circ \quad (P \text{ is the centre of circle } BED) \\
= 45^\circ + 90^\circ \\
= 135^\circ.
\]

Since $\angle ACB = 45^\circ$, it follows that $ACBE$ is cyclic. Hence $E$ lies on $\Gamma$.  \qed
Solution 2  (Based on the solution of Michelle Chen, year 12, Methodist Ladies’ College, VIC)

Step 1  Let $D$ be the foot of the angle bisector from $A$.

As in solution 1, $DPAQ$ is a cyclic kite having $AD$ as an axis of symmetry, and points $E$ and $A$ lie on the same side of $PQ$.

Hence $E$ lies on the ray $DA$. But since $D$ lies inside $\Gamma$, the ray $DA$ intersects $\Gamma$ only at one point, namely the point $A$. Since $E \in \Gamma$, it follows that $E = A$. Thus

$$\angle PAQ = \angle PEQ \quad (E = A)$$
$$= \angle QDP \quad \text{(reflection)}$$
$$= 180^\circ - \angle PAQ. \quad \text{($DPAQ$ cyclic)}$$

From this we deduce $\angle BAC = \angle PAQ = 90^\circ$.

Step 2  Let $D$ be the midpoint of $BC$. Without loss of generality suppose that $E$ lies on the minor arc $AC$ of $\Gamma$, that does not contain point $B$.

Observe that $DPAQ$ is a rectangle, and its circumcircle has $PQ$ as a diameter. From the reflection we have $\angle PEQ = \angle PDQ = 90^\circ$. Hence $E$ also lies on the circle with diameter $PQ$. Thus $DPAEQ$ is cyclic in that order.

Since $D$ is the circumcentre of $\triangle ABC$, we have $DP$ and $DQ$ are the perpendicular bisectors of $AB$ and $AC$, respectively. Using the rectangle $DPAQ$ and the reflection, we have

$$BP = PA = DQ = EQ. \quad (1)$$

Similarly, we have

$$QC = AQ = PD = PE. \quad (2)$$

Furthermore, since $PEAQ$ is cyclic in that order, we have $\angle EPA = \angle EQA$. Thus

$$\angle BPE = \angle CQE. \quad (3)$$

It follows from (1), (2), and (3) that $\triangle BPE \equiv \triangle EQC$ (SAS). Hence

$$\angle QEC = \angle EBP \quad (\triangle BPE \equiv \triangle EQC)$$
$$= \angle EBA$$
$$= \angle ECA \quad (E \in \Gamma)$$
$$= \angle ECQ.$$ 

Thus $QC = QD$ from which it follows that

$$AB = 2AP = 2QD = 2QC = AC.$$

Step 3  Without loss of generality we may assume $BD \leq DC$. As in solution 1, we deduce that $DPEAQ$ is cyclic in that order, and that $PB = PE$. Therefore,

$$\angle AQE = \angle APE = \angle PBE + \angle BEP = 2\angle ABE. \quad (4)$$

From the reflection we have

$$\angle EQD = \angle EQP + \angle PQD = 2\angle EQP = 2\angle EQP = 2\angle EAB. \quad (5)$$

Combining (4) and (5) we find

$$\angle ABE + \angle EAB = \frac{1}{2} (\angle AQE + \angle EQD) = \frac{1}{2} \angle AQD = 45^\circ.$$

The angle sum in $\triangle ABE$ then yields $\angle BEA = 135^\circ$. Since $\angle ACB = 45^\circ$, it follows that $ACBE$ is cyclic. Hence $E \in \Gamma$. 

□
2. **Solution 1** (Kevin Xian, year 12, James Ruse Agricultural High School, NSW)

Answer: \( n = 2^{101} - 1 \)

Let us call a number \( k \)-fantastic if it has a multiple that can be expressed as a sum of \( k \) (not necessarily distinct) non-negative powers of two. We seek the smallest positive integer \( n \) such that \( n \) is not 100-fantastic.

**Lemma** If a number is \( k \)-fantastic, then it is also \((k + 1)\)-fantastic.

**Proof** Suppose \( m \) is \( k \)-fantastic. That is, \( m \) is a factor of \( 2^{a_1} + 2^{a_2} + \cdots + 2^{a_k} \). Then \( m \) is also a factor of
\[
2(2^{a_1} + 2^{a_2} + \cdots + 2^{a_k}) = 2^{a_1+1} + 2^{a_2+1} + \cdots + 2^{a_k-1+1} + 2^{a_k+1}
\]
\[
= 2^{a_1+1} + 2^{a_2+1} + \cdots + 2^{a_k-1+1} + 2^{a_k} + 2^{a_k}.
\]

Thus \( m \) is also \((k + 1)\)-fantastic. \( \square \)

We claim that every \( n \in \{1, 2, \ldots, 2^{101} - 2\} \) is 100-fantastic. To see this, write \( n \) in binary. The digit 1 occurs \( k \leq 100 \) times. Thus \( n \) is \( k \)-fantastic. Applying the lemma \( 100 - k \) times, we deduce that \( n \) is 100-fantastic.

Finally we prove that \( n = 2^{101} - 1 \) is not 100-fantastic. Suppose, for the sake of contradiction, that \( n \) is 100-fantastic. Then there exists a smallest positive integer \( k \leq 100 \) such that \( n \) is \( k \)-fantastic. Thus
\[
2^{a_1} + 2^{a_2} + \cdots + 2^{a_k} = cn,
\]
for some positive integer \( c \). Since \( 2^{101} \equiv 2^0 \pmod n \), we may assume \( 0 \leq a_i \leq 100 \) for each \( i \in \{1, 2, \ldots, k\} \).

If \( a_i = a_j = r \) for some \( i \neq j \), then we could combine the two terms \( 2^r + 2^r \) into the single term \( 2^{r+1} \). But this would imply that \( n \) is \((k - 1)\)-fantastic, contradicting the minimality of \( k \). Hence all the \( a_i \) are pairwise different. Since \( k \leq 100 \), it follows that \( \{a_1, a_2, \ldots, a_k\} \) is a proper subset of \( \{0, 1, \ldots, 100\} \). Thus
\[
2^{a_1} + 2^{a_2} + \cdots + 2^{a_k} < 2^0 + 2^1 + \cdots + 2^{100}
\]
\[
= n.
\]

However this contradicts (1). This completes the proof. \( \square \)
Solution 2  (Seyoon Ragavan, year 12, Knox Grammar School, NSW)

Case 1  \(100 \leq n \leq 2^{101} - 2\)

From the binary representation of \(n\), we have

\[ n = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_k}, \]

for non-negative integers \(a_1, a_2, \ldots, a_k\), where \(k \leq 100\).

Let \(S\) be the multiset \(\{a_1, a_2, \ldots, a_k\}\). Consider the following operation which we apply repeatedly to \(S\).

(O1) Take \(x \in S\) with \(x \geq 1\), and replace it with \(x - 1\) and \(x - 1\) in \(S\).

Observe that O1 increases \(|S|\) by 1 and preserves \(n = \sum_{s \in S} 2^s\). Apply the operation until all elements of \(S\) are equal to 0. In the end we have \(|S| = n \geq 100\). Since in the beginning we have \(|S| \leq 100\), by the discrete intermediate value theorem there is a point at which \(|S| = 100\). Hence \(n\) itself is fancy.

Case 2  \(1 \leq n \leq 99\)

There is a multiple of \(n\), say \(cn\), in the range from 100 to 198 inclusive. Since \(100 < cn < 2^{101} - 2\), case 1 guarantees that \(cn\) is fancy.

Case 3  \(n = 2^{101} - 1\)

We shall be done if we can show that \(n = 2^{101} - 1\) has no fancy multiple. Suppose, for the sake of contradiction, that \(2^{a_1} + 2^{a_2} + \cdots + 2^{a_{100}}\) is a multiple of \(n\).

Let \(S\) be the multiset \(\{a_1, a_2, \ldots, a_k\}\). Consider the following operations which we apply repeatedly to \(S\).

(O2) If an element \(x \geq 101\) occurs in \(S\), replace it with \(x - 101\).

(O3) If an element \(x\) occurs (at least) twice in \(S\), replace \(x, x\) with \(x+1\).

Observe that since \(2^x \equiv 2^{x-101} \pmod{n}\), and \(2^x + 2^x = 2^{x+1}\), both O2 and O3 retain the property that

\[ \sum_{s \in S} 2^s \equiv 0 \pmod{n}. \]

Furthermore, note that O3 can only be applied finitely many times because each application decreases \(|S|\) by 1. Also O2 can only be applied finitely many times in a row. Therefore, let us apply operations (O2) and (O3) until neither can be applied anymore. At this stage we have \(S = \{x_1, x_2, \ldots, x_k\}\) where \(x_1 < x_2 < \cdots < x_k\) and \(k \leq 100\). Since \(S\) is a proper subset of \(\{0, 1, \ldots, 100\}\) we have

\[ \sum_{s \in S} 2^s < \sum_{i=0}^{100} 2^i = n. \]

Hence \(\sum_{s \in S} 2^s\) is not a multiple of \(n\). This contradiction finishes the proof. \(\square\)
3. Solution 1  (Seyoon Ragavan, year 12, Knox Grammar School, NSW)

Let $N'$ be the point on segment $AE$ such that $MN'$ is tangent to $\omega$. It is sufficient to prove that $N' = N$. For this it suffices to show that points $P$, $R$ and $N'$ are collinear. Since $MR \parallel AN'$, this is equivalent to showing that $\triangle PMR \sim \triangle PAN'$. Since $MR \parallel AN'$, this is equivalent to showing that

$$\frac{PM}{PA} = \frac{MR}{AN'}.$$  \hspace{1cm} (1)

We introduce some variables and some new points that will enable us to calculate both of these ratios.

In $\triangle AMN'$ let $a = MN'$, $b = AN'$, $c = AM$, and $s = \frac{a + b + c}{2}$. Note that since $\omega$ is the excircle opposite $A$ in $\triangle AMB'$, a standard calculation yields $AE = AF = s$.

Let $I$ be the incentre of $\triangle AMN'$ and let $X$ be on $AM$ such that $IX \perp AX$. Observe that due to symmetry, $I$ lies on segment $AO$, and $AO \perp EF$. Hence $AO \perp OP$, and so $XIOP$ is cyclic.

First we compute $\frac{MR}{AN'}$ in terms of $a, b, c, s$.

$$\frac{MR}{AN'} = \frac{MR}{AE} \cdot \frac{AE}{AN'}$$

$$= \frac{FM}{FA} \cdot \frac{AE}{AN'} \quad (MR \parallel AE)$$

$$= \frac{FM}{AN'} \quad (AE = AF)$$

$$= \frac{AF - AM}{AN'}$$

$$= \frac{s - c}{b}.$$  \hspace{1cm} (2)
Second we compute $\frac{PM}{PA}$ in terms of $a, b, c, s$. Since $PM = PA - c$, it remains to compute $AP$ in terms of $a, b, c, s$. Since $XIOP$ is cyclic, by power of a point we have

$$AX \cdot AP = AI \cdot AO. \quad (3)$$

From the incircle substitution we know $AX = s - a$. Therefore it remains to compute $AI \cdot AO$ in terms of $a, b, c, s$.

Let $\angle MAN = 2\alpha$, $\angle N'MA = 2\beta$, and $\angle AN'M = 2\gamma$. Note $\alpha + \beta + \gamma = 90^\circ$. Then

$$\angle FMO = 90^\circ - \beta \quad \Rightarrow \quad \angle OMA = 90^\circ + \beta = 180^\circ - \alpha - \gamma.$$

From the angle sum in $\triangle AIN'$, we have

$$\angle N'IA = 180^\circ - \alpha - \gamma.$$

Since also $\angle MAO = \angle IAN' = \alpha$, it follows that $\triangle AMO \sim \triangle AIN'$ (AA). Thus

$$\frac{AI}{AM} = \frac{AN'}{AO} \quad \Rightarrow \quad AI \cdot AO = AM \cdot AN' = bc.$$

Putting this into (3) yields $AP = \frac{bc}{s-a}$. Hence

$$\frac{PM}{PA} = \frac{AP - c}{AP} = \frac{\frac{bc}{s-a} - c}{\frac{bc}{s-a}} = \frac{a + b - s}{b} = \frac{s - c}{b}.$$

Comparing (2) and (4) yields (1), as required. \qed
Solution 2  (Ilia Kucherov, year 12, Westall Secondary College, VIC)

Let $D$ ($D \neq E$) be on $\omega$ such that $ND$ is tangent to $\omega$. Let $M' = ND \cap AF$. It is sufficient to prove that $M' = M$. For this it suffices to show that $M'R \parallel AC$.

Let $G$ ($G \neq F$) be on $\omega$, such that $PG$ is tangent to $\omega$. Hence we have

$$\angle GPO = \angle OPF = \angle EFA = \angle AEF.$$  

Since also $OP \parallel EF$, it follows that $GP \parallel EN$.

With this result in mind, let us now focus only on the following part of diagram. Note that for a point $X$ on the circle, $XX$ denotes the tangent at $X$. Points $E, D, F, G$ are cyclic in that order around the circle and satisfy $EE \parallel GG$. The other points satisfy $N = EE \cap DD$, $M' = DD \cap FF$, $P = FF \cap GG$, and $R = EF \cap NP$. It suffices to prove that $M'R \parallel EE$. 

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Apply a central projection that sends the circle to a circle and the point \( R \) to the centre of the circle.\(^2\) For a point \( X \), we shall let \( X^* \) denote the image of \( X \) under this transformation. Since \( R^* \) is the centre of the circle, \( E^*F^* \) is a diameter. Thus \( E^*E^* \parallel F^*F^* \).

We also have

\[
\angle D^*N^*R^* = \angle R^*N^*E^* = \angle R^*P^*F^* = \angle G^*P^*R^*.
\]

Hence \( D^*D^* \parallel G^*G^* \), and so \( M''N^*J^*P^* \) is a parallelogram where \( J = E^*E^* \cap G^*G^* \). Since \( M''N^*L^*P^* \) is a parallelogram with an incircle, it must be a rhombus. Since the diagonals of a rhombus intersect at its centre, the lines \( M''R^* \), \( E^*N^* \), and \( G^*P^* \) are concurrent. It follows that in the original diagram, the three lines \( M'R \), \( EN \), and \( GP \) are concurrent or parallel. Since we earlier proved that \( EN \parallel GP \), we have \( M'R \parallel EN \parallel GP \), as desired. \( \square \)

\(^2\) A central projection that sends the polar of \( R \) to the line at infinity, and the circle to circle, will achieve this.
4. **Solution 1** (Ilia Kucherov, year 12, Westall Secondary College, VIC)

Answer: $k = 57$

Consider the case where the flight paths of Starways consist of two disjoint directed cycles: one with 57 cities and one with the remaining 1959 cities. Let $A$ and $B$ be any two cities in the first directed cycle. Suppose that $m$ flights are required to get from $A$ to $B$ and $n$ flights are required to get from $B$ to $A$. Since $m + n = 57$ we have $\min\{m, n\} \leq 28$. Hence $A$ and $B$ must be in different groups. Since this is true for any two cities in the first circuit, it follows that $k \geq 57$.

To prove that at most 57 groups are sufficient, we generalise the result to $n$ cities and prove it by induction. Our result will be the special case where $n = 2016$.

The result is true if $n \leq 57$ because we can put each city into its own group.

Suppose that the result is true for all configurations of valid flights for $n$ cities. Consider any situation with $n + 1$ cities.

**Case 1** There exists a city $C$ with no flights into it.

Remove $C$. Then there are left $n$ cities with exactly one flight out of each. By the inductive assumption they can be partitioned into 57 groups.

Observe that one can reach no more than 28 cities from $C$ using at most 28 flights. This is because there is only one flight out of each city. Also city $C$ is not reachable from any other city because $C$ has no flights into it. Now add $C$ back and simply assign $C$ to a group that contains no city reachable from $C$ in at most 28 flights.

**Case 2** Each city has a flight into it.

There are exactly $n + 1$ flights and $n + 1$ cities. Hence each city has exactly one flight into it and one flight out of it. Thus the system of flights may be partitioned into disjoint directed cycles. Since cities from different directed cycles cannot reach each other, it is enough to prove that each directed cycle can be partitioned into 57 groups.

If a directed cycle has at most 57 cities, then each city can be put into its own group.

If a directed cycle has $m \geq 58$ cities, we may write $m = 29x + r$ where $x \geq 2$ and $0 \leq r \leq 28$. Let the directed cycle of cities be $A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_m \rightarrow A_1$. For each $i \in \{1, 2, \ldots, 29\}$ let group $i$ consist of all cities $A_j$ where $j \equiv i \pmod{29}$ and $2 \leq j \leq x$. Note that any two cities in group $i$ require at least 29 flights to connect them. Finally, we put each of the remaining $r$ cities into its own group.

This completes the inductive step, and hence also the proof. 

\[\square\]
Solution 2  (Hadyn Tang, year 7, Trinity Grammar School, VIC)

As in solution 1, we prove that \( k \geq 57 \).

It suffices to show that at most 57 groups are sufficient. We represent the situation in graph theoretical language in the canonical way. Vertices of the graph \( G \) correspond to cities, and there is a directed edge between two vertices of \( G \) if and only if there is a directed flight between the corresponding two cities. It is enough to prove that each connected component can be partitioned into 57 groups. From here on \( H \) shall denote a connected component of \( G \).

The following three lemmas help us to determine the structure of \( H \).

Lemma 1  Let us say that vertex \( P \) is connected to vertex \( Q \) if there is a directed edge \( P \rightarrow Q \) or a directed edge \( Q \rightarrow P \). Suppose that \( A_1, A_2, A_3, \ldots, A_m \) is a sequence of vertices (the \( A_i \) need not be distinct) such that \( A_i \) is connected to \( A_{i+1} \) for \( i = 1, 2, \ldots, m - 1 \). If the edge connecting \( A_1 \) to \( A_2 \) is directed \( A_2 \rightarrow A_1 \), then we have \( A_{i+1} \rightarrow A_i \) for \( i = 1, 2, \ldots, m - 1 \).

Lemma 2  If \( H \) contains an undirected cycle when we ignore edge directions, then that cycle is a directed cycle when we remember edge directions.

Lemma 3  \( H \) contains at most one cycle.

Proofs  Lemma 1 follows inductively from the fact that each vertex has out-degree at most 1. Lemma 2 is a simple corollary of lemma 1. Thus from here on all cycles may be assumed to be directed. For lemma 3, suppose that \( H \) contains at least two cycles. Then at least one of the following three situations occurs for \( H \).

(i) \( H \) contains a cycle with a chord, that is, a cycle plus a path that splits the cycle into two smaller cycles. (The splitting path is part of both cycles.)

(ii) \( H \) contains two cycles that have exactly one vertex in common.

(iii) \( H \) contains two cycles with no vertices in common.

Applying lemmas 1 and 2 shows that any of the above situations leads to a vertex having out-degree at least 2. This contradiction establishes the lemma. \( \square \)

Lemma 3 shows that \( H \) is either a directed tree or else contains a directed cycle possibly with strands leading in to it. Note that following any such strands away from the cycle may result in forking.

Case 1  \( H \) contains a cycle \( A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_m \rightarrow A_1 \), where \( 3 \leq m \leq 29 \).

First put city \( A_i \) (\( 1 \leq i \leq m \)) into group \( i \). This effectively deals with the cycle.

Suppose there is some other city in \( H \) that is not part of the cycle. By lemmas 1–3, there is a unique directed path from that city to the cycle. Suppose that \( u \) directed edges are required to reach the cycle from that city. We write this as \( B_u \rightarrow B_{u-1} \rightarrow \cdots \rightarrow B_1 \rightarrow A_j \), where \( A_j \) is on the cycle, but no \( B_i \) is on the cycle. Put city \( B_i \) into group \( 29 + i \) (mod 57) for each \( i \in \{1, 2, \ldots, u\} \). (Here, group 0 is the same as group 57.) By lemmas 1–3, this effectively deals with the rest of \( H \). An example illustrating the construction for \( m = 6 \) is shown below.
Case 2 \( H \) contains a cycle \( A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_m \rightarrow A_1 \), where \( m \geq 30 \).

Write \( m = 29x + r \), where \( x \) is a non-negative integer and \( 0 \leq r \leq 28 \). For each \( i \in \{1, 2, \ldots, 29\} \), let group \( i \) consist of all cities \( A_j \), where \( j \equiv i \pmod{29} \), and \( 1 \leq j \leq 29x \). For each number of the form \( 29x + s \) where \( 1 \leq s \leq r \), put city \( A_{29x+s} \) into group \( 29 + s \). Note that this deals effectively with the cycle.

Suppose some other city in \( H \) that is not part of the cycle. As in case 1, there is a unique directed path from that city to the cycle. Suppose that \( u \) directed edges are required to reach the cycle from that city. We write this as \( B_u \rightarrow B_{u-1} \rightarrow \cdots \rightarrow B_1 \rightarrow A_j \), where \( A_j \) is on the cycle, but no \( B_i \) is on the cycle. Put city \( B_i \) in group \( j - i \pmod{57} \) for each \( i \in \{1, 2, \ldots, u\} \). (Here, group 0 is the same as group 57.) By lemmas 1–3, this effectively deals with the rest of \( H \). The diagram below illustrates the construction.

![Diagram](attachment:diagram.png)

Case 3 \( H \) contains no cycle.

Choose any vertex and from that vertex walk around the graph following the directions of its edges. Since the graph contains no cycle, this walk must eventually terminate at a vertex which we shall call \( A_1 \). We now use the same construction as in case 1, except that we treat the single vertex \( A_1 \) as being a cycle of length 1. This effectively deals with the rest of \( H \).

Since we have shown how to deal with all cases, the proof is complete. \( \square \)
5. **Solution 1** (Seyoon Ragavan, year 12, Knox Grammar School, NSW)

Clearly \( f(x) = x \) satisfies the functional equation. We show this is the only answer.

For reference, the functional equation is \( f: \mathbb{R}^+ \to \mathbb{R}^+ \)

\[
(z + 1)f(x + y) = f(xf(z) + y) + f(yf(z) + x). \tag{1}
\]

Suppose that \( f(z_1) = f(z_2) \). Then

\[
(z + 1)f(x + y) = f(xf(z_1) + y) + f(yf(z_1) + x)
= f(xf(z_2) + y) + f(yf(z_2) + x)
= (z_2 + 1)f(x + y).
\]

It follows that \( z_1 = z_2 \). Hence \( f \) is injective.

Set \( x = y = \frac{u}{2} \) into (1) to find for all \( x, y \in \mathbb{R}^+ \)

\[
(z + 1)f(u) = 2f\left( \frac{u}{2}(f(z) + 1) \right)
\Rightarrow f\left( \left( \frac{f(z) + 1}{2} \right) u \right) = \left( \frac{z + 1}{2} \right) f(u). \tag{2}
\]

Applying (2) repeatedly, we find

\[
f\left( \prod_{i=1}^{n} \left( \frac{f(z_i) + 1}{2} \right) \right) = \left( \frac{z_1 + 1}{2} \right) f\left( \prod_{i=2}^{n} \left( \frac{f(z_i) + 1}{2} \right) \right)
= \left( \frac{z_1 + 1}{2} \right) \left( \frac{z_2 + 1}{2} \right) f\left( \prod_{i=3}^{n} \left( \frac{f(z_i) + 1}{2} \right) \right)
= \cdots
= \left( \prod_{i=1}^{n} \left( \frac{z_i + 1}{2} \right) \right) f(1) \tag{3}
\]

Let \( z_1, z_2, \ldots, z_n \) satisfy \( \prod_{i=1}^{n} \left( \frac{z_i + 1}{2} \right) = 1 \). Using (3) and the fact that \( f \) is injective, we deduce

\[
\prod_{i=1}^{n} \left( \frac{z_i + 1}{2} \right) = 1 \Rightarrow \prod_{i=1}^{n} \left( \frac{f(z_i) + 1}{2} \right) = 1. \tag{4}
\]

Suppose that \( \left( \frac{u_1 + 1}{2} \right) \left( \frac{u_2 + 1}{2} \right) = \frac{u_3 + 1}{2} \) for some \( u_1, u_2, u_3 > 0 \). For sufficiently large \( n \), there exist \( u_4, u_5, \ldots, u_n > 0 \) such that

\[
\left( \frac{u_3 + 1}{2} \right) \prod_{i=4}^{n} \left( \frac{u_i + 1}{2} \right) = \left( \frac{u_1 + 1}{2} \right) \left( \frac{u_2 + 1}{2} \right) \prod_{i=4}^{n} \left( \frac{u_i + 1}{2} \right) = 1.
\]

Applying (4) to this, it follows that

\[
\left( \frac{f(u_3) + 1}{2} \right) \prod_{i=4}^{n} \left( \frac{f(u_i) + 1}{2} \right) = \left( \frac{f(u_1) + 1}{2} \right) \left( \frac{f(u_2) + 1}{2} \right) \prod_{i=4}^{n} \left( \frac{f(u_i) + 1}{2} \right) = 1.
\]
Hence we have deduced that
\[
\left( \frac{u_1 + 1}{2} \right) \left( \frac{u_2 + 1}{2} \right) = \frac{u_3 + 1}{2} \Rightarrow \left( \frac{f(u_1) + 1}{2} \right) \left( \frac{f(u_2) + 1}{2} \right) = \frac{f(u_3) + 1}{2}. \tag{5}
\]

Define a new function \( g: \left( \frac{1}{2}, \infty \right) \to \left( \frac{1}{2}, \infty \right) \) by
\[
g(x) = \frac{f(2x - 1) + 1}{2}.
\]

If we set \( v_i = \frac{u_i + 1}{2} \) for \( i = 1, 2, 3 \) in (5) then we deduce that
\[
g(v_1)g(v_2) = g(v_1v_2). \tag{6}
\]

Note that (6) is true for any \( v_1, v_2 \) satisfying \( v_1, v_2 > \frac{1}{2} \) and \( v_1v_2 > \frac{1}{2} \). In particular, (6) is true for any \( v_1, v_2 > 1 \).

Define another new function \( h: \mathbb{R}^+ \to \mathbb{R}^+ \) by
\[
h(x) = \log_2(g(2^x)).
\]

From (6), it follows that for any \( x, y > 0 \), we have
\[
h(x + y) = h(x) + h(y). \tag{7}
\]

Define yet another new function \( H: \mathbb{R} \to \mathbb{R} \) by
\[
H(x) = \begin{cases} 
  h(x) & \text{if } x > 0 \\
  0 & \text{if } x = 0 \\
 -h(-x) & \text{if } x < 0.
\end{cases}
\]

It is easily shown that for any \( x, y \in \mathbb{R} \), we have
\[
H(x + y) = H(x) + H(y). \tag{8}
\]

We now apply the following theorem.

**Theorem** Suppose that \( F: \mathbb{R} \to \mathbb{R} \) satisfies
\[
F(x + y) = F(x) + F(y),
\]

for all \( x, y \in \mathbb{R} \). Then one of the following two situations is true.

(i) There exists \( k \in \mathbb{R} \) such \( F(x) = kx \) for all \( x \in \mathbb{R} \).

(ii) Every circle in the plane contains a point of the graph of \( F \).

Let us apply the above theorem to \( H \). If (ii) is true, then any circle that lies strictly to the right of the line \( x = 0 \) contains a point of the graph of \( H(x) \). But \( H(x) = h(x) \) in this region, and the graph of \( h(x) \) lies completely above the line \( y = -1 \), a contradiction.

\[^3\text{See the subheading Properties of other solutions in the Wikipedia article Cauchy’s functional equation.} \]

https://en.wikipedia.org/wiki/Cauchy%27s_functional_equation
Hence (i) is true, and \( H(x) = kx \) for some \( k \in \mathbb{R} \). From this we successively derive

\[
\begin{align*}
    h(x) &= kx & \text{for all } x > 0 \\
    g(2^x) &= 2^{h(x)} = 2^{kx} = (2^x)^k \\
    g(y) &= y^k & \text{for all } y > 1 \\
    f (2y - 1) + 1 &= y^k \\
    f(z) &= 2 \left( \frac{z + 1}{2} \right)^k - 1 & \text{for all } z > 1. \quad \text{(9)}
\end{align*}
\]

Substituting (9) into the functional equation for \( z > 1 \) and \( x = y = 1 \) (note that all of the arguments in all function expressions are greater than 1), we have

\[
(z + 1) \left( 2 \left( \frac{3}{2} \right)^k - 1 \right) = 4 \left( \frac{2 \left( \frac{z+1}{2} \right)^k + 1}{2} \right)^k - 2. \quad \text{(10)}
\]

If \( k < 0 \), then from (9), \( f(z) \to -1 \) as \( z \to \infty \), which is a contradiction because \( f(z) > 0 \) for all \( z \in \mathbb{R}^+ \).

If \( k = 0 \), then \( \text{RHS(10)} \) is constant, while \( \text{LHS(10)} \) varies with \( z \), a contradiction.

If \( k > 0 \), then \( \text{LHS(10)} \) is asymptotic\(^4\) to \( cz \), where \( c = 2^{k-1} \cdot 3^k \), while the \( \text{RHS(10)} \) is asymptotic to \( dx^2 \), where \( d = 4 \cdot 2^{-k^2} \). This implies that \( k^2 = 1 \), and so \( k = 1 \). Thus (9) now becomes \( f(z) = z \) for all \( z > 1 \).

Finally, for any real number \( r > 0 \), let us set \( x = y = \frac{r}{2} \) in (1), and choose \( z > 1 \), so that \( rz > 2 \). Then (1) easily implies that \( f(r) = r \), which completes the proof. \( \square \)

**Comment** The theorem referred to after equation (8) is both very interesting and useful. Therefore, we provide the following proof of it.

**Lemma** Suppose that \( F: \mathbb{R} \to \mathbb{R} \) satisfies \( F(x+y) = F(x) + F(y) \) for all \( x, y \in \mathbb{R} \). If \( P \) and \( Q \) are any two different points on the graph of \( F \), then the points of the graph of \( F \) that lie on the line \( PQ \) form a dense\(^5\) subset of the line \( PQ \).

**Proof** It is a standard exercise to show that \( F(q_1x + q_2y) = q_1F(x) + q_2F(y) \) for any \( q_1, q_2 \in \mathbb{Q} \). Thus if \( P \) and \( Q \) are any points on the graph of \( F \) then so is \( q_1P + q_2Q \). In particular \( qP + (1-q)Q \) is on the graph for any \( q \in \mathbb{Q} \), and this is a dense subset of the line \( PQ \). \( \square \)

Now suppose there is no \( k \in \mathbb{R} \) such that \( F(x) = kx \) for all \( x \in \mathbb{R} \). It follows that there exist two points \( P \) and \( Q \) on the graph of \( F \) such that the lines \( OP \) and \( OQ \) are different, where \( O \) is the origin. (Note that \( O \) is on the graph of \( F \).)

Let \( \Gamma \) be any circle in the plane. Consider the cone of lines through \( P \) that also intersect the interior of \( \Gamma \). This cone also intersects the line \( OQ \) in a subset that contains at least an interval, \( I \) say. By the lemma the graph of \( F \) is dense in the line \( OQ \). Hence the interval \( I \) also contains a point, \( R \) say, of the graph. Again by the lemma, the the graph of \( F \) is dense in the line \( PR \). Since \( PR \) intersects the interior of \( \Gamma \), the result follows. \( \square \)

\(^4\)We say that \( A(x) \) is asymptotic to \( B(x) \) if \( \lim_{x \to \infty} \frac{A(x)}{B(x)} = C \) for some nonzero constant \( C \).

\(^5\)Suppose that \( X \) is subset of \( Y \). We say that \( X \) is dense in \( Y \) if for each \( y \in Y \) there exists a sequence of points \( x_1, x_2, \ldots \) all in \( X \) which get arbitrarily close to \( y \).
Solution 2  (Problem Selection Committee)

Clearly \( f(x) = x \) satisfies the functional equation. We show this is the only answer.

For reference, the functional equation is \( f: \mathbb{R}^+ \to \mathbb{R}^+ \)
\[
(z + 1)f(x + y) = f(xf(z) + y) + f(yf(z) + x). \quad (1)
\]

Set \( x = y = 1 \) in (1) to deduce \( 2f(f(z) + 1) = (z + 1)f(2) \). Thus we have

All sufficiently large real numbers are in the range of \( f \). \quad (2)

With the RHS of (1) in mind, we attempt to make the following change of variables
\[
a = xf(z) + y \quad \text{and} \quad b = yf(z) + x.
\]

Solving the above for \( x \) and \( y \) yields
\[
x = \frac{af(z) - b}{f(z)^2 - 1} \quad \text{and} \quad y = \frac{bf(z) - a}{f(z)^2 - 1}.
\]

This change of variables is valid if the expressions given for \( x \) and \( y \) above are positive. From (2) we can ensure this by choosing \( z \) so that \( f(z) > \max\{1, \frac{b}{a}, \frac{a}{b}\} \).

Putting this change of variables into (1) yields
\[
f(a) + f(b) = (z + 1)f\left(\frac{(a + b)(f(z) - 1)}{f(z)^2 - 1}\right).
\]

The above equation is true for any \( a, b \in \mathbb{R}^+ \) and \( f(z) > \max\{1, \frac{b}{a}, \frac{a}{b}\} \). Now suppose that \( a, b, c, d \in \mathbb{R}^+ \) satisfy \( a + b = c + d \). If we choose \( f(z) > \max\{1, \frac{c}{d}, \frac{d}{c}\} \), then we also have
\[
f(c) + f(d) = (z + 1)f\left(\frac{(c + d)(f(z) - 1)}{f(z)^2 - 1}\right).
\]

Thus if \( f(z) > \max\{1, \frac{b}{a}, \frac{a}{b}, \frac{c}{d}, \frac{d}{c}\} \), we can combine the previous two equations to deduce the following very useful result for any \( a, b, c, d \in \mathbb{R}^+ \) with \( a + b = c + d \).

\[
a + b = c + d \quad \Rightarrow \quad f(a) + f(b) = f(c) + f(d) \quad (3)
\]

Next put \( x = y = \frac{u}{2} \) into (1), and then apply (3) with \( a = b = \frac{u}{2}f(z) + \frac{u}{2}, c = uf(z) \), and \( d = u \) to derive
\[
(z + 1)f(u) = f\left(\frac{u}{2}f(z) + \frac{u}{2}\right) + f\left(uf(z) + u\right)
\]
\[
= f(uf(z)) + f(u).
\]

It follows that
\[
zf(u) = f(uf(z)) \quad \text{for all } u, z \in \mathbb{R}^+.
\quad (4)
\]

Setting \( u = 1 \) in (4) easily implies that \( f \) is injective. Setting \( z = 1 \) in (4) and using the fact that \( f \) is injective yields \( f(1) = 1 \). Then setting \( u = 1 \) in (4) yields
\[
f(f(z)) = z \quad \text{for all } z \in \mathbb{R}^+.
\quad (5)
For any $x, y \in \mathbb{R}^+$ we may use (3) twice to find
\[ f(x + y) + f(1) = f(x) + f(y + 1) \]
and
\[ f(y + 1) + f(1) = f(y) + f(2). \]
Combining these yields
\[ f(x + y) = f(x) + f(y) + f(2) - 2 \quad \text{for all } x, y \in \mathbb{R}^+. \quad (6) \]
Putting $x = y = f(2)$ into (6), and using (5) and then (4) yields
\[ f(2f(2)) = 2f(f(2)) + f(2) - 2 = f(2) + 2 \]
\[ \Rightarrow \quad 2f(2) = f(2) + 2 \]
\[ \Rightarrow \quad f(2) = 2. \]
This shows that (6) may be improved to
\[ f(x + y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}^+. \quad (7) \]
Since $f(y) > 0$ for all $y \in \mathbb{R}^+$, equation (7) implies that $f$ is strictly increasing.
Suppose that $f(x) > x$ for some $x \in \mathbb{R}^+$. Since $f$ is strictly increasing, it would follow that $f(f(x)) > f(x)$. Thus from (5) we would have $x > f(x)$, a contradiction.
A similar contradiction is reached if $f(x) < x$ for some $x \in \mathbb{R}^+$.
Thus $f(x) = x$ for all real numbers $x$. \qed
### Top 10 Australian scores

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### Country scores

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</table>
AMOC SELECTION SCHOOL

The 2016 IMO Selection School was held 10–19 April at Robert Menzies College, Macquarie University, Sydney. The main qualifying exams are the AMO and the APMO from which 25 students are selected for the school.

The routine is similar to that for the December School of Excellence; however, there is the added interest of the actual selection of the Australian IMO team. This year the IMO would be held in Hong Kong.

The students are divided into a junior group and a senior group. This year there were 10 juniors and 15 seniors. It is from the seniors that the team of six for the IMO plus one reserve team member is selected.

Until this year, the AMO, the APMO and the final three senior exams at the school were the official selection exams. However this was modified as follows. Once a student has qualified as a senior for the school, only the senior exams at the school are used to select the team. The official team selection exams at this stage are now the final three senior exams at the school. In case the results are too close to effectively separate students, then we use the remaining two senior exams at the school as tie-breakers.

My thanks go to Adrian Agisilaou, Alexander Chua, Andrew Elvey Price, and Andy Tran, who assisted me as live-in staff members. Also to Alexander Babidge, Vaishnavi Calisa, Paul Cheung, Mike Clapper, Victor Khou, Vickie Lee, Allen Lu, John Papantoniou, Yang Song, Gareth White, Sampson Wong, and Jonathan Zheng, all of whom came in to give lectures or help with the marking of exams.

Angelo Di Pasquale
Director of Training, AMOC
2016 Australian IMO Team

<table>
<thead>
<tr>
<th>Name</th>
<th>School</th>
<th>Year</th>
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</thead>
<tbody>
<tr>
<td>Michelle Chen</td>
<td>Methodist Ladies’ College VIC</td>
<td>Year 12</td>
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<tr>
<td>Ilia Kucherov</td>
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</tr>
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<td>Jongmin Lim</td>
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<td>Reserve</td>
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<tr>
<td>Matthew Cheah</td>
<td>Penleigh and Essendon Grammar School VIC</td>
<td>Year 11</td>
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2016 Australian IMO Team, above from left, Ilia Kucherov, Jongmin Lim, Kevin Xian, and Wilson Zhao. Top right, Michelle Chen and bottom right, Seyoon Ragavan.
Participants at the 2015 IMO Selection School

<table>
<thead>
<tr>
<th>Name</th>
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<tr>
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<td>Linus Cooper</td>
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<td>Yong See Foo</td>
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<td>Ilia Kucherov</td>
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<td>Charles Li</td>
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<td>Jack Liu</td>
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<td>Seyoon Ragavan</td>
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<td>Kevin Xian</td>
<td>James Ruse Agricultural High School Guowen Zhang</td>
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<td>Yasiru Jayasooriya</td>
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<td>Jeff (Zefeng) Li</td>
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<td>William Li</td>
<td>Barker College NSW</td>
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<td>Adrian Lo</td>
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<td>Forbes Mailler</td>
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<td>Oliver Papillo</td>
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<td>Anthony Tew</td>
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<td>Ruquian Tong</td>
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For the week preceding the IMO, our team met with our British counterparts for a final dose of training and acclimatisation to the IMO conditions. This year we stayed at a lovely hotel in Tagaytay, about 50km south of central Manila. I was accompanied by Mike Clapper and his wife Jo Cockwill for the entire camp, which was very helpful, especially given Angelo’s absence. The British delegation at the camp consisted of the 6 team members, their deputy leader, Dominic, and observer, Jill.

The Seven of us who came directly from Australia (myself, Mike, Jo and four of our team members) all had early morning flights, but otherwise nothing to complain about. Our other two team members came directly from MOP, the training camp in the US, so they were understandably a bit more jet lagged than the rest of us. Unfortunately, their flight to Manila was delayed, so we didn’t arrive at the Hotel until about midnight local time. There was no stopping the training program however, we started the first of five four and a half hour training exams the following morning.

The next four days all started the same way, with an IMO style exam for all 12 students. After the exam each morning, they were given free time for the rest of the day, during which they played games or, weather permitting, visited some local attractions. For most of the week we got constant, heavy rain. On one of the days with relatively light rain, we all visited Peoples Park in the Sky, a location from which we could apparently get a good view of the area. At first the surrounding clouds only allowed us a view of a few metres in each direction, much like our first visit to Table Mountain in 2014. Eventually the clouds cleared up enough to take a few decent photos of both of our teams.

As tradition dictates, the fifth and final exam was designated the mathematical ashes, in which our teams compete for glory and an urn containing burnt remains of some British scripts from 2008. The UK had a very strong team this year, and sadly it showed as they defeated us 82–74, thereby retaining the ashes for the 8th year in a row. This was an impressive performance from both teams on what could have easily been a real IMO paper. I’m sure that our team will fight valiantly to reclaim the urn next year.

Once again, this camp was a great way to wind up both teams’ training before the IMO. The co-training experience was hugely beneficial for all involved, as the students learnt a lot from each other, and Dominic and I learnt from each other’s training techniques as well. This tradition will hopefully continue well into the future. Many thanks to everyone who made this a success, in particular, the UKMT, the entire UK delegation as well as Dr Simon Chua and Dr Joseph Wu, who helped with local organisation and dessert suggestions.

Andrew Elvey Price
IMO Deputy Leader
The 2016 Mathematical Ashes: AUS v UK

Exam

1. Let $ABC$ be an acute triangle with orthocentre $H$. Let $G$ be the point such that the quadrilateral $ABGH$ is a parallelogram ($AB \parallel GH$ and $BG \parallel AH$). Let $I$ be the point on the line $GH$ such that $AC$ bisects $HI$. Suppose that the line $AC$ intersects the circumcircle of the triangle $GCI$ at $C$ and $J$.

Prove that $IJ = AH$.

2. Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{ka_k}{a_k^2 + k - 1}$$

for every positive integer $k$.

Prove that

$$a_1 + a_2 + \cdots + a_n \geq n$$

for every $n \geq 2$.

3. Let $S$ be a nonempty set of positive integers. We say that a positive integer $n$ is clean if it has a unique representation as a sum of an odd number of distinct elements of $S$.

Prove that there exist infinitely many positive integers that are not clean.
THE MATHEMATICS ASHES RESULTS

The 9th Mathematics Ashes competition at the joint pre-IMO training camp in Tagaytay, was won by Britain; the results for the two teams were as follows, with Australia scoring a total of 74 and Britain scoring a total of 82:

**Australia**

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**United Kingdom**

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<td>42</td>
<td>27</td>
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The 57th International Mathematical Olympiad (IMO) was held 6–16 July in Hong Kong. This was easily the largest IMO in history with a record number of 602 high school students from 109 countries participating. Of these, 71 were girls. Each participating country may send a team of up to six students, a Team Leader and a Deputy Team Leader. At the IMO the Team Leaders, as an international collective, form what is called the Jury. This Jury was chaired by Professor Kar-Ping Shum who was ably assisted by the much younger Andy Loo.¹

The first major task facing the Jury is to set the two competition papers. During this period the Leaders and their observers are trusted to keep all information about the contest problems completely confidential. The local Problem Selection Committee had already shortlisted 32 problems from 121 problem proposals submitted by 40 of the participating countries from around the world. During the Jury meetings three of the shortlisted problems had to be discarded from consideration due to being too similar to material already in the public domain. Eventually, the Jury finalised the exam questions and then made translations into the more than 50 languages required by the contestants.

The six questions that ultimately appeared on the IMO contest are described as follows.

1. An easy classical geometry problem proposed by Belgium.
2. A medium chessboard style problem proposed by Australia.
3. A difficult number theory problem with a hint of geometry proposed by Russia.
4. An easy number theory problem proposed by Luxembourg.
5. A medium polynomial problem proposed by Russia.
6. A difficult problem in combinatorial geometry proposed by the Czech Republic.

These six questions were posed in two exam papers held on Monday 11 July and Tuesday 12 July. Each paper had three problems. The contestants worked individually. They were allowed four and a half hours per paper to write their attempted proofs. Each problem was scored out of a maximum of seven points.

For many years now there has been an opening ceremony prior to the first day of competition. A highlight was the original music performances specially composed for the IMO by composer Dr Kwong-Chiu Mui. Following the formal speeches there was the parade of the teams and the 2016 IMO was declared open.

After the exams the Leaders and their Deputies spent about two days assessing the work of the students from their own countries, guided by marking schemes, which been agreed to earlier. A local team of markers called Coordinators also assessed the papers. They too were guided by the marking schemes but are allowed some flexibility if, for example, a Leader brought something to their attention in a contestant’s exam script that is not covered by the marking scheme. The Team Leader and Coordinators have to agree on scores for each student of the Leader’s country in order to finalise scores. Any disagreements that cannot be resolved in this way are ultimately referred to the Jury.

Problem 1 turned out to be the most accessible with an average score of 5.27. At the other end, problem 3 ended up being one of the most difficult problems at the IMO averaging only 0.25. Just 10 students managed to score full marks on it, while 548 students were unable to score a single point.

The medal cuts were set at 29 for Gold, 22 for Silver and 16 for Bronze. Consequently, there were 280 (=46.5%) medals awarded. The medal distributions² were 44 (=7.3%) Gold, 101 (=16.8%) Silver and 135 (=22.4%) Bronze. These awards were presented at the closing ceremony. Of those who did not get a medal, a further 162 contestants received an Honourable Mention for solving at least one question perfectly.

¹ Not to be confused with the well known Canadian mathematician Andy Liu.
² One must go back to IMO 2009 to find a problem that scored lower. Problem 6 of that year averaged 0.17 out of 7.
³ The total number of medals must be approved by the Jury and should not normally exceed half the total number of contestants. The numbers of gold, silver and bronze medals must be approximately in the ratio 1:2:3.
The following six students achieved the most excellent feat of a perfect score of 42.

Yuan Yang  China
Jaewon Choi  South Korea
Eui Cheon Hong  South Korea
Junghun Ju  South Korea
Allen Liu  United States
Yuan Yao  United States

They were given a standing ovation during the presentation of medals at the closing ceremony.

Congratulations to the Australian IMO team on their solid performance this year. They finished equal 25th in the rankings, bringing home two Silver and four Bronze medals.

The two Silver medallists were Jongmin Lim, year 12, Killara High School, NSW, and Kevin Xian, year 12, James Ruse Agricultural High School, NSW.

The Bronze medallists were Michelle Chen, year 12, Methodist Ladies College, VIC, Ilia Kucherov, year 12, Westall Secondary College, VIC, Seyoon Ragavan, year 12, Knox Grammar School, NSW, and Wilson Zhao, year 12, Killara High School, NSW.

All members of the 2016 Australian IMO will finish year 12 this year. So it will be interesting to see how we do in 2017 with a completely new team.

Congratulations also to Trevor Tao, himself a former IMO medallist. He was the author of IMO problem 2.

The 2016 IMO was organised by: The International Mathematical Olympiad Hong Kong Committee Limited with support from the Hong Kong University of Science and Technology and the Education Bureau of the Hong Kong SAR Government.

The 2017 IMO is scheduled to be held July 12–24 in Rio de Janeiro, Brazil. Venues for future IMOs have been secured up to 2021 as follows:

- 2018 Romania
- 2019 United Kingdom
- 2020 Russia
- 2021 United States

Much of the statistical information found in this report can also be found at the official website of the IMO.

www.imo-official.org

Angelo Di Pasquale
IMO Team Leader, Australia

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4 The ranking of countries is not officially part of the IMO general regulations. However, countries are ranked each year on the IMO's official website according to the sum of the individual student scores from each country.

5 Trevor was a Bronze medallist with the 1995 Australian IMO team. He is also a brother of world renowned mathematician Terence Tao.
Monday, July 11, 2016

**Problem 1.** Triangle $BCF$ has a right angle at $B$. Let $A$ be the point on line $CF$ such that $FA = FB$ and $F$ lies between $A$ and $C$. Point $D$ is chosen such that $DA = DC$ and $AC$ is the bisector of $\angle DAB$. Point $E$ is chosen such that $EA = ED$ and $AD$ is the bisector of $\angle EAC$. Let $M$ be the midpoint of $CF$. Let $X$ be the point such that $AMXE$ is a parallelogram (where $AM || EX$ and $AE || MX$). Prove that lines $BD$, $FX$, and $ME$ are concurrent.

**Problem 2.** Find all positive integers $n$ for which each cell of an $n \times n$ table can be filled with one of the letters $I$, $M$ and $O$ in such a way that:

- in each row and each column, one third of the entries are $I$, one third are $M$ and one third are $O$; and
- in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are $I$, one third are $M$ and one third are $O$.

**Note:** The rows and columns of an $n \times n$ table are each labelled 1 to $n$ in a natural order. Thus each cell corresponds to a pair of positive integers $(i, j)$ with $1 \leq i, j \leq n$. For $n > 1$, the table has $4n - 2$ diagonals of two types. A diagonal of the first type consists of all cells $(i, j)$ for which $i + j$ is a constant, and a diagonal of the second type consists of all cells $(i, j)$ for which $i - j$ is a constant.

**Problem 3.** Let $P = A_1A_2\ldots A_k$ be a convex polygon in the plane. The vertices $A_1$, $A_2$, $\ldots$, $A_k$ have integral coordinates and lie on a circle. Let $S$ be the area of $P$. An odd positive integer $n$ is given such that the squares of the side lengths of $P$ are integers divisible by $n$. Prove that $2S$ is an integer divisible by $n$. 

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points
**Problem 4.** A set of positive integers is called *fragrant* if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let \( P(n) = n^2 + n + 1 \). What is the least possible value of the positive integer \( b \) such that there exists a non-negative integer \( a \) for which the set \( \{P(a + 1), P(a + 2), \ldots, P(a + b)\} \) is fragrant?

**Problem 5.** The equation
\[
(x - 1)(x - 2) \cdots (x - 2016) = (x - 1)(x - 2) \cdots (x - 2016)
\]
is written on the board, with 2016 linear factors on each side. What is the least possible value of \( k \) for which it is possible to erase exactly \( k \) of these 4032 linear factors so that at least one factor remains on each side and the resulting equation has no real solutions?

**Problem 6.** There are \( n \geq 2 \) line segments in the plane such that every two segments cross, and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it, facing the other endpoint. Then he will clap his hands \( n - 1 \) times. Every time he claps, each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

(a) Prove that Geoff can always fulfil his wish if \( n \) is odd.

(b) Prove that Geoff can never fulfil his wish if \( n \) is even.
1. **Solution 1** (Kevin Xian, year 12, James Ruse Agricultural High School, NSW. Kevin was a Silver medallist with the 2016 Australian IMO team.)

From the problem statement it easily follows that \( \triangle AFB, \triangle ADC, \) and \( \triangle AED \) are similar isosceles triangles. Thus, as shown in the diagram, we may let

\[
\alpha = \angle FAB = \angle ABF = \angle DAC = \angle ACD = \angle EAD = \angle ADE.
\]

From this it follows that \( ED \parallel AC. \) But \( AC \parallel EX. \) Hence \( D \) lies on the line \( EX. \)

From \( \triangle ADC \sim \triangle AFB, \) we have \( AD/AC = AF/AB. \) Since also \( \angle DAF = \angle CAB, \) it follows that \( \triangle ADF \sim \triangle ACB \) (PAP). Thus

\[
\angle AFD = \angle ABC = 90^\circ + \alpha \Rightarrow \angle DFC = 90^\circ - \alpha \Rightarrow \angle CDF = 90^\circ. \quad \text{(angle sum } \triangle CDF)\]

Thus the circle with diameter \( CF \) passes through points \( B \) and \( D. \) Since \( M \) is the midpoint of \( CF, \) it follows that \( M \) is the centre of this circle. Therefore

\[
MB = MC = MD = MF.
\]

We also have

\[
\angle MXD = \angle EAM \quad \text{(parallelogram } AMXE) = 2\alpha = 2\angle FCD = \angle FMD \quad \text{(angle at centre of circle } BCDF) = \angle XDM. \quad \text{(AM } \parallel EX)\]

Hence \( MX = MD. \)

From parallelogram \( AMXE, \) we have \( MX = AE. \) We are also given \( AE = DE. \) Hence

\[
MB = MC = MD = MF = MX = AE = DE.
\]
From $MF = DE$ and $MF \parallel DE$, it follows that $FMDE$ is a parallelogram.

From parallelograms $AMXE$ and $FMDE$ we have $\overrightarrow{XE} = \overrightarrow{MA}$ and $\overrightarrow{DE} = \overrightarrow{MF}$.

Thus $XD = FA$, and so $AFXD$ is also a parallelogram.

Also since $CM = DE$ and $CM \parallel DE$, we have $MCDE$ is also a parallelogram.

Let $K$ be the intersection of lines $FX$ and $ME$. We have

$$\angle FKE = \angle FMK + \angle KFM \quad (\text{exterior angle sum } \triangle KMF)$$
$$= \angle ACD + \angle DAC \quad (ME \parallel CD \text{ and } FX \parallel AD)$$
$$= \angle ABF + \angle FAB$$
$$= \angle MFB \quad (\text{exterior angle sum } \triangle AFB)$$
$$= \angle FBM. \quad (MF = MB)$$

Therefore quadrilateral $BMKF$ is cyclic.

From here we compute

$$\angle FBK = \angle FMK \quad (BMKF \text{ cyclic})$$
$$= \angle FCD \quad (ME \parallel CD)$$
$$= \angle FBD. \quad (BCDF \text{ cyclic})$$

Since $\angle FBK = \angle FBD$, it follows that points $B$, $K$, and $D$ are collinear. Hence $BD$, $FX$, and $ME$ all pass through $K$. \qed
Solution 2  (Michelle Chen, year 12, Methodist Ladies’ College, VIC. Michelle was a Bronze medallist with the 2016 Australian IMO team.)

Our strategy is to find three circles for which $BD$, $FX$, and $ME$ are the three radical axes associated which each pair of those circles. The result will then follow from the radical axis theorem.

As in solution 1, we have points $E$, $D$, and $X$ are collinear, and

$$MB = MC = MD = MF = MX = AE.$$  

Thus the pentagon $BCXDF$ is cyclic.

From parallelogram $AMXE$, we have

$$\angle EAM = \angle MXD = \angle XDM.$$  

Hence $AMDE$ is cyclic.

We know $FB = FA$ and $BM = AE$. Also

$$\angle FBM = \angle MFB = \angle ABF + \angle FAB = 2\alpha = \angle EAF.$$  

Thus $\triangle FBM \cong \triangle FAE$ (SAS).

From this we have $\angle MFB = \angle EFA$. Since $AFM$ is a straight line, it follows that $BFE$ is also a straight line.

A further consequence is

$$\angle BMA = \angle BMF = \angle FEA = \angle BEA.$$  

Thus $ABME$ is cyclic.

Since $AMDE$ is also cyclic, we conclude that the pentagon $ABMDE$ is cyclic.

From $EF = MF = EA$, and parallelogram $AMXE$, we have

$$\angle AFE = \angle EAF = \angle MXE.$$  

Hence $EFMX$ is cyclic.

Applying the radical axis theorem to circles $BCXDF$, $ABMDE$, and $EFMX$, yields that $BD$, $FX$, and $ME$ are concurrent, as desired. $\square$
Solution 3  (Based on the solution of Ilia Kucherov, year 12, Westall Secondary College, VIC. Ilia was a Bronze medallist with the 2016 Australian IMO team.)

Refer to the diagram in solution 2. As in solution 1 we deduce the following facts.

(i) $BCDF$ is cyclic with centre $M$.
(ii) $EFMD$ is a parallelogram.
(iii) $AFXD$ is a parallelogram.

From (i) we have
\[ MX = MB. \]  
\[ (1) \]

From (i) and (ii), it follows that $EFMD$ is a rhombus. Thus
\[ ED = EF. \]  
\[ (2) \]

From (iii), we have $DX = AF$. Since we are given $AF = FB$, it follows that $DX = FB$. Since also $ED = EF$, we have
\[ EX = EB. \]  
\[ (3) \]

From (1) and (3) we see that $EXMB$ is a kite that is symmetric in the line $EM$. From (2), the points $D$ and $F$ are also symmetric in $EM$. Therefore lines $DB$ and $FX$ are symmetric in $EM$. It follows that the intersection of $DB$ and $FX$ is symmetric in $EM$, and hence lies on $EM$. This means that $DB$, $FX$, and $EM$ are concurrent. \[ \square \]
2. Answer: All positive integers \( n \) that are multiples of 9.

All solutions to this problem can be divided into the following two steps.

**Step 1** Prove that a valid table is possible whenever \( n \) is a multiple of 9.

**Step 2** Prove that if there is a valid \( n \times n \) table, then \( n \) is a multiple of 9.

**Solution 1** (Ilia Kucherov, year 12, Westall Secondary College, VIC. Ilia was a Bronze medallist with the 2016 Australian IMO team.)

**Step 1** The following configuration shows that \( n = 9 \) is possible.

Next, let us consider the above table to be a single tile. Take an \( m \times m \) array of these tiles. This yields a working \( 9m \times 9m \) table for any positive integer \( m \).

**Step 2** From the problem statement it is obvious that \( 3 \mid n \), so let us write \( n = 3k \). We shall count the total the number of occurrences of the letter \( M \) in the table in two different ways.

Partition the \( 3k \times 3k \) table into \( k^2 \) boxes, each of dimensions \( 3 \times 3 \), in the natural way. Colour each unit square blue that lies at the centre of any such \( 3 \times 3 \) box. The diagram below illustrates the case for \( n = 12 \).

Call a row, column, or diagonal *good* if it contains at least one blue square. Note that the good diagonals are those whose number of squares is a multiple of three.
Let $A$ be the number of Ms that lie in north-east facing good diagonals. The union of these diagonals intersects each $3 \times 3$ box in exactly three squares for a total of $3k^2$ squares. Since one third of these contain the letter $M$, we have $A = k^2$.

Let $B$ be the number of Ms that lie in north-west facing good diagonals. Let $C$ be the number of Ms that lie in good columns. Let $D$ be the number of Ms that lie in good rows. By similar arguments we deduce that $B = C = D = k^2$.

Summing yields $A + B + C + D = 4k^2$. In this way, we have accounted for every square in the table exactly once except for the blue squares that have been accounted for exactly four times. It follows that the number of Ms in the table is $4k^2 - 3c$, where $c$ is the number of blue squares containing the letter $M$.

On the other hand, the number of Ms in the table is simply one third of the total number of squares, that is, $3k^2$. Equating these two expressions, we have

$$4k^2 - 3c = 3k^2$$
$$\Rightarrow \quad k^2 = 3c.$$

From this it follows that $3 \mid k$. Thus $9 \mid n$, as claimed. \qed
Solution 2  (Seyoon Ragavan, year 12, Knox Grammar School, NSW. Seyoon was a Bronze medallist with the 2016 Australian IMO team.)

Step 1  The following configuration shows that $n = 9$ is possible.

\[
\begin{array}{cccccccc}
I & M & O & O & I & M & M & O & I \\
M & M & I & I & I & O & O & O & I \\
I & M & O & O & I & M & M & O & I \\
O & I & M & M & O & I & I & M & O \\
I & I & I & O & O & M & M & M & M \\
O & I & M & M & O & I & I & M & O \\
M & M & I & I & M & O & O & I & M \\
O & O & O & M & M & I & I & I & I \\
M & O & I & I & M & O & O & I & M \\
\end{array}
\]

As in solution 1, we consider the above table to be a single tile and take an $m \times m$ array of these tiles to obtain a working $9m \times 9m$ table for any positive integer $m$.

The problem statement refers to the following four families.

(i) Rows
(ii) Columns
(iii) North-east facing diagonals whose number of squares is a multiple of three
(iv) North-west facing diagonals whose number of squares is a multiple of three

This motivates the following. Write the number $j$ in a square if it is accounted for in $j$ of the above families. For visual enhancement, we have coloured the squares. The 2s are in white squares, the 3s are in red squares, and the 4s are in blues squares.

\[
\begin{array}{cccccccc}
3 & 2 & 3 & 3 & 2 & 3 & 3 & 2 & 3 \\
2 & 4 & 2 & 2 & 4 & 2 & 2 & 4 & 2 \\
3 & 2 & 3 & 3 & 2 & 3 & 3 & 2 & 3 \\
3 & 2 & 3 & 3 & 2 & 3 & 3 & 2 & 3 \\
2 & 4 & 2 & 2 & 4 & 2 & 2 & 4 & 2 \\
3 & 2 & 3 & 3 & 2 & 3 & 3 & 2 & 3 \\
3 & 2 & 3 & 3 & 2 & 3 & 3 & 2 & 3 \\
2 & 4 & 2 & 2 & 4 & 2 & 2 & 4 & 2 \\
3 & 2 & 3 & 3 & 2 & 3 & 3 & 2 & 3 \\
\end{array}
\]

Let $x$, $y$, and $z$ be the number of Ms on white, red, and blue squares, respectively.
To find the number of Ms in a family we simply count the total number of squares in the family and divide the result by three. Thus the total number of Ms in family
(i) is $3k^2$. For family (ii) it is $3k^2$. For family (iii) it is $k^2$. For family (iv) it is $k^2$. If we add all of these together we see that the white squares have been accounted for twice, the red squares three times, and the blue squares four times. Hence

$$2x + 3y + 4z = 8k^2. \tag{1}$$

On the other hand, the total number of Ms in rows $2, 5, 8, \ldots, 3k - 1$ is $k^2$, and the total number of Ms in columns $2, 5, 8, \ldots, 3k - 1$ is also $k^2$. Adding these together we find that the white squares have been accounted for once, the red squares are completely missed, and the blue squares have been accounted for twice. This yields

$$x + 2z = 2k^2. \tag{2}$$

Computing $(1) - 2 \times (2)$ yields $3y = 4k^2$. Hence $k \mid 3$, as claimed. \hfill \Box
3. This was the hardest problem of the 2016 IMO. Only 10 of the 602 contestants were able to solve this problem completely.

**Solution 1** (Adapted from the solution provided by the Problem Selection Committee by Angelo Di Pasquale, Leader of the 2016 Australian IMO team)

By Pick’s theorem, the area of any polygon whose vertices are lattice points has area that is equal to an integer multiple of \( \frac{1}{2} \). Therefore, \( 2S \) is an integer. Note that the squares of the lengths of all sides and diagonals of \( P \) are positive integers.

If the conclusion of the problem is true for \( n = a \) and \( n = b \) where \( \gcd(a, b) = 1 \), then it is also true for \( n = ab \). Thus it is enough to provide a proof for the cases \( n = p^j \) where \( p \) is an odd prime and \( t \) is a positive integer. We shall prove by strong induction on \( k \geq 3 \) that \( 2S \) is divisible by \( n = p^k \).

For the base case \( k = 3 \), let the side lengths of \( P \) be \( \sqrt{ab}, \sqrt{nb}, \sqrt{nc} \) where \( a, b, c \) are positive integers. By Heron’s formula we have

\[
16S^2 = n^2(2ab + 2bc + 2ac - a^2 - b^2 - c^2).
\]

Thus \( n^2 \mid 16S^2 \). Since \( n \) is odd, we have \( n \mid 2S \), as desired.

For the inductive step, let us assume that the conclusion of the problem is true for \( k = 3, 4, \ldots, j - 1 \), where \( j \geq 4 \). Let \( P \) be a cyclic lattice polygon with \( j \) sides.

**Case 1** There is a diagonal, the square of whose length is divisible by \( p^j \).

The aforesaid diagonal divides \( P \) into two smaller polygons \( P_1 \) and \( P_2 \). Applying the inductive assumption to \( P_1 \) and \( P_2 \) yields that the areas of \( P_1 \) and \( P_2 \) are each divisible by \( p^j \). Hence the area of \( P \) is divisible by \( p^j \), as required.

**Case 2** There is no diagonal, the square of whose length is divisible by \( p^j \).

Let \( u \) be the largest non-negative integer such that \( p^u \) divides the squares of the lengths of all diagonals. Thus there is a diagonal, without loss of generality let it be \( A_1A_i \) where \( 3 \leq i \leq j - 1 \), with \( \nu_p(A_1A_i^2) = u \).

**Lemma** Suppose that \( a + b = c \), where \( a^2, b^2, \) and \( c^2 \) are positive integers. Then we have \( \nu_p(c^2) \geq \nu_p(a^2), \nu_p(b^2) \).

**Proof** We have \( (a + b + c)(b + c - a)(c + a - b)(a + b - c) = 0 \), and so

\[
2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 = c^4.
\]

If \( \nu_p(c^2) < \nu_p(a^2), \nu_p(b^2) \), then \( \nu_p(\text{RHS}) < \nu_p(\text{LHS}) \), a contradiction. \( \square \)

Applying Ptolemy’s theorem to the quadrilateral \( A_1A_2A_iA_j \), we have

\[
A_1A_2 \cdot A_iA_j + A_1A_j \cdot A_2A_i = A_1A_i \cdot A_2A_j.
\]

But the above equation contradicts the lemma with \( a = A_1A_2 \cdot A_iA_j, b = A_1A_j \cdot A_2A_i, \) and \( c = A_1A_i \cdot A_2A_j \). Indeed, we have

\[
\nu_p(A_1A_2^2 \cdot A_iA_j^2) \geq t + u, \quad \nu_p(A_1A_j^2 \cdot A_2A_i^2) \geq t + u, \quad \text{and} \quad \nu_p(A_1A_i^2 \cdot A_2A_j^2) < u + t.
\]

This shows that case 2 does not occur, and thus completes the proof. \( \square \)

\(^1\)For a prime \( p \) and an integer \( m \), \( \nu_p(m) \) denotes the exponent of \( p \) in the prime factorisation of \( m \).
Solution 2  (Angelo Di Pasquale, Leader of the 2016 Australian IMO team)

As in solution 1, it suffices to prove the problem for the cases \( n = p^t \) where \( p \) is an odd prime and \( t \) is a positive integer.

**Lemma**  For \( n = p^t \) as above, it is possible to triangulate \( P \) so that the squares of the side lengths of each triangle are divisible by \( p^t \).

**Proof**  We shall prove this by induction on \( k \geq 3 \).

The base case \( k = 3 \) is immediate. For the inductive step, assume the lemma is true for \( k = 3, 4, \ldots, j - 1 \), where \( j \geq 4 \).

Let \( u \) be the greatest integer such that the square of the length of at least one diagonal is divisible by \( p^u \). Consider one such diagonal, such that the square of its length is divisible by \( p^u \). This diagonal divides \( P \) into two smaller polygons \( P_1 \) and \( P_2 \). We may apply the inductive assumption to \( P_1 \) and \( P_2 \) to deduce that each of \( P_1 \) and \( P_2 \) can be triangulated so that the squares of the side lengths of each triangle are divisible by \( p^\min(t,u) \). This yields a triangulation of \( P \) itself such that the squares of the side lengths of each triangle are divisible by \( p^\min(t,u) \).

If \( u \geq t \), then the inductive step is complete.

If \( u < t \), then all the diagonals involved in the triangulation of \( P \) have the squares of their lengths divisible by \( p^u \) but not \( p^{u+1} \). It is well known that any triangulation of a polygon contains at least one ear.\(^2\) Without loss of generality let \( \triangle A_jA_1A_2 \) be such an ear. Let \( \triangle A_2A_iA_j \) be the other triangle in the triangulation that shares the side \( A_2A_j \). Applying Ptolemy’s theorem to the quadrilateral \( A_1A_2A_iA_j \), we have

\[
A_1A_2 \cdot A_iA_j + A_1A_j \cdot A_2A_i = A_1A_i \cdot A_2A_j.
\]

But the above equation is incompatible with the lemma proved in solution 1 with \( a = A_1A_2 \cdot A_iA_j \), \( b = A_1A_j \cdot A_2A_i \), and \( c = A_1A_i \cdot A_2A_j \). Indeed, we have

\[
\nu_p(A_1A_2^2 \cdot A_iA_j^2) \geq t + u, \quad \nu_p(A_1A_i^2 \cdot A_2A_j^2) \geq t + u, \quad \text{and} \quad \nu_p(A_1A_i^2 \cdot A_2A_j^2) \leq 2u.
\]

This shows that \( u < t \) is impossible, completing the proof of the lemma. \( \square \)

Finally, to complete the proof of the problem, triangulate \( P \) as postulated in the lemma. By Heron’s formula, as in solution 1, twice the area of each triangle is divisible by \( n \). Summing over all triangles yields that \( 2S \) is divisibly by \( n \). \( \square \)

\(^2\)For a triangulated polygon, an *ear* is a triangle that contains two consecutive sides of the polygon.
Comment  For any cyclic $n$-gon (not just one whose vertices are lattice points), there exists a multivariable polynomial $f$ with integer coefficients such that

$$f(16S^2, a_1^2, a_2^2, \ldots, a_n^2) = 0.$$  

Here $S$ is the area of the $n$-gon, and $a_1, a_2, \ldots, a_n$ are the lengths of its sides. Any such polynomial $f$ is called a *generalised Heron polynomial*. It is known that if $S$ is assigned degree 2 and the $a_i$ are each assigned degree 1, then $f$ is homogeneous. Furthermore it is known that $f$ is monic in its first variable.$^3$

$^3$These results are non-trivial. See *Comments on generalized Heron polynomials and Robbins’ conjectures* by Robert Connelly. (2009)
Lemma  For any integer \( n \) we have the following.

\[
\begin{align*}
gcd(P(n), P(n + 1)) &= 1 \quad (1) \\
gcd(P(n), P(n + 2)) > 1 &\implies n \equiv 2 \pmod{7} \quad (2) \\
gcd(P(n), P(n + 3)) > 1 &\implies n \equiv 1 \pmod{3} \quad (3)
\end{align*}
\]

Proof  First we note that \( P(n) = n^2 + n + 1 \) is always odd because \( n^2 + n = n(n+1) \) is even. Thus any prime factor of \( P(n) \) is odd.

For (1), suppose that \( P(n) = n^2 + n + 1 \) and \( P(n + 1) = n^2 + 3n + 3 \) are both divisible by \( p \) for some odd prime \( p \). Then \( p \mid P(n + 1) - P(n) = 2n + 2 \). Since \( p \) is odd, we have \( p \mid n + 1 \). But \( p \mid P(n + 1) = (n + 1)^2 + (n + 1) + 1 \). Hence \( p \mid 1 \). This contradiction establishes (1).

For (2), suppose that \( P(n) = n^2 + n + 1 \) and \( P(n + 2) = n^2 + 5n + 7 \) are both divisible by \( p \) for some odd prime \( p \). Then \( p \mid P(n + 2) - P(n) = 4n + 6 \). Since \( p \) is odd, we have \( p \mid 2n + 3 \), and so

\[
2n \equiv -3 \pmod{p}. \tag{*}
\]

However, since \( p \mid P(n) \), we have

\[
\begin{align*}
n^2 + n + 1 &\equiv 0 \pmod{p} \\
2n^2 + 2n + 2 &\equiv 0 \pmod{p} \\
n(-3) + (-3) + 2 &\equiv 0 \pmod{p} \quad \text{(from \(*\))} \\
3n + 1 &\equiv 0 \pmod{p} \\
6n + 2 &\equiv 0 \pmod{p} \\
3(-3) + 2 &\equiv 0 \pmod{p} \quad \text{(from \(*\))} \\
p &\equiv 7.
\end{align*}
\]

Furthermore, from \((*)\) we have

\[
2n \equiv -3 \pmod{7} \implies 4 \pmod{7} \implies n \equiv 2 \pmod{7}.
\]

Thus we have established (2).

For (3), suppose that \( P(n) = n^2 + n + 1 \) and \( P(n + 3) = n^2 + 7n + 13 \) are both divisible by \( p \) for some odd prime \( p \). Then \( p \mid P(n + 3) - P(n) = 6n + 12 \).

If \( p \neq 3 \), this would imply \( p \mid n + 2 \), so that \( n \equiv -2 \pmod{p} \). But then we would have \( P(n) \equiv (-2)^2 + (-2) + 1 \equiv 3 \pmod{p} \). Since \( p \mid P(n) \), this implies \( p = 3 \), which is a contradiction.

Thus \( p = 3 \), and so \( n^2 + n + 1 \equiv 0 \pmod{3} \). Observe that \( P(n) \equiv 1 \pmod{3} \) for \( n \equiv 0, 2 \pmod{3} \), and \( P(n) \equiv 0 \pmod{3} \) for \( n \equiv 1 \pmod{3} \). Thus \( n \equiv 1 \pmod{3} \). Hence (3) is also true, completing the proof of the lemma.

Returning to the problem, we first show that no fragrant set exists for \( b \leq 5 \).

Answer: \( b = 6 \)
For reference, let 

\[ S = \{P(a + 1), P(a + 2), \ldots, P(a + b)\}. \]

**Case 1** \( b = 2 \) or \( 3 \)

From (1), we have \( \gcd(P(a + 2), P(a + 1)) = \gcd(P(a + 2), P(a + 3)) = 1 \). Thus \( S \) is not fragrant.

**Case 2** \( b = 4 \)

From (1), we have \( \gcd(P(a + 2), P(a + 1)) = \gcd(P(a + 3), P(a + 4)) = 1 \).

If \( S \) were fragrant then we would require \( \gcd(P(a + 2), P(a + 4)) > 1 \). From (2), this implies \( a + 2 \equiv 2 \pmod{7} \).

From (1) we also have \( \gcd(P(a + 3), P(a + 2)) = \gcd(P(a + 3), P(a + 4)) = 1 \).

So for \( S \) to be fragrant we would also require \( \gcd(P(a + 3), P(a + 1)) > 1 \). Thus from (2), we have \( a + 1 \equiv 2 \pmod{7} \). This is incompatible with \( a + 2 \equiv 2 \pmod{7} \) found earlier. Thus \( S \) is not fragrant.

**Case 3** \( b = 5 \)

From (1), we have \( \gcd(P(a + 3), P(a + 2)) = \gcd(P(a + 3), P(a + 4)) = 1 \).

If \( S \) were fragrant, we would require \( P(a + 3) \) to share a common factor with at least one of \( P(a + 1) \) or \( P(a + 5) \). From (2) this implies that either \( a + 1 \equiv 2 \pmod{7} \) or \( a + 3 \equiv 2 \pmod{7} \).

If \( \gcd(P(a + 2), P(a + 4)) > 1 \), then from (2) we would have \( a + 2 \equiv 2 \pmod{7} \). This is incompatible with \( a + 1 \equiv 2 \pmod{7} \) found in the previous paragraph. Therefore \( \gcd(P(a + 2), P(a + 4)) = 1 \).

From (1), we have \( \gcd(P(a + 2), P(a + 1)) = \gcd(P(a + 2), P(a + 3)) = 1 \). Recall also \( \gcd(P(a + 2), P(a + 4)) = 1 \). Hence for \( S \) to be fragrant, we would require \( \gcd(P(a + 2), P(a + 5)) > 1 \). From (3) this implies \( a + 2 \equiv 1 \pmod{3} \).

But from (1), we also have \( \gcd(P(a + 4), P(a + 3)) = \gcd(P(a + 4), P(a + 5)) = 1 \). Recall also \( \gcd(P(a + 4), P(a + 2)) = 1 \). Hence for \( S \) to be fragrant, we would require \( \gcd(P(a + 4), P(a + 1)) > 1 \). From (3) this implies \( a + 1 \equiv 1 \pmod{3} \). This is incompatible with \( a + 2 \equiv 1 \pmod{3} \) found earlier. Thus \( S \) is not fragrant.

Finally, we show that a fragrant set exists for \( b = 6 \).

Consider the following system of simultaneous congruences.

\[
\begin{align*}
a & \equiv 1 \pmod{3} \\
& a \equiv 0 \pmod{7} \\
& a \equiv 6 \pmod{19}
\end{align*}
\]

Since 3, 7, and 19 are pairwise coprime, it follows by the Chinese remainder theorem that the above system has a solution modulo \( 3 \times 7 \times 19 = 399 \). By adding multiples of 399 to any such solution we can ensure that \( a \) is a positive integer. It is now a simple matter to compute the following.

\[
\begin{align*}
P(a + 1) & \equiv P(a + 5) \equiv 0 \pmod{19} \\
P(a + 2) & \equiv P(a + 4) \equiv 0 \pmod{7} \\
P(a + 3) & \equiv P(a + 6) \equiv 0 \pmod{3}
\end{align*}
\]

Hence \( \{P(a + 1), P(a + 2), P(a + 3), P(a + 4), P(a + 5), P(a + 6)\} \) is fragrant, as desired. \( \square \)
Solution 2  (Based on the solution of Jongmin Lim, year 12, Killara High School, NSW. Jongmin was a Silver medallist with the 2016 Australian IMO team.)

Lemma  For any integer \( n \) we have the following.

\[
\begin{align*}
\gcd(P(n), P(n+1)) &= 1 \quad (1) \\
\gcd(P(n), P(n+2)) > 1 &\iff n \equiv 2 \pmod{7} \quad (2) \\
\gcd(P(n), P(n+3)) > 1 &\iff n \equiv 1 \pmod{3} \quad (3) \\
\gcd(P(n), P(n+4)) > 1 &\iff n \equiv 7 \pmod{19} \quad (4)
\end{align*}
\]

Proof  The proof of (1), and the proofs of the “\( \Rightarrow \)” directions of (2) and (3) can be carried out as in solution 1. The proof of the “\( \Rightarrow \)” direction of (4) is as follows. Suppose that \( P(n) = n^2 + n + 1 \) and \( P(n+4) = n^2 + 9n + 21 \) are both divisible by \( p \) for some odd prime \( p \). Then \( |P(n+4) - P(n)| = 8n + 20 \). Since \( p \) is odd, we have \( p | 2n + 5 \). It follows that \( p | 4(n^2 + n + 1) - (2n + 5)(2n - 3) = 19 \). Thus \( p = 19 \). Furthermore since \( p | 2n + 5 \), we have \( 19 | 2n + 5 \). Thus \( 19 | 2n - 14 \) from which we deduce \( 19 | n - 7 \).

The proofs of the “\( \Leftarrow \)” directions of (2), (3), and (4) are straightforward computations. For example, for (2) we verify that \( P(n) \equiv P(n+2) \equiv 0 \pmod{7} \) whenever \( n \equiv 2 \pmod{7} \). Similar computations apply for (3) and (4). \( \square \)

The proof that no fragrant set exists for \( b \leq 5 \) can be carried out as in solution 1. To find a fragrant set for \( b = 6 \), it is sufficient to observe that if we can partition the set \( \{a+1, a+2, a+3, a+4, a+5, a+6\} \) into three disjoint pairs so that the respective differences between elements of each pair are 2, 3, and 4, then we can apply (2), (3), and (4) and the Chinese remainder theorem to find a working value of \( x \). By inspection there are only two different ways to do this.

Way 1  \((a+2, a+4), (a+3, a+6), \) and \((a+1, a+5)\).

These pairs respectively correspond to the following congruences.

\[
\begin{align*}
a &\equiv 0 \pmod{7} \\
a &\equiv 1 \pmod{3} \\
a &\equiv 6 \pmod{19}
\end{align*}
\]

Solving yields \( a \equiv 196 \pmod{399} \).

Way 2  \((a+3, a+5), (a+1, a+4), \) and \((a+2, a+6)\).

These pairs respectively correspond to the following congruences.

\[
\begin{align*}
a &\equiv 6 \pmod{7} \\
a &\equiv 0 \pmod{3} \\
a &\equiv 5 \pmod{19}
\end{align*}
\]

Solving yields \( a \equiv 195 \pmod{399} \).

Thus \( a = 195 \) and \( a = 196 \) are two possible solutions for \( b = 6 \). \( \square \)

Comment  With a just a little more work it can be shown that the full set of solutions for \( b = 6 \) are \( a = 195 + 399k \) and \( a = 196 + 399k \), where \( k \) ranges over the set of positive integers.
5. **Solution 1** (Based on the presentation of Jongmin Lim, year 12, Killara High School, NSW. Jongmin was a Silver medallist with the 2016 Australian IMO team.)

**Answer:** 2016

If \( k \leq 2015 \), then there exists a term \((x - a)\) that is still present on both sides of the equation. This would allow \( x = a \) to be a real solution, which is a contradiction.

It suffices to show that \( k = 2016 \) is possible. From the LHS of the equation let us erase all the terms of the form \((x - a)\) with \( a \equiv 0 \) or 1 (mod 4). From the RHS of the equation let us erase all the terms of the form \((x - a)\) with \( a \equiv 2 \) or 3 (mod 4).

We are left with the following equation

\[
\prod_{i=1}^{504} (x - (4i - 2))(x - (4i - 1)) = \prod_{i=1}^{504} (x - (4i - 3))(x - 4i)
\]

Clearly none of \( x = 1, 2, \ldots, 2016 \) satisfy the above equation. It suffices to show that the equation \( f(x) = 1 \) has no real solutions where

\[
f(x) = \prod_{i=1}^{504} \frac{(x - (4i - 2))(x - (4i - 1))}{(x - (4i - 3))(x - 4i)}
\]

\[
= \prod_{i=1}^{504} \left(1 + \frac{2}{(x - (4i - 3))(x - 4i)}\right).
\]

The zeros of \( f(x) \) are given by \( x = 4i - 2 \) and \( x = 4i - 1 \) for \( i = 1, 2, \ldots, 504 \). The vertical asymptotes of \( f(x) \) are the lines \( x = 4i - 3 \) and \( x = 4i \) for \( i = 1, 2, \ldots, 504 \). Moreover \( f(x) \) is a continuous function everywhere except at these asymptotes.

Although the following picture does not represent a scale diagram of the graph of \( f(x) \), we claim that it does represent the essential features of the graph.
In the subsequent analysis we make repeated use of the following. Suppose that
\( y = (x - a)(x - b) \) for real numbers \( a < b \). Then we have the following situations.

- If \( t \) lies inside the interval \((a, b)\), then \( y < 0 \).
- If \( t \) lies outside the interval \([a, b]\), then \( y > 0 \).
- If \( t \) is constrained to lie in a closed interval \([c, d]\) that is disjoint from \([a, b]\), then the minimum value of \( y \) occurs when \( t \) is at the endpoint of the interval that is closest to \([a, b]\).

**Case 1** \( x < 1 \) or \( x > 2016 \) or \( 4r < x < 4r + 1 \) for some \( r \in \{1, 2, \ldots, 503\} \).
For each \( i \in \{1, 2, \ldots, 504\} \), we have \( x \not\in [4i - 3, 4i] \). Thus \( (x - (4i - 3))(x - 4i) > 0 \). Hence \( f(x) > 1 \) in this case.

**Case 2** \( 4r - 3 < x < 4r - 2 \) or \( 4r - 1 < x < 4r \) for some \( r \in \{1, 2, \ldots, 504\} \).
For \( i \neq r \), we have \( x \not\in [4i - 3, 4i] \). Thus \( (x - (4i - 3))(x - 4i) > 0 \). Hence

\[
1 + \frac{2}{(x - (4i - 3))(x - 4i)} > 1 \quad \text{for all } i \neq r.
\]

For \( i = r \), we have

\[
\lim_{x \to 4r-3^+} 1 + \frac{2}{(x - (4r - 3))(x - 4r)} = -\infty
\]

and

\[
\lim_{x \to 4r^-} 1 + \frac{2}{(x - (4r - 3))(x - 4r)} = -\infty.
\]

Consequently

\[
\lim_{x \to 4r-3^+} f(x) = \lim_{x \to 4r^-} f(x) = -\infty.
\]

Since \( f(x) \) has no further zeros apart from \( x = 4r - 2 \) and \( x = 4r - 1 \) in the interval \([4r - 3, 4r]\), it follows by the intermediate value theorem that \( f(x) < 0 \) in this case.

**Case 3** \( 4r - 2 < x < 4r - 1 \) for some \( r \in \{1, 2, \ldots, 504\} \).

The product for \( f(x) \) may be split into the following three portions.

\[
A = \prod_{i=1}^{r-1} \left(1 + \frac{2}{(x - (4i - 3))(x - 4i)}\right)
\]

\[
B = 1 + \frac{2}{(x - (4r - 3))(x - 4r)}
\]

\[
C = \prod_{i=r+1}^{504} \left(1 + \frac{2}{(x - (4i - 3))(x - 4i)}\right)
\]

We estimate each of \( A \), \( B \) and \( C \).

**Estimation of \( B \)**

Consider the parabola \( y = (x - (4r - 3))(x - 4r) \). Note that \( y < 0 \) for all \( x \) in the interval \((4r - 2, 4r - 1)\). Hence the maximum value of \( B \) occurs at the minimum value of \( y \). This occurs at the parabola’s turning point. By symmetry this is at \( x = 4r - \frac{3}{2} \). Substituting this in yields \( B \leq \frac{1}{9} \).
Estimation of $A$

(The remainder of this proof is by Angelo Di Pasquale, Leader of the 2016 Australian IMO team.)

Since $r > i$, the interval $[4r - 2, 4r - 1]$ lies to the right of the interval $[4i - 3, 4i]$. Hence $y = (x - (4i - 3))(x - 4i) > 0$ for all $x \in [4r - 2, 4r - 1]$, and the minimum value of $y$ occurs when $x = 4r - 2$. It follows that

$$A \leq \prod_{i=1}^{r-1} \left(1 + \frac{2}{(4r - 2 - (4i - 3))(4r - 2 - 4i)} \right)$$

$$= \prod_{j=1}^{r-1} \left(1 + \frac{2}{(4j + 1)(4j - 2)} \right)$$

$$\leq \prod_{j=1}^{503} \left(1 + \frac{1}{j(j + 2)} \right)$$

$$= \prod_{j=1}^{503} \left(\frac{j + 1}{j}\right) \cdot \prod_{j=1}^{503} \left(\frac{j + 1}{j + 2}\right)$$

$$= \frac{504}{1} \cdot \frac{2}{505} \quad (\text{each product telescopes})$$

$$< 2.$$

Estimation of $C$

Since $r < i$, the interval $[4r - 2, 4r - 1]$ lies to the left of the interval $[4i - 3, 4i]$. Hence $y = (x - (4i - 3))(x - 4i) > 0$ for all $x \in [4r - 2, 4r - 1]$, and the minimum value of $y$ occurs when $x = 4r - 1$. It follows that

$$C \leq \prod_{i=r+1}^{504} \left(1 + \frac{2}{(4r - 1 - (4i - 3))(4r - 1 - 4i)} \right)$$

$$= \prod_{i=r+1}^{504} \left(1 + \frac{2}{(4i - 3 - (4r - 1))(4i - (4r - 1))} \right)$$

$$= \prod_{j=1}^{504-r} \left(1 + \frac{2}{(4j - 2)(4j + 1)} \right)$$

$$\leq \prod_{j=1}^{503} \left(1 + \frac{1}{j(j + 2)} \right).$$

This last expression is the same as that found on the RHS of (*). Hence $C < 2$.

We may now combine our estimates for $A$, $B$, and $C$, to find $f(x) < \frac{4}{9}$.

Having shown that $f(x) \neq 1$ in all cases, the proof is complete. □
Solution 2 (Based on the solution provided by the Problem Selection Committee)

As in solution 1, it suffices to show that the equation

\[
\prod_{i=1}^{504} (x - (4i - 3))(x - (4i - 1)) = \prod_{i=1}^{504} (x - (4i - 2))(x - 4i)
\]

\(1\)

does not have any real solutions.

Case 1 \(x = 1, 2, \ldots, 2016\).

One side of (1) is zero while the other is not. So there are no solutions in this case.

Case 2 \(4r - 3 < x < 4r - 2 \) or \(4r - 1 < x < 4r\) for some \(r \in \{1, 2, \ldots, 504\}\).

For every \(i \in \{1, 2, \ldots, 504\}\) we have \((x - (4i - 2))(x - (4i - 1)) > 0\). Hence the LHS of (1) is positive.

For every \(i \in \{1, 2, \ldots, 504\}\) with \(i \neq r\) we have \((x - (4i - 3))(x - 4i) > 0\). However, for \(i = r\), we have \((x - (4r - 3))(x - 4r) < 0\). Hence the RHS of (1) is negative.

It follows that there are no solutions in this case.

Case 3 \(x < 1\) or \(x > 2016\) or \(4r < x < 4r + 1\) for some \(r \in \{1, 2, \ldots, 503\}\).

Equation (1) can be rewritten as

\[
1 = \prod_{i=1}^{504} \frac{(x - (4i - 2))(x - (4i - 1))}{(x - (4i - 3))(x - 4i)}
\]

\[
= \prod_{i=1}^{504} \left(1 + \frac{2}{(x - (4i - 3))(x - 4i)}\right). \quad (2)
\]

Observe that \((x - (4i - 3))(x - 4i) > 0\). This is because any \(x\) satisfying the conditions given by case 3 lies outside of the interval \((4i - 3, 4i)\). Therefore the LHS of (2) is greater than 1. Hence there are no solutions in this case.

Case 4 \(4r - 2 < x < 4r - 1\), for some \(r \in \{1, 2, \ldots, 504\}\).

Equation (1) can be rewritten as

\[
1 = \prod_{i=1}^{504} \frac{(x - (4i - 3))(x - 4i)}{(x - (4i - 2))(x - (4i - 1))}
\]

\[
= \frac{x - 1}{x - 2} \cdot \frac{x - 2016}{x - 2015} \cdot \prod_{i=1}^{503} \frac{(x - 4i)(x - (4i + 1))}{(x - (4i - 1))(x - (4i + 2))}
\]

\[
= \frac{x - 1}{x - 2} \cdot \frac{x - 2016}{x - 2015} \cdot \prod_{i=1}^{503} \left(1 + \frac{2}{(x - (4i + 2))(x - (4i - 1))}\right). \quad (3)
\]

Note that \((x - 1)(x - 2016) < (x - 2)(x - 2015)\). Since \(2 < x < 2015\), it follows that \(\frac{x - 2}{x - 1} \cdot \frac{x - 2016}{x - 2015} > 1\). Furthermore \((x - (4i + 2))(x - (4i - 1)) > 0\). This is because any \(x\) satisfying the conditions given by case 4 lies outside of the interval \((4i - 1, 4i + 2)\).

Hence the RHS of (3) is greater than 1. So there are no solutions in this case either.

From the four cases we may conclude that equation (1) has no real solutions.  \(\square\)
6 (a). **Solution**  (Adapted from the solution of Wilson Zhao, year 12, Killara High School, NSW. Wilson was a Bronze medallist with the 2016 Australian IMO team.)

Let $\Gamma$ be any circle that contains all of the segments. Extend each segment so that both of its endpoints lie on $\Gamma$. Observe that no further crossing points of segments are introduced because every segment already crossed every other segment. So from now on we may assume that each segment’s endpoints lie on $\Gamma$.

There are $2n$ endpoints of segments on $\Gamma$. Hence we may colour them in two colours, say black and white, alternating around the circle. We claim that Geoff can fulfil his wish by placing a frog at each black endpoint.

Suppose that $P$ and $P'$ are opposite endpoints of any segment. Between each of the two arcs around $\Gamma$ from $P$ to $P'$ there are $n - 1$ other endpoints. Since $n$ is odd, this implies that $P$ and $P'$ have opposite colours. So exactly one endpoint of each segment has a frog placed at it.

Let $PP'$ and $QQ'$ be any two segments, where $P$ and $Q$ are both black. Let $X$ be the intersection point of $PP'$ and $QQ'$. Let $p$ denote the number of intersection points on the interior of segment $PX$, and let $q$ denote the number of intersection points on the interior of segment $QX$. We wish to prove that $p \neq q$. For this it suffices to show that $s = p + q$ is odd.

All segments apart from $PP'$ and $QQ'$ are covered by one of the two following cases.

**Case 1** Segments that have an endpoint lying on the arc $PQ$ not containing $P'$.

Since the colours of endpoints alternate around the circle, there are an odd number of segments covered by this case. Each such segment intersects exactly one of $PX$ or $QX$. Hence the total contribution to $s$ from this case is an odd number.

**Case 2** Segments that have an endpoint lying on the arc $PQ'$ not containing $P'$.

Each of these segments either intersects both $PX$ and $QX$, or it intersects neither of them. Hence the total contribution to $s$ from this case is an even number.

Combining the results from cases 1 and 2 shows that $s$ is odd, as required. \(\square\)
6(b). Solution (Problem Selection Committee)

As in part (a) we may assume that all of the segment’s endpoints lie on a single circle $\Gamma$. Let us colour all the endpoints with frogs black, and all the endpoints without frogs white. So we have $2n$ coloured points around the circle.

If no two adjacent points are black, then the colours must alternate around the circle. Since $n$ is even, this would imply that the endpoints of any given segment have the same colour. This contradicts that the endpoints of each segment must have opposite colours.

Hence it is possible to find two adjacent black points, $P$ and $Q$ say, on $\Gamma$. Let the segments containing $P$ and $Q$ intersect at $X$.

![Diagram](image)

Observe that each other segment either intersects both $PX$ and $QX$ or neither of $PX$ or $QX$. It follows that the number of intersection points on $PX$ is equal to the number of intersection points on $QX$. Hence the frogs will meet at $X$. \hfill $\square$
INTERNATIONAL MATHEMATICAL OLYMPIAD
RESULTS

Mark distribution by question

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The medal cuts were set at 29 for Gold, 22 for Silver and 16 for Bronze.
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### Distribution of awards at the 2016 IMO

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ORIGIN OF SOME QUESTIONS

AMOC Senior Contest 2016
Questions 1, 2 and 4 were submitted by Norman Do.
Question 3 was submitted by Alan Offer.
Question 5 was submitted by Angelo Di Pasquale.

Australian Mathematical Olympiad 2016
Question 1 was submitted by Mike Clapper.
Questions 2 and 8 were submitted by Angelo Di Pasquale.
Questions 3, 4, 5 and 6 were submitted by Norman Do.
Question 7 was submitted by Andrei Storozhev.

Asian Pacific Mathematical Olympiad 2016
Questions 1 and 2 were composed by Angelo Di Pasquale and submitted by the AMOC Senior Problems Committee.

International Mathematical Olympiad 2016
Question 2 was composed by Trevor Tao and submitted by the AMOC Senior Problems Committee. Trevor Tao has a PhD in Applied Mathematics and is currently a research scientist working for the Department of Defence. He is a keen Chess and Scrabble player, while his other hobbies include mathematics and music. Trevor won a Bronze medal at the 1995 International Mathematical Olympiad. He is the brother of world-renowned mathematician Terence Tao.

Trevor provided the following background information on the problem.

The inspiration for this problem came when I was futzing around with variations on Sudoku. When I discovered $n = 12$ was impossible, I thought this might make a worthy IMO problem. It is not hard to imagine my joy upon discovering the sum of digits of 2016 was a multiple of 9. The rest, if you pardon the cliché, is history.

A natural question is what happens when we generalise this problem for all integers $c$ (assuming the given problem corresponds to $c = 3$). Suppose an $E \times E$ table satisfying the given conditions exists for some $E$ (physicists can probably guess where this is heading). It turns out that:

- If $c = 2$ or 3, then there are enough diagonals of length $kc$ to enable the double-counting technique to work. Therefore $E = mc^2$ for some integer $m$.
- If $c > 3$, then the diagonals of length $kc$ cover at most half the squares, so double-counting fails and we are left with the trivial solution $E = mc$. This is somewhat unfortunate for students who had a bet on winning at least one special prize.
MATHS CHALLENGE FOR YOUNG AUSTRALIANS
HONOUR ROLL

Because of changing titles and affiliations, the most senior title achieved and later affiliations are generally used, except for the Interim committee, where they are listed as they were at the time.

Problems Committee for Challenge

Dr K McAvaney Victoria, (Director) 11 years; 2006–2016
  Member  1 year; 2005–2006
Mr B Henry Victoria (Director) 17 years; 1990–2006
  Member  11 years; 2006–2016
Prof P J O’Halloran University of Canberra, ACT 5 years; 1990–1994
Dr R A Bryce Australian National University, ACT 23 years; 1990–2012
Adj Prof M Clapper Australian Mathematics Trust, ACT 4 years; 2013–2016
Ms L Corcoran Australian Capital Territory 3 years; 1990–1992
Ms B Denney New South Wales 7 years; 2010–2016
Mr J Dowsey University of Melbourne, VIC 8 years; 1995–2002
Mr A R Edwards Department of Education, Qld 27 years; 1990–2016
Dr M Evans Scotch College, VIC 6 years; 1990–1995
Assoc Prof H Lausch Monash University, VIC 24 years; 1990–2013
Ms J McIntosh AMSI, VIC 15 years; 2002–2016
Mrs L Mottershead New South Wales 25 years; 1992–2016
Miss A Nakos Temple Christian College, SA 24 years; 1993–2016
Dr M Newman Australian National University, ACT 27 years; 1990–2016
Ms F Peel St Peter’s College, SA 2 years; 1999, 2000
Dr I Roberts Northern Territory 4 years; 2013–2016
Ms T Shaw SCEGGS, NSW 4 years; 2013–2016
Ms K Sims New South Wales 18 years; 1999–2016
Dr A Storozhev Attorney General’s Department, ACT 23 years; 1994–2016
Prof P Taylor Australian Mathematics Trust, ACT 20 years; 1995–2014
Mrs A Thomas New South Wales 18 years; 1990–2007
Dr S Thornton South Australia 19 years; 1998–2016

Visiting members

Prof G Berzsenyi Rose Hulman Institute of Technology, USA 1993, 2002
Dr L Burjan Department of Education, Slovakia 1993
Dr V Burjan Institute for Educational Research, Slovakia 1993
Mrs A Ferguson Canada 1992
Prof B Ferguson University of Waterloo, Canada 1992, 2005
Dr D Fomin St Petersburg State University, Russia 1994
Prof F Holland University College, Ireland 1994
Dr A Liu University of Alberta, Canada 1995, 2006, 2009
Prof Q Zhonghu Academy of Science, China 1995
Dr A Gardiner University of Birmingham, United Kingdom 1996
Prof P H Cheung Hong Kong 1997
Prof R Dunkley University of Waterloo, Canada 1997
Dr S Shirali India 1998
Mr M Starck New Caledonia 1999
Dr R Geretschlager Austria 1999, 2013
Dr A Soifer United States of America 2000
Prof M Falk de Losada Colombia 2000
Mr H Groves United Kingdom 2001
Prof J Tabov Bulgaria 2001, 2010
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<td>Prof Dr H-D Gronau</td>
<td>University of Rostock, Germany</td>
<td>2003</td>
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<tr>
<td>Prof J Webb</td>
<td>University of Cape Town, South Africa</td>
<td>2003, 2011</td>
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<tr>
<td>Mr A Parris</td>
<td>Lynwood High School, New Zealand</td>
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<td>Dr A McBride</td>
<td>University of Strathclyde, United Kingdom</td>
<td>2007</td>
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<td>Prof P Vaderlind</td>
<td>Stockholm University, Sweden</td>
<td>2009, 2012</td>
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<td>Prof A Jobbings</td>
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<td>Assoc Prof D Wells</td>
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<td>Dr P Neumann</td>
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**Moderators for Challenge**

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<td>ACT Department of Education, ACT</td>
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<tr>
<td>Dr E Casling</td>
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<td>Mr B Darcy</td>
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<td>Br K Friel</td>
<td>Trinity Catholic College, NSW</td>
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<td>Dr D Fomin</td>
<td>St Petersburg University, Russia</td>
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<td>Mrs P Forster</td>
<td>Penrhos College, WA</td>
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<td>Mr T Freiberg</td>
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<tr>
<td>Mr W Galvin</td>
<td>University of Newcastle, NSW</td>
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<tr>
<td>Mr M Gardner</td>
<td>North Virginia, USA</td>
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<td>Mr S Gardiner</td>
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<td>Ms P Graham</td>
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<tr>
<td>Mr B Harridge</td>
<td>University of Melbourne, VIC</td>
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<td>Ms J Hartnett</td>
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<tr>
<td>Mr G Harvey</td>
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<td>Ms I Hill</td>
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<td>Ms N Hill</td>
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<td>Dr N Hoffman</td>
<td>Edith Cowan University, WA</td>
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<tr>
<td>Prof F Holland</td>
<td>University College, Ireland</td>
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<tr>
<td>Mr D Jones</td>
<td>Coff's Harbour High School, NSW</td>
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<td>Ms R Jorgenson</td>
<td>Australian Capital Territory</td>
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<tr>
<td>Dr T Kalinowski</td>
<td>University of Newcastle, NSW</td>
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<tr>
<td>Assoc Prof H Lausch</td>
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<tr>
<td>Mr J Lawson</td>
<td>St Pius X School, NSW</td>
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<td>Mr R Longmuir</td>
<td>China</td>
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<td>Ms K McAsey</td>
<td>Victoria</td>
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<td>Dr K McAvanir</td>
<td>Victoria</td>
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<tr>
<td>Ms J McIntosh</td>
<td>AMSI, VIC</td>
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<td>Ms N McKinnon</td>
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<td>Ms T McNamara</td>
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<tr>
<td>Mr G Meiklejohn</td>
<td>Queensland School Curriculum Council, QLD</td>
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Moderators for Challenge continued

Mr M O’Connor  AMSI, VIC
Mr J Oliver  Northern Territory
Mr S Palmer  New South Wales
Dr W Palmer  University of Sydney, NSW
Mr G Pointer  South Australia
Prof H Reiter  University of North Carolina, USA
Mr M Richardson  Yarraville Primary School, VIC
Mr G Samson  Nedlands Primary School, WA
Mr J Sattler  Parramatta High School, NSW
Mr A Saund  Victoria
Mr W Scott  Seven Hills West Public School, NSW
Mr R Shaw  Hale School, WA
Ms T Shaw  New South Wales
Dr B Sims  University of Newcastle, NSW
Dr H Sims  Victoria
Ms K Sims  New South Wales
Prof J Smit  The Netherlands
Mrs M Spandler  New South Wales
Mr G Spyker  Curtin University, WA
Ms C Stanley  Queensland
Dr E Strzelecki  Monash University, VIC
Mr P Swain  Ivanhoe Girls Grammar School, VIC
Dr P Swedosh  The King David School, VIC
Prof J Tabov  Academy of Sciences, Bulgaria
Mrs A Thomas  New South Wales
Ms K Trudgian  Queensland
Ms J Vincent  Melbourne Girls Grammar School, VIC
Prof J Webb  University of Capetown, South Africa
Dr D Wells  USA

Mathematics Enrichment Development

Enrichment Committee — Development Team (1992–1995)

Mr B Henry  Victoria (Chairman)
Prof P O’Halloran  University of Canberra, ACT (Director)
Mr G Ball  University of Sydney, NSW
Dr M Evans  Scotch College, VIC
Mr K Hamann  South Australia
Assoc Prof H Lausch  Monash University, VIC
Dr A Storozhev  Australian Mathematics Trust, ACT


Mr G Ball  University of Sydney, NSW (Editor)
Mr K Hamann  South Australia (Editor)
Prof J Burns  Australian Defence Force Academy, ACT
Mr J Carty  Merici College, ACT
Dr H Gastineau-Hill  University of Sydney, NSW
Mr B Henry  Victoria
Assoc Prof H Lausch  Monash University, VIC
Prof P O’Halloran  University of Canberra, ACT
Dr A Storozhev  Australian Mathematics Trust, ACT


Dr M Evans  Scotch College, VIC (Editor)
Mr B Henry  Victoria (Editor)
Mr L Doolan  Melbourne Grammar School, VIC
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<td>Mr K Hamann</td>
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<td>Mr W Atkins</td>
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<td>Australian Defence Force Academy, ACT</td>
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<td>Mr L Doolan</td>
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<td>Mr A Edwards</td>
<td>Mildura High School, VIC</td>
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<td>Mr N Gale</td>
<td>Hornby High School, New Zealand</td>
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<td>Dulwich Hill High School, NSW</td>
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<td><strong>Newton Development Team</strong></td>
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<td>Mr J Dowsey</td>
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<td>Mrs L Mottershead</td>
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<td>Ms G Vardaro</td>
<td>Annesley College, SA</td>
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<td>Ms A Nakos</td>
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<td>Mr A Edwards</td>
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<td>Ms A Nakos</td>
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<td>Mrs K Sims</td>
<td>Chapman Primary School, ACT</td>
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<td>New South Wales</td>
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<td><strong>Australian Intermediate Mathematics Olympiad Committee</strong></td>
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<tr>
<td>Dr K McAvaney</td>
<td>Victoria (Chair)</td>
<td>10 years; 2007–2016</td>
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<tr>
<td>Adj Prof M Clapper</td>
<td>Australian Mathematics Trust, ACT</td>
<td>3 years; 2014–2016</td>
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<tr>
<td>Mr J Dowsey</td>
<td>University of Melbourne, VIC</td>
<td>18 years; 1999–2016</td>
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<td>Dr M Evans</td>
<td>AMSI, VIC</td>
<td>18 years; 1999–2016</td>
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<td>Mr B Henry</td>
<td>Victoria (Chair)</td>
<td>8 years; 1999–2006</td>
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<td>Member</td>
<td>10 years; 2007–2016</td>
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<tr>
<td>Assoc Prof H Lausch</td>
<td>Monash University, VIC</td>
<td>17 years; 1999–2015</td>
</tr>
<tr>
<td>Mr R Longmuir</td>
<td>China</td>
<td>2 years; 1999–2000</td>
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AUSTRALIAN MATHEMATICAL OLYMPIAD COMMITTEE
HONOUR ROLL

Because of changing titles and affiliations, the most senior title achieved and later affiliations are generally used, except for the Interim committee, where they are listed as they were at the time.

Interim Committee 1979–1980
Mr P J O’Halloran Canberra College of Advanced Education, ACT, Chair
Prof A L Blakers University of Western Australia
Dr J M Gani Australian Mathematical Society, ACT,
Prof B H Neumann Australian National University, ACT,
Prof G E Wall University of Sydney, NSW
Mr J L Williams University of Sydney, NSW

The Australian Mathematical Olympiad Committee was founded at a meeting of the Australian Academy of Science at its meeting of 2–3 April 1980.

* denotes Executive Position

Chair*
Prof B H Neumann Australian National University, ACT 7 years; 1980–1986
Prof G B Preston Monash University, VIC 10 years; 1986–1995
Prof A P Street University of Queensland 6 years; 1996–2001
Prof C Praeger University of Western Australia 14 years; 2002–2015

Deputy Chair*
Prof P J O’Halloran University of Canberra, ACT 15 years; 1980–1994
Prof A P Street University of Queensland 1 year; 1995
Prof C Praeger, University of Western Australia 6 years; 1996–2001
Assoc Prof D Hunt University of New South Wales 14 years; 2002–2015

Executive Director*
Prof P J O’Halloran University of Canberra, ACT 15 years; 1980–1994
Prof P J Taylor University of Canberra, ACT 18 years; 1994–2012
Adj Prof M G Clapper University of Canberra, ACT 4 years; 2013–2016

Secretary
Prof J C Burns Australian Defence Force Academy, ACT 9 years; 1980–1988
Vacant 4 years; 1989–1992
Mrs K Doolan Victorian Chamber of Mines, VIC 6 years; 1993–1998

Treasurer*
Prof J C Burns Australian Defence Force Academy, ACT 8 years; 1981–1988
Prof P J O’Halloran University of Canberra, ACT 2 years; 1989–1990
Ms J Downes CPA 5 years; 1991–1995
Dr P Edwards Monash University, VIC 8 years; 1995–2002
Prof M Newman Australian National University, ACT 6 years; 2003–2008
Dr P Swedosh The King David School, VIC 8 years; 2009–2016

Director of Mathematics Challenge for Young Australians*
Mr J B Henry Deakin University, VIC 17 years; 1990–2006
Dr K McAvaney Deakin University, VIC 11 years; 2006–2016

Chair, Senior Problems Committee
Prof B C Rennie James Cook University, QLD 1 year; 1980
Mr J L Williams University of Sydney, NSW 6 years; 1981–1986
Assoc Prof H Lausch Monash University, VIC 27 years; 1987–2013
Dr N Do Monash University, VIC 3 years; 2014–2016
Director of Training*
Mr J L Williams University of Sydney, NSW 7 years; 1980–1986
Mr G Ball University of Sydney, NSW 3 years; 1987–1989
Dr D Paget University of Tasmania 6 years; 1990–1995
Dr M Evans Scotch College, VIC 3 months; 1995
Assoc Prof D Hunt University of New South Wales 5 years; 1996–2000
Dr A Di Pasquale University of Melbourne, VIC 16 years; 2001–2016

Team Leader
Mr J L Williams University of Sydney, NSW 5 years; 1981–1985
Dr E Strzelecki Monash University, VIC 2 years; 1987, 1988
Dr D Paget University of Tasmania 5 years; 1991–1995
Dr A Di Pasquale University of Melbourne, VIC 14 years; 2002–2010, 2012–2016
Dr I Guo University of New South Wales 1 year; 2011

Deputy Team Leader
Prof G Szekeres University of New South Wales 2 years; 1981–1982
Mr G Ball University of Sydney, NSW 7 years; 1983–1989
Dr D Paget University of Tasmania 1 year; 1990
Dr J Graham University of Sydney, NSW 3 years; 1991–1993
Dr M Evans Scotch College, VIC 3 years; 1994–1996
Dr A Di Pasquale University of Melbourne, VIC 5 years; 1997–2001
Dr D Mathews University of Melbourne, VIC 3 years; 2002–2004
Dr N Do University of Melbourne, VIC 4 years; 2005–2008
Dr I Guo University of New South Wales 4 years; 2009–10, 2012–2013
Mr G White University of Sydney, NSW 1 year; 2011
Mr A Elvey Price Melbourne University, VIC 3 years; 2014–2016

State Directors
Australian Capital Territory
Prof M Newman Australian National University 1 year; 1980
Mr D Thorpe ACT Department of Education 2 years; 1981–1982
Dr R A Bryce Australian National University 7 years; 1983–1989
Mr R Welsh Canberra Grammar School 1 year; 1990
Mrs J Kain Canberra Grammar School 5 years; 1991–1995
Mr J Carty ACT Department of Education 17 years; 1995–2011
Mr J Hassall Burgmann Anglican School 2 years; 2012–2013
Dr C Wetherell Radford College 3 years; 2014–2016

New South Wales
Dr M Hirschhorn University of New South Wales 1 year; 1980
Mr G Ball University of Sydney, NSW 16 years; 1981–1996
Dr W Palmer University of Sydney, NSW 20 years; 1997–2016

Northern Territory
Dr I Roberts Charles Darwin University 3 years; 2014–2016

Queensland
Dr N H Williams University of Queensland 21 years; 1980–2000
Dr G Carter Queensland University of Technology 10 years; 2001–2010
Dr V Scharaschkin University of Queensland 4 years; 2011–2014
Dr A Offer Queensland 2 years; 2015 –2016

South Australia/Northern Territory
Mr V Treilbs SA Department of Education 8 years; 1983–1990
Dr M Peake Adelaide 8 years; 2006–2013
Dr D Martin Adelaide 3 years; 2014–2016
Tasmania
Mr J Kelly Tasmanian Department of Education 8 years; 1980–1987
Dr D Paget University of Tasmania 8 years; 1988–1995
Mr W Evers St Michael’s Collegiate School 9 years; 1995–2003
Dr K Dharmadasa University of Tasmania 13 years; 2004–2016

Victoria
Dr D Holton University of Melbourne 3 years; 1980–1982
Mr B Harridge Melbourne High School 1 year; 1982
Ms J Downes CPA 6 years; 1983–1988
Mr L Doolan Melbourne Grammar School 9 years; 1989–1998
Dr P Swedosh The King David School 19 years; 1998–2016

Western Australia
Dr N Hoffman WA Department of Education 3 years; 1980–1982
Assoc Prof W Bloom Murdoch University 2 years; 1989–1990
Dr E Stoyanova WA Department of Education 7 years; 1995, 2000–2005
Dr G Gamble University of Western Australia 11 years; 2006–2016

Editor
Prof P J O’Halloran University of Canberra, ACT 1 year; 1983
Dr A W Plank University of Southern Queensland 11 years; 1984–1994
Dr A Storozhev Australian Mathematics Trust, ACT 15 years; 1994–2008

Editorial Consultant
Dr O Yevdokimov University of Southern Queensland 8 years; 2009–2016

Other Members of AMOC (showing organisations represented where applicable)
Mr W J Atkins Australian Mathematics Foundation 18 years; 1995–2012
Dr S Britton University of Sydney, NSW 8 years; 1990–1998
Prof G Brown Australian Academy of Science, ACT 10 years; 1980, 1986–1994
Dr R A Bryce Australian Mathematical Society, ACT 10 years; 1991–1998
Mathematics Challenge for Young Australians 13 years; 1999–2012
Mr G Cristofani Department of Education and Training 2 years; 1993–1994
Ms L Davis IBM Australia 4 years; 1991–1994
Dr W Franzsen Australian Catholic University, ACT 9 years; 1990–1998
Dr J Gani Australian Mathematical Society, ACT 1980
Assoc Prof T Gagen ANU AAMT Summer School 6 years; 1993–1998
Ms P Gould Department of Education and Training 2 years; 1995–1996
Prof G M Kelly University of Sydney, NSW 6 years; 1982–1987
Prof R B Mitchell University of Canberra, ACT 5 years; 1991–1995
Ms Anna Nakos Mathematics Challenge for Young Australians 14 years; 2003–2016
Mr S Neal Department of Education and Training 4 years; 1990–1993
Prof M Newman Australian National University, ACT 15 years; 1986–1998
Mathematics Challenge for Young Australians 10 years; 1999–2002, (Treasurer during the interim) 2009–2014
Prof R B Potts University of Adelaide, SA 1 year; 1980
Mr H Reeves Australian Association of Maths Teachers 12 years; 1988–1998
Mr N Reid IBM Australia 3 years; 1988–1990
Mr R Smith Telecom Australia 5 years; 1990–1994
Prof N S Trudinger Australian Mathematical Society, ACT 3 years; 1986–1988
Assoc Prof I F Vivian University of Canberra, ACT 1 year; 1990
Dr M W White IBM Australia 9 years; 1980–1988
Associate Membership (inaugurated in 2000)

Ms S Britton 17 years; 2000–2016
Dr M Evans 17 years; 2000–2016
Dr W Franzsen 17 years; 2000–2016
Prof T Gagen 17 years; 2000–2016
Mr H Reeves 17 years; 2000–2016
Mr G Ball 17 years; 2000–2016

AMOC Senior Problems Committee

Current members

Dr N Do Monash University, VIC (Chair) 3 years; 2014–2016
Adj Prof M Clapper Australian Mathematics Trust 11 years; 2003–2013
Dr A Devillers University of Western Australia, WA 4 years; 2013–2016
Dr A Di Pasquale University of Melbourne, VIC 1 year; 2016
Dr I Guo University of Sydney, NSW 16 years; 2001–2016
Dr J Kupka Monash University, VIC 14 years; 2003–2016
Dr K McAvaney Deakin University, VIC 21 years; 1996–2016
Dr D Mathews Monash University, VIC 16 years; 2001–2016
Dr A Offer Queensland 5 years; 2012–2016
Dr C Rao Telstra, VIC 17 years; 2000–2016
Dr B B Saad Monash University, VIC 23 years; 1994–2016
Dr J Simpson Curtin University, WA 18 years; 1999–2016
Dr I Wanless Monash University, VIC 17 years; 2000–2016

Previous members

Mr G Ball University of Sydney, NSW 16 years; 1982–1997
Mr M Brazil LaTrobe University, VIC 5 years; 1990–1994
Dr M S Brooks University of Canberra, ACT 8 years; 1983–1990
Dr G Carter Queensland University of Technology 10 years; 2001–2010
Dr M Evans Australian Mathematical Sciences Institute, VIC 27 years; 1990–2016
Dr J Graham University of Sydney, NSW 1 year; 1992
Dr M Herzberg Telecom Australia 1 year; 1990
Assoc Prof D Hunt University of New South Wales 29 years; 1986–2014
Dr L Kovacs Australian National University, ACT 5 years; 1981–1985
Assoc Prof H Lausch Monash University, VIC (Chair) 27 years; 1987–2013
Dr D Paget University of Tasmania 7 years; 1989–1995
Prof P Schultz University of Western Australia 8 years; 1993–2000
Dr L Stoyanov University of Western Australia 5 years; 2001–2005
Dr E Strzelecki Monash University, VIC 5 years; 1986–1990
Dr E Szekeres University of New South Wales 7 years; 1981–1987
Prof G Szekeres University of New South Wales 7 years; 1981–1987
Em Prof P J Taylor Australian Capital Territory 1 year; 2013
Dr N H Williams University of Queensland 20 years; 1981–2000

Mathematics School of Excellence

Dr S Britton University of Sydney, NSW (Coordinator) 2 years; 1990–1991
Dr L Doolan Melbourne Grammar, VIC (Coordinator) 6 years; 1992, 1993–1997
Mr W Franzsen Australian Catholic University, ACT (Coordinator) 2 years; 1990–1991
Dr D Paget University of Tasmania (Director) 5 years; 1990–1994
Dr M Evans Scotch College, VIC 1 year; 1995
Assoc Prof D Hunt University of New South Wales (Director) 4 years; 1996–1999
Dr A Di Pasquale University of Melbourne, VIC (Director) 17 years; 2000–2016
### International Mathematical Olympiad Selection School

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<td>2 years; 1982–1983</td>
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<td>University of Sydney, NSW (Director)</td>
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<td>5 years; 1996–2000</td>
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<td>University of Melbourne, VIC (Director)</td>
<td>16 years; 2001–2016</td>
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