1. 2014 S6
If \(x, x^2\) and \(x^3\) lie on a number line in the order shown below, which of the following could be the value of \(x\)?

\[
\begin{array}{c}
\bullet & \bullet & \bullet \\
\text{x³} & \text{x²} & \text{x}
\end{array}
\]

(A) \(-2\) (B) \(-\frac{1}{2}\) (C) \(\frac{3}{4}\) (D) 1 (E) \(\frac{3}{2}\)

We have \(0 < x^2 < x\) so that \(x\) is positive and \(x < 1\). The only possibility is \(x = \frac{3}{4}\),
and \(x^3 = \frac{27}{64}, x^2 = \frac{9}{16} = \frac{36}{64}\) and \(x = \frac{3}{4} = \frac{48}{64}\),
hence (C).

2. 2014 S10
If \(\frac{p}{p - 2q} = 3\) then \(\frac{p}{q}\) equals

(A) \(-3\) (B) 3 (C) \(\frac{1}{3}\) (D) \(\frac{2}{3}\) (E) 2

We have \(p = 3(p - 2q)\), so \(6q = 2p\) and \(p = 3q\). Then \(\frac{p}{q} = 3\),
hence (B).

3. 2014 S15
In the diagram, \(PS = 5, PQ = 3, \triangle PQS\) is right-angled at \(Q, \angle QSR = 30^\circ\) and \(QR = RS\).
The length of \(RS\) is

(A) \(\frac{\sqrt{3}}{2}\) (B) \(\sqrt{3}\) (C) 2 (D) \(\frac{4\sqrt{3}}{3}\) (E) 4
Due to the right-angled triangle $\triangle PQS$, Pythagoras’ theorem gives $QS = 4$. Then $\triangle QRS$ is isosceles, so its altitude $RT$ bisects $QS$. 

Now, $\triangle SRT$ is standard $30^\circ, 60^\circ, 90^\circ$ triangle with $RT : RS : ST = 1 : 2 : \sqrt{3}$ so that $x = RS = \frac{2}{\sqrt{3}}ST = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$, hence (D).

Comment
This problem can also be solved using trigonometry: $x = \frac{2}{\cos 30^\circ} = \frac{4}{\sqrt{3}}$.

4. 2014 S20

Given that $f_1(x) = \frac{x}{x + 1}$ and $f_{n+1}(x) = f_1(f_n(x))$, then $f_{2014}(x)$ equals

(A) $\frac{x}{2014x + 1}$  (B) $\frac{2014x}{2014x + 1}$  (C) $\frac{x}{x + 2014}$  (D) $\frac{2014x}{x + 1}$  (E) $\frac{x}{2014(x + 1)}$

Alternative 1

$f_2(x) = f\left(\frac{x}{x + 1}\right) = \frac{x}{x + 1} = \frac{x}{x + x + 1} = \frac{x}{2x + 1}$

$f_3(x) = \frac{x}{2x + 1} + 1 = \frac{x + 2x + 1}{3x + 1}$

and in general, by induction

$f_n(x) = \frac{x}{nx + 1} \Rightarrow f_{n+1}(x) = \frac{x}{nx + 1} + 1 = \frac{x}{x + nx + 1} = \frac{x}{(n + 1)x + 1}$,

so $f_{2014}(x) = \frac{x}{2014x + 1}$, hence (A).

Alternative 2

Consider $\frac{1}{f_n(x)}$.

$$\frac{1}{f_1(x)} = 1 + \frac{1}{x} \quad \Rightarrow \quad \frac{1}{f_{n+1}(x)} = f_1(f_n(x)) = 1 + \frac{1}{f_n(x)}$$

$$\Rightarrow \quad \frac{1}{f_{2014}(x)} = 1 + \frac{1}{f_{2013}(x)} = 2 + \frac{1}{f_{2012}(x)} = \cdots$$

$$\cdots = 2013 + \frac{1}{f_1(x)} = 2014 + \frac{1}{x} = \frac{2014x + 1}{x}$$

Hence $f_{2014}(x) = \frac{x}{2014x + 1}$, hence (A).
5. **2014 S25**

The sequence

\[ 2, 2^2, 2^{2^2}, 2^{2^{2^2}}, \ldots \]

is defined by \( a_1 = 2 \) and \( a_{n+1} = 2^{a_n} \) for all \( n \geq 1 \). What is the first term in the sequence greater than 1000? 

(A) \( a_4 = 2^{2^{2^2}} \)  
(B) \( a_5 = 2^{2^{2^{2^2}}} \)  
(C) \( a_6 = 2^{2^{2^{2^{2^2}}}} \)  
(D) \( a_7 = 2^{2^{2^{2^{2^{2^2}}}}} \)  
(E) \( a_8 = 2^{2^{2^{2^{2^{2^{2^{2^2}}}}}}} \)

We want \( a_n > 1000^{1000} = 10^{3000} \). We know that \( a_1 = 2, a_2 = 2^2 = 4, a_3 = 2^4 = 16 \) and \( a_4 = 2^{16} = 65536 \), all less than \( 10^{3000} \). Also \( 2^{10} = 1024 > 10^3 \), so that we can estimate \( a_5 \),

\[ a_5 = 2^{65536} = (2^{10})^{6553} 2^6 > (10^3)^{6553} 2^6 = 64 \times 10^{19659} \]

This is greater than \( 10^{3000} \), hence (B).

6. **2014 S26**

What is the largest three-digit number with the property that the number is equal to the sum of its hundreds digit, the square of its tens digit and the cube of its units digit?

**Alternative 1**

Let the number be \( abc \).

Then

\[
\begin{align*}
100a + 10b + c &= a + b^2 + c^3 \\
99a + 10b - b^2 &= c(c^2 - 1) \\
99a + b(10 - b) &= (c - 1)c(c + 1)
\end{align*}
\]

Consider the possibilities:

<table>
<thead>
<tr>
<th>99a</th>
<th>b(10 - b)</th>
<th>(c - 1)c(c + 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>99 × 1 = 99</td>
<td>1 × 9 = 9</td>
<td>1 × 2 × 3 = 6</td>
</tr>
<tr>
<td>99 × 2 = 198</td>
<td>2 × 8 = 16</td>
<td>2 × 3 × 4 = 24</td>
</tr>
<tr>
<td>99 × 3 = 297</td>
<td>3 × 7 = 21</td>
<td>3 × 4 × 5 = 60</td>
</tr>
<tr>
<td>99 × 4 = 396</td>
<td>4 × 6 = 24</td>
<td>4 × 5 × 6 = 120</td>
</tr>
<tr>
<td>99 × 5 = 495</td>
<td>5 × 5 = 25</td>
<td>5 × 6 × 7 = 210</td>
</tr>
<tr>
<td>99 × 6 = 594</td>
<td>6 × 4 = 24</td>
<td>6 × 7 × 8 = 336</td>
</tr>
<tr>
<td>99 × 7 = 693</td>
<td>7 × 3 = 21</td>
<td>7 × 8 × 9 = 504</td>
</tr>
<tr>
<td>99 × 8 = 792</td>
<td>8 × 2 = 16</td>
<td>8 × 9 × 10 = 720</td>
</tr>
<tr>
<td>99 × 9 = 891</td>
<td>9 × 1 = 9</td>
<td></td>
</tr>
</tbody>
</table>

Looking at the possibilities for \( 99a + b(10 - b) = (c - 1)c(c + 1) \), we have two:

\begin{align*}
99 + 21 &= 120 \implies a = 1, b = 3 \text{ or } 7, c = 5 \implies n = 135 \text{ or } n = 175. \\
495 + 9 &= 504 \implies a = 5, b = 1 \text{ or } 9, c = 8 \implies n = 518 \text{ or } n = 598.
\end{align*}

So, there are four 3-digit numbers which satisfy the requirements and the largest of these four numbers is 598,

hence (598).
Alternative 2

The number $abc$ is equal to $a + b^2 + c^3$, and these are the possible values of $b^2$ and $c^3$:

<table>
<thead>
<tr>
<th>Digit</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
<td>36</td>
<td>49</td>
<td>64</td>
<td>81</td>
</tr>
<tr>
<td>Cube</td>
<td>0</td>
<td>1</td>
<td>8</td>
<td>27</td>
<td>64</td>
<td>125</td>
<td>216</td>
<td>343</td>
<td>512</td>
<td>729</td>
</tr>
</tbody>
</table>

We try these numbers in an addition grid, trying the large values of $c$ first, then filling in possible values for $a$ and $b$. This trial-and-error search is presented here as a tree.

The largest solution found is 598, and any solutions on branches $c = 7, c = 6, \ldots, c = 1$ must be less than this, hence (598).