AUSTRALIAN MATHS TRUST

AMC PRACTICE QUESTIONS AND SOLUTIONS

Senior

1. 2014 S6

If x, x^2 and x^3 lie on a number line in the order shown below, which of the following could be the value of x?

(A) -2 (B) $-\frac{1}{2}$ (C) $\frac{3}{4}$ (D) 1 (E) $\frac{3}{2}$

▸ We have $0 < x^2 < x$ so that x is positive and $x < 1$. The only possibility is $x = \frac{3}{4}$, and $x^3 = \frac{27}{64}$, $x^2 = \frac{9}{16} = \frac{36}{64}$ and $x = \frac{3}{4} = \frac{48}{64}$,

hence (C).

2. 2014 S10

If $\dfrac{p}{p-2q} = 3$ then $\dfrac{p}{q}$ equals

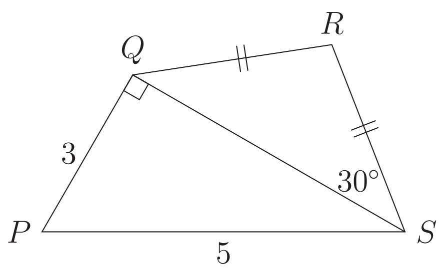
(A) -3 (B) 3 (C) $\frac{1}{3}$ (D) $\frac{2}{3}$ (E) 2

▸ We have $p = 3(p-2q)$, so $6q = 2p$ and $p = 3q$. Then $\frac{p}{q} = 3$,

hence (B).

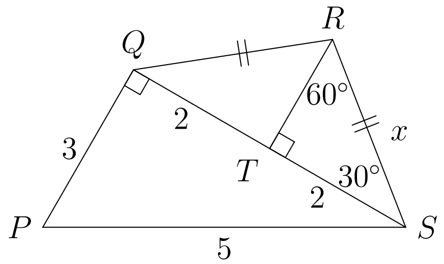
3. 2014 S15

In the diagram, $PS = 5$, $PQ = 3$, $\triangle PQS$ is right-angled at Q, $\angle QSR = 30^\circ$ and $QR = RS$. The length of RS is



(A) $\frac{\sqrt{3}}{2}$ (B) $\sqrt{3}$ (C) 2 (D) $\frac{4\sqrt{3}}{3}$ (E) 4

▶ Due to the right-angled triangle $\triangle PQS$, Pythagoras' theorem gives $QS = 4$. Then $\triangle QRS$ is isosceles, so its altitude RT bisects QS.



Now, $\triangle SRT$ is standard $30°, 60°, 90°$ triangle with $RT : RS : ST = 1 : 2 : \sqrt{3}$ so that $x = RS = \frac{2}{\sqrt{3}} ST = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$,

hence (D).

Comment

This problem can also be solved using trigonometry: $x = \dfrac{2}{\cos 30°} = \dfrac{4}{\sqrt{3}}$.

4. 2014 S20

Given that $f_1(x) = \dfrac{x}{x+1}$ and $f_{n+1}(x) = f_1(f_n(x))$, then $f_{2014}(x)$ equals

(A) $\dfrac{x}{2014x+1}$ (B) $\dfrac{2014x}{2014x+1}$ (C) $\dfrac{x}{x+2014}$ (D) $\dfrac{2014x}{x+1}$ (E) $\dfrac{x}{2014(x+1)}$

▶ *Alternative 1*

$$f_2(x) = f\left(\frac{x}{x+1}\right) = \frac{\frac{x}{x+1}}{\frac{x}{x+1}+1} = \frac{x}{x+x+1} = \frac{x}{2x+1}$$

$$f_3(x) = \frac{\frac{x}{2x+1}}{\frac{x}{2x+1}+1} = \frac{x}{x+2x+1} = \frac{x}{3x+1}$$

and in general, by induction

$$f_n(x) = \frac{x}{nx+1} \implies f_{n+1}(x) = \frac{\frac{x}{nx+1}}{\frac{x}{nx+1}+1} = \frac{x}{x+nx+1} = \frac{x}{(n+1)x+1},$$

so $f_{2014}(x) = \dfrac{x}{2014x+1}$,

hence (A).

Alternative 2

Consider $\dfrac{1}{f_n(x)}$.

$$\frac{1}{f_1(x)} = 1 + \frac{1}{x} \implies \frac{1}{f_{n+1}(x)} = f_1(f_n(x)) = 1 + \frac{1}{f_n(x)}$$

$$\implies \frac{1}{f_{2014}(x)} = 1 + \frac{1}{f_{2013}(x)} = 2 + \frac{1}{f_{2012}(x)} = \cdots$$

$$\cdots = 2013 + \frac{1}{f_1(x)} = 2014 + \frac{1}{x} = \frac{2014x+1}{x}$$

Hence $f_{2014}(x) = \dfrac{x}{2014x+1}$,

hence (A).

5. 2014 S25

The sequence

$$2\ ,\ 2^2\ ,\ 2^{2^2}\ ,\ 2^{2^{2^2}}\ ,\ldots$$

is defined by $a_1 = 2$ and $a_{n+1} = 2^{a_n}$ for all $n \geq 1$. What is the first term in the sequence greater than 1000^{1000} ?

(A) $a_4 = 2^{2^{2^2}}$ (B) $a_5 = 2^{2^{2^{2^2}}}$ (C) $a_6 = 2^{2^{2^{2^{2^2}}}}$ (D) $a_7 = 2^{2^{2^{2^{2^{2^2}}}}}$ (E) $a_8 = 2^{2^{2^{2^{2^{2^{2^2}}}}}}$

▶ We want $a_n > 1000^{1000} = 10^{3000}$. We know that $a_1 = 2$, $a_2 = 2^2 = 4$, $a_3 = 2^4 = 16$ and $a_4 = 2^{16} = 65536$, all less than 10^{3000}. Also $2^{10} = 1024 > 10^3$, so that we can estimate a_5,

$$a_5 = 2^{65536} = (2^{10})^{6553}\, 2^6 > (10^3)^{6553}\, 2^6 = 64 \times 10^{19659}$$

This is greater than 10^{3000},

hence (B).

6. 2014 S26

What is the largest three-digit number with the property that the number is equal to the sum of its hundreds digit, the square of its tens digit and the cube of its units digit?

▶ *Alternative 1*

Let the number be abc.

Then

$$\begin{aligned} 100a + 10b + c &= a + b^2 + c^3 \\ 99a + 10b - b^2 &= c(c^2 - 1) \\ 99a + b(10 - b) &= (c-1)c(c+1) \end{aligned}$$

Consider the possibilities:

$99a$	$b(10-b)$	$(c-1)c(c+1)$
$99 \times 1 = 99$	$1 \times 9 = 9$	$1 \times 2 \times 3 = 6$
$99 \times 2 = 198$	$2 \times 8 = 16$	$2 \times 3 \times 4 = 24$
$99 \times 3 = 297$	$3 \times 7 = 21$	$3 \times 4 \times 5 = 60$
$99 \times 4 = 396$	$4 \times 6 = 24$	$4 \times 5 \times 6 = 120$
$99 \times 5 = 495$	$5 \times 5 = 25$	$5 \times 6 \times 7 = 210$
$99 \times 6 = 594$	$6 \times 4 = 24$	$6 \times 7 \times 8 = 336$
$99 \times 7 = 693$	$7 \times 3 = 21$	$7 \times 8 \times 9 = 504$
$99 \times 8 = 792$	$8 \times 2 = 16$	$8 \times 9 \times 10 = 720$
$99 \times 9 = 891$	$9 \times 1 = 9$	

Looking at the possibilities for $99a + b(10-b) = (c-1)c(c+1)$, we have two:
$99 + 21 = 120 \implies a = 1,\ b = 3$ or $7, c = 5 \implies n = 135$ or $n = 175$.
$495 + 9 = 504 \implies a = 5,\ b = 1$ or $9, c = 8 \implies n = 518$ or $n = 598$.
So, there are four 3-digit numbers which satisfy the requirements and the largest of these four numbers is 598,

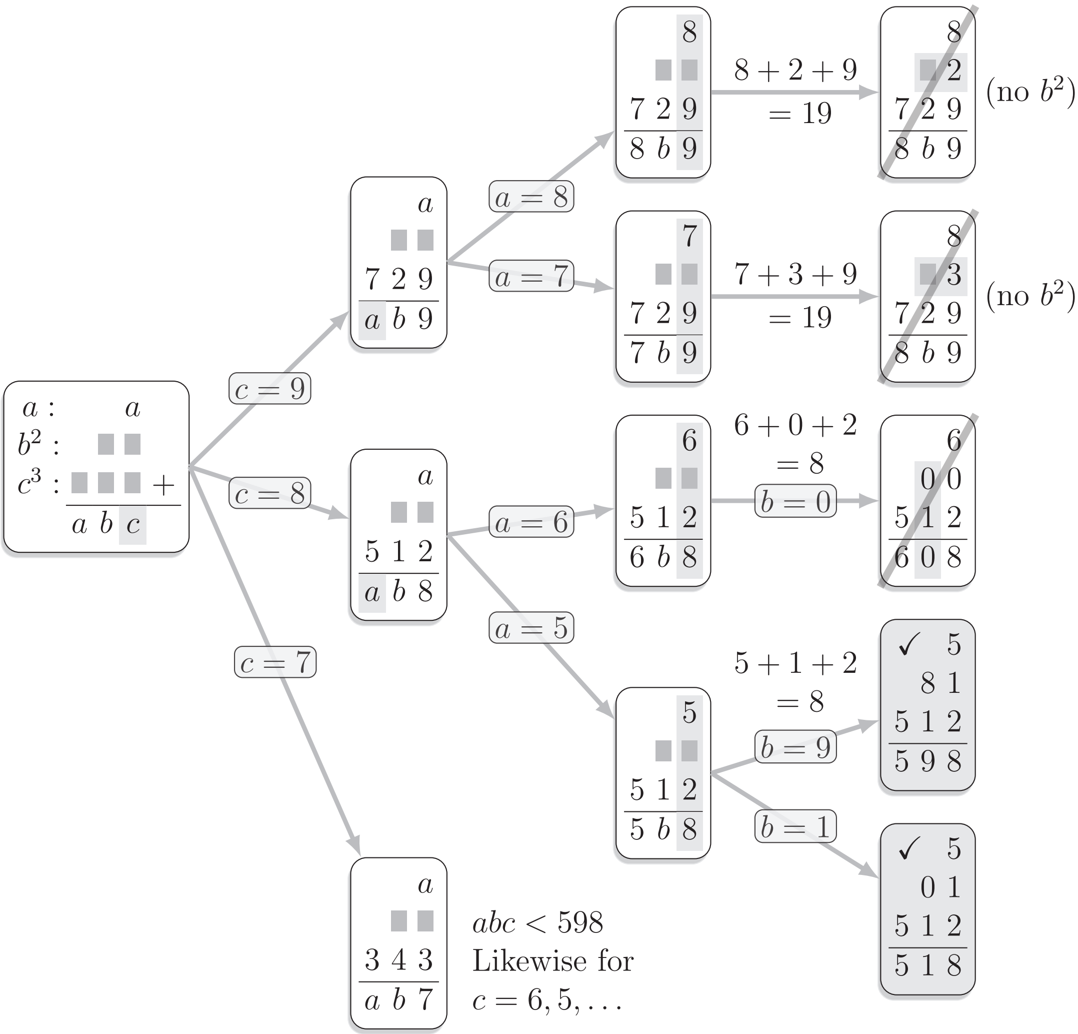
hence (598).

Alternative 2

The number abc is equal to $a + b^2 + c^3$, and these are the possible values of b^2 and c^3:

Digit	0	1	2	3	4	5	6	7	8	9
Square	0	1	4	9	16	25	36	49	64	81
Cube	0	1	8	27	64	125	216	343	512	729

We try these numbers in an addition grid, trying the large values of c first, then filling in possible values for a and b. This trial-and-error search is presented here as a tree.



The largest solution found is 598, and any solutions on branches $c = 7$, $c = 6$, ..., $c = 1$ must be less than this,

hence (598).