AUSTRALIAN MATHS TRUST

AMC PRACTICE QUESTIONS AND SOLUTIONS

Intermediate

1. 2014 I5

If $\frac{5}{6}$ of a number is 30, what is $\frac{3}{4}$ of the number?

(A) 22.5 (B) 24 (C) 25 (D) 27 (E) 40

▶ One-sixth of the number is 6, so the number is 36. Then $\frac{1}{4}$ of the number is 9 and $\frac{3}{4}$ of the number is 27,

hence (D).

2. 2014 I10

Each May a farmer plants barley seed and then in October he harvests 12 times the weight of seed planted. From each harvest, he sells 50 tonnes and the rest he keeps as seed for the next year's crop. This year he has planted enough to harvest 120 tonnes. How many tonnes did he plant last year?

(A) 5 (B) 10 (C) 20 (D) 30 (E) 60

▶ This year he planted 10 tonnes. So last year he harvested 60 tonnes. Thus last year he planted 5 tonnes,

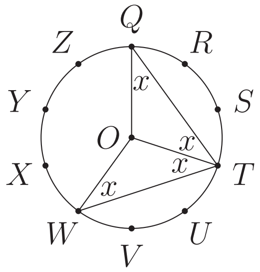
hence (A).

3. 2014 I15

Ten points Q, R, S, T, U, V, W, X, Y and Z are equally and consecutively spaced on a circle. What is the size, in degrees, of the angle $\angle QTW$?

(A) 36 (B) 54 (C) 60 (D) 72 (E) 75

▶ *Alternative 1*

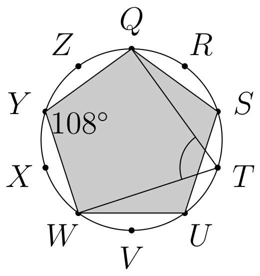


Let O be the centre of the circle and draw congruent isosceles triangles $\triangle QOT$ and $\triangle TOW$, and let x be the acute angle in these two triangles.

Each of the ten angles $\angle QOR$, $\angle ROS$, ... are equal, so that $\angle QOT = \frac{3}{10} \times 360° = 108°$. In the triangle $\triangle QOT$, $108° + 2x = 180°$, so that $x = 36°$. Therefore $\angle QTW = 2x = 72°$,

hence (D).

Alternative 2



The pentagon $QSUWY$ is regular, so that $\angle QYW = 108°$. The quadrilateral $QTWY$ is cyclic, so opposite angles add to 180°. Thus $\angle QTW = 180° - 108° = 72°$,

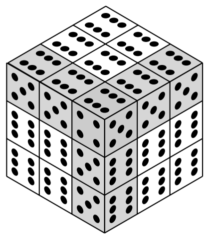
hence (D).

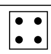
4. 2014 I21

Standard six-sided dice have their dots arranged so that the opposite faces add up to 7. If 27 standard dice are arranged in a $3 \times 3 \times 3$ cube on a solid table what is the maximum number of dots that can be seen from one position?

(A) 90 (B) 94 (C) 153 (D) 154 (E) 189

▶ There are at most three 3×3 faces of the cube visible, and the maximum will occur with exactly three cube faces visible. A total of 153 is possible.



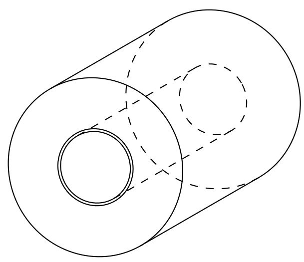
dice face	count	pips
	1	4
	7	35
	19	114
		153

To see that no greater total is possible, of the 19 visible dice, one has 3 faces visible, six have 2 faces visible and 12 have one face visible. So the sum of all visible faces cannot exceed $1 \times 15 + 6 \times 11 + 12 \times 6 = 153$,

hence (C).

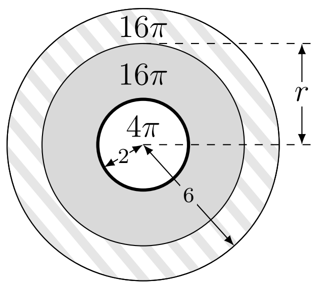
5. 2014 I25

Thanom has a roll of paper consisting of a very long sheet of thin paper tightly rolled around a cylindrical tube, forming the shape indicated in the diagram.



Initially, the diameter of the roll is 12 cm and the diameter of the tube is 4 cm. After Thanom uses half of the paper, the diameter of the remaining roll is closest to

(A) 6 cm (B) 8 cm (C) 8.5 cm (D) 9 cm (E) 9.5 cm

▶ 

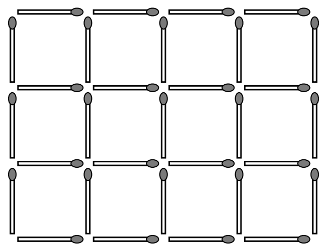
Working in centimetres, let the half-roll's radius be r. From end on, the full roll had area $\pi(6^2 - 2^2) = 32\pi$, so half the roll has area 16π. Including the tube, the end of the half-roll has area $20\pi = \pi r^2$.

Then $r^2 = 20$, but $4.5^2 = 20.25$ and $4.4^2 = 19.36$, so that $4.4 < r < 4.5$, and the diameter is twice that,

hence (D).

6. 2014 I29

As shown in the diagram, you can create a grid of squares 3 units high and 4 units wide using 31 matches. I would like to make a grid of squares a units high and b units wide, where $a < b$ are positive integers. Determine the sum of the areas of all such rectangles that can be made, each using exactly 337 matches.



▶ *Alternative 1*

There are $a + 1$ rows of horizontal matches and each row contains b matches. There are $b + 1$ columns of vertical matches and each column contains a matches. So the total number of matches is $(a + 1)b + (b + 1)a = 2ab + a + b$.

We would like to solve the equation $2ab + a + b = 337$, where $a < b$ are positive integers. By multiplying the equation by 2 and adding 1 to both sides, we obtain

$$4ab + 2a + 2b + 1 = 675 \qquad \Rightarrow \qquad (2a + 1)(2b + 1) = 675$$

The only ways to factorise 675 into two positive integers are

$$1 \times 675, \quad 3 \times 225, \quad 5 \times 135, \quad 9 \times 75, \quad 15 \times 45, \quad 25 \times 27$$

We must have $2a + 1$ correspond to the smaller factor and $2b + 1$ to the larger factor. So the solutions we obtain for (a, b) are

$$(0, 337), \quad (1, 112), \quad (2, 67), \quad (4, 37), \quad (7, 22), \quad (12, 13)$$

We must disregard the first solution, but one can check that the remaining ones are all valid. So the sum of the areas of all such rectangles is

$$1 \times 112 + 2 \times 67 + 4 \times 37 + 7 \times 22 + 12 \times 13 = 704$$

hence (704).

Alternative 2

There are $(a + 1)b$ horizontal and $a(b + 1)$ vertical matches. Then

$$(a + 1)b + a(b + 1) = 2ab + a + b = 337 \qquad \Longrightarrow \qquad b = \frac{337 - a}{2a + 1}$$

In this table, for each a we work out b and the area ab, keeping only those where b is a whole number and $b > a$.

a	1	2	3	4	5	6	7	8	9	10	11	12	13	⋯
b	$\frac{336}{3}$ 112	$\frac{335}{5}$ 67	$\frac{334}{7}$ ×	$\frac{333}{9}$ 37	$\frac{331}{11}$ ×	$\frac{332}{13}$ ×	$\frac{330}{15}$ 22	$\frac{329}{17}$ ×	$\frac{328}{19}$ ×	$\frac{327}{21}$ ×	$\frac{326}{23}$ ×	$\frac{325}{25}$ 13	$\frac{324}{27}$ (12)	⋯
ab	112	134		148			154					156		

Then the total area is $112 + 134 + 148 + 154 + 156 = 704$ square units,

hence (704).