SUPPLEMENTARY NOTES

The Frobenius Coin Problem

AUSTRALIAN MATHS TRUST

Learning areas: Number, patterns, algebra, algorithms, linear equations, Diophantine equations, proof by contradiction, mathematical induction, strong induction, divisibility, greatest common divisor, Extended Euclidean Algorithm

These supplementary notes on the *Frobenius Coin Problem* are intended for school teachers, year 11 or 12 students studying Specialist Mathematics, students who have studied the Noether or Polya books of the AMT's *Enrichment* series, or other interested readers with a strong background in secondary school mathematics. It is recommended that you first attempt parts (a) to (f) of the introductory activity and also explore the accompanying linear graphing tool, both available at http://www.amt.edu.au/mathspack.

For further hints and tips, contact mail@amt.edu.au.

1 Pythagistan revisited: the 5c and 8c problem

In the introductory activity we were able to argue, somewhat informally, that 27 cents is the largest total that cannot be made from combinations of 5c and 8c coins. In this section we will work through two different proofs of this fact using the Principle of Mathematical Induction.

The first follows the same line of reasoning as the solution of part (a) vi. in the introductory activity, where sufficiently many consecutive cases are known to work. As such, it is actually an application of strong induction, with more than one initial case needing to be checked manually. While mathematically sound and reasonably intuitive to follow, the drawback of this approach is that the number of consecutive initial cases to be checked depends on the value of the two coins themselves. It is therefore not obvious how this might be generalised to coins with arbitrary values.

The second proof is more nuanced since it relies on the analysis of several cases, but it is more readily adaptable to other coin values since only one initial case is required to kickstart the inductive argument.

1.1 The 'adding extra coins' strategy

We aim to prove that every total of at least 28 cents can be made from a combination of 5c and 8c coins. This statement can be written more formally as follows, where x and y represent the numbers of each type of coin.

Theorem: *For every integer* $n \geq 28$*, there exist non-negative integers* x *and* y *such that* $5x+8y=n$.

Proof: We begin with the observation that the statement is true for all n satisfying $28 \leq n \leq 32$, since

$$28 = 5\times 4 + 8\times 1, \quad 29 = 5\times 1 + 8\times 3, \quad 30 = 5\times 6 + 8\times 0,$$
$$31 = 5\times 3 + 8\times 2, \quad 32 = 5\times 0 + 8\times 4.$$

Next we assume, by strong induction, that for some $k \geq 32$, the statement is true for all n satisfying $28 \leq n \leq k$. We are required to prove that the statement is also true for $n = k+1$.

Since $k \geq 32$, we have $28 \leq k-4 < k$, so the statement is true for $n = k-4$ by the inductive assumption. That is, there exist non-negative integers x' and y' such that $5x'+8y' = k-4$. Letting $x = x'+1$ and $y = y'$, which are both non-negative, we have

$$\begin{aligned} 5x + 8y &= 5(x'+1) + 8y' \\ &= (5x' + 8y') + 5 \\ &= (k-4) + 5 \\ &= k+1, \end{aligned}$$

so the statement is true for $n = k+1$. This completes the proof by strong induction. □

Exercise A: Adapt the proof above to show that every total of 50c or more can be made from a combination of the Eulish 6c and 11c coins.

1.2 The 'coin replacement' strategy

Here we give an alternative proof based on the following strategy: if you have enough 5c coins, replace some of them with 8c coins to increase the total value by 1 cent; if you have enough 8c coins, you can achieve the same effect by replacing some of those instead; if you don't have enough of either coin, then not to worry, your total must have been less than the range of values we are interested in anyway. The precise replacement strategy relies on the values found in part (e) of the introductory activity.

Theorem: *For every integer $n \geq 28$, there exist non-negative integers x and y such that $5x+8y = n$.*

Proof: We begin by observing that the statement is true for $n = 28$, since $28 = 5 \times 4 + 8 \times 1$. Next we assume, by induction, that for some $k \geq 28$ the statement is true for $n = k$. That is, there exist non-negative integers x' and y' such that $5x' + 8y' = k$. We are required to prove that the statement is also true for $n = k+1$.

If $x' \geq 3$, then we may let $x = x'-3$ and $y = y'+2$, so that x and y are both non-negative. Then

$$\begin{aligned} 5x + 8y &= 5(x'-3) + 8(y'+2) \\ &= (5x' + 8y') - 15 + 16 \\ &= k+1, \end{aligned}$$

so the statement is also true for $n = k+1$, as required. (This argument amounts to replacing three 5c coins with two 8c coins, provided there are enough 5c coins available.)

If $y' \geq 3$, then we may let $x = x'+5$ and $y = y'-3$, so that x and y are both non-negative. Then

$$\begin{aligned} 5x + 8y &= 5(x'+5) + 8(y'-3) \\ &= (5x' + 8y') + 25 - 24 \\ &= k+1, \end{aligned}$$

so the statement is also true for $n = k+1$, as required. (This is the reverse situation, replacing three 8c coins with five 5c coins.)

Suppose, on the other hand, that both $x' < 3$ and $y' < 3$. Then, given that they are integers, we have $x' \leq 2$ and $y' \leq 2$ so that

$$k = 5x' + 8y' \leq 5 \times 2 + 8 \times 2 = 26.$$

However, this contradicts the inductive assumption that $k \geq 28$. (That is, if the total is large enough, you always have enough 5c or 8c coins available to apply at least one of the replacement strategies above.) This completes the proof by induction. □

Exercise B: Adapt the proof above to show that every total of 50c or more can be made from a combination of the Eulish 6c and 11c coins.

1.3 Consolidation

Exercise C: Choose two other coin values a and b, with no common factors greater than 1. Experiment to find c, the largest impossible total (it will be less than ab). Use both the 'adding extra coins' and 'coin replacement' strategies described above to prove the following result:

For every integer $n \geq c+1$, there exist non-negative integers x and y such that $ax + by = n$.

2 Finding the coin replacement strategy

In Section 1.2, the two coin replacement strategies required for the Pythagistani 5c and 8c coins were as follows: replace three 5c coins with two 8c coins, or replace three 8c coins with five 5c coins. These strategies can be summarised by the following numerical observations:

$$8 \times 2 = 5 \times 3 + 1 \qquad \text{and} \qquad 5 \times 5 = 8 \times 3 + 1,$$

or equivalently

$$1 = 5 \times (-3) + 8 \times 2 = 5 \times 5 + 8 \times (-3).$$

Thus we have found two integer solutions (s, t) of the equation $5s + 8t = 1$, one with $s < 0$ and $t > 0$, and the other with $s > 0$ and $t < 0$.

You may not have found it too difficult to stumble upon the required replacement strategies for the Eulish 6c and 11c coins, or your chosen coin values in Exercise C. However, for arbitrary pairs of coin values it is useful to have a method for finding those replacement strategies more systematically.

2.1 The Euclidean Algorithm

In primary or secondary school, you have most likely been introduced to the following strategy for finding the *greatest common divisor* (otherwise known as the *highest common factor*) of two positive integers: determine the factorisation of each number as a product of primes (via a factor tree, for example) and then identify the common primes. For example, $90 = 2 \times 3 \times 3 \times 5$ and $24 = 2 \times 2 \times 2 \times 3$, so $\gcd(90, 24) = 2 \times 3 = 6$.

In practice this method is terribly inefficient. The *Euclidean Algorithm* provides an alternative method which does not rely on prime factorisation; in fact, in its simplest form it relies only on repeated subtraction.

Given positive integers a and b, with $a > b$, $\gcd(a, b)$ is found with the Euclidean Algorithm as follows:

- Divide a by b, taking note of the quotient q and remainder r, and rewrite the division statement as a multiplication with remainder, namely $a = bq + r$.
- Replace the pair of integers (a, b) with the new pair (b, r) and repeat the above step.
- Keep repeating until the remainder is zero. The last non-zero remainder is the greatest common divisor.

Example: Calculate $\gcd(90, 24)$.

$$\begin{aligned} 90 \div 24 &= 3 \,\text{rem}\, 18 \\ 24 \div 18 &= 1 \,\text{rem}\, 6 \\ 18 \div 6 &= 3 \,\text{rem}\, 0 \end{aligned} \qquad\qquad \begin{aligned} 90 &= \boxed{24} \times 3 + \boxed{18} \\ \boxed{24} &= \boxed{18} \times 1 + 6 \\ 18 &= 6 \times 3 + 0 \end{aligned}$$

so $\gcd(24, 90) = 6$ (highlighted).

It is not necessary to write the division statement each time, as seen here.

Example: Calculate $\gcd(184, 54)$.

$$\begin{aligned} 184 &= 54 \times 3 + 22 \\ 54 &= 22 \times 2 + 10 \\ 22 &= 10 \times 2 + 2 \\ 10 &= 2 \times 5 + 0 \end{aligned}$$

so $\gcd(130, 54) = 2$.

Exercise D: Use the Euclidean Algorithm to calculate the following:

i. $\gcd(30, 17)$ ii. $\gcd(96, 27)$ iii. $\gcd(425, 245)$ iv. $\gcd(201, 88)$

2.2 The Extended Euclidean Algorithm

Given two positive integers a and b, the *Extended Euclidean Algorithm* provides a method for writing $\gcd(a, b)$ as a combination of a and b. That is, it finds integer solutions (s, t) of the equation $as + bt = \gcd(a, b)$ (known as *Bezout's identity*).

The Extended Euclidean Algorithm uses the lines of working generated when calculating the greatest common divisor. Starting with the second last line of working, write $\gcd(a, b)$ as the subject of the equation. Then manipulate the right-hand side by substituting alternative expressions from each preceding line, simplifying much as you would with algebraic like terms, but resisting the temptation to simply evaluate the numerical values involved. Working from bottom to top in this way results in a combination of the original numbers a and b, as desired.

Example: Find an integer solution (s, t) for the equation $184s + 54t = 2$.

Recall the following lines of working to calculate $\gcd(184, 54)$:

$$\begin{aligned} 184 &= 54 \times 3 + 22 && (1) \\ 54 &= 22 \times 2 + 10 && (2) \\ 22 &= 10 \times 2 + 2 && (3) \\ 10 &= 2 \times 5 + 0 \end{aligned}$$

Then

$$\begin{aligned} 2 &= 22 - 10 \times 2 && \text{from } (3) \\ &= 22 - (54 - 22 \times 2) \times 2 && \text{from } (2) \\ &= 22 - 54 \times 2 + 22 \times 4 \\ &= 22 \times 5 - 54 \times 2 \\ &= (184 - 54 \times 3) \times 5 - 54 \times 2 && \text{from } (1) \\ &= 184 \times 5 - 54 \times 15 - 54 \times 2 \\ &= 184 \times 5 + 54 \times (-17), \end{aligned}$$

so $(s, t) = (5, -17)$.

Exercise E: Apply the Extended Euclidean Algorithm to find integer solutions of $as+bt = \gcd(a, b)$ with the values from Exercise D. Answers can be checked at the following website: http://people.math.sc.edu/sumner/numbertheory/euclidean/euclidean.html

2.3 Application to Pythagistani coin replacement

Since $\gcd(5, 8) = 1$, we can use the Extended Euclidean Algorithm to find integer solutions of $5s + 8t = 1$, as follows:

$$
\begin{aligned}
8 &= 5 \times 1 + 3 && (4)\\
5 &= 3 \times 1 + 2 && (5)\\
3 &= 2 \times 1 + 1 && (6)\\
2 &= 1 \times 2 + 0
\end{aligned}
$$

$$
\begin{aligned}
1 &= 3 - 2 && \text{from } (6)\\
&= 3 - (5 - 3) && \text{from } (5)\\
&= 3 \times 2 - 5\\
&= (8 - 5) \times 2 - 5 && \text{from } (4)\\
&= 8 \times 2 - 5 \times 3\\
&= 5 \times (-3) + 8 \times 2,
\end{aligned}
$$

hence $(s, t) = (-3, 2)$ is a solution.

The solution above happens to correspond to the first coin replacement strategy for the induction argument in Section 1.2: replacing three 5c coins with two 8c coins results in an overall increase of 1 cent.

Finding the alternative replacement strategy amounts to observing that it is possible to make a total of 40c in two different ways. That is, replacing five 8c coins with eight 5c coins has no net effect:

$$
\begin{aligned}
1 &= 5 \times (-3) + 8 \times 2\\
&= 5 \times (-3) + [5 \times 8 - 8 \times 5] + 8 \times 2\\
&= 5 \times 5 + 8 \times (-3),
\end{aligned}
$$

so $(s, t) = (5, -3)$ is the other required solution.

Exercise F: Use the Extended Euclidean Algorithm to establish two replacement strategies for the Eulish 6c and 11c coins, and also for the pair of coins you chose for Exercise C.

Note that repeatedly applying the final coin swap, it becomes clear that there are in fact infinitely many strategies, as shown in the next exercise.

Exercise G: Suppose that (s_1, t_1) is a known integer solution for the equation $as + bt = \gcd(a, b)$. Show that $(s_1 + kb, t_1 - ka)$ is also a solution for any integer k.

Exercise H: State the general solution for each equation in Exercise E.

While the general solution provides an unlimited number of potential coin replacement strategies for the induction argument, the larger the value of n (either positive or negative), the larger the number of coins required to effect the replacement. Hence the smallest solutions, namely when $n = 0$ and $n = 1$ or -1, should still be used since they are the most widely applicable.

2.4 Consolidation

Exercise I: Use the coin replacement strategy to prove that every total greater than or equal to 7788 cents can be made from a combination of 119c and 67c coins. How many initial cases would you need to check in order to apply the 'adding extra coins' strategy of Section 1.1?

3 Finding the largest impossible total

The induction argument for Pythagistani 5c and 8c coins starts with the initial total of 28 cents. Verifying that this case is possible is simply a matter of finding a combination of coins that works. But how can we be certain that making 27 cents is impossible? After all, the Extended Euclidean Algorithm provides infinitely many solutions for the equation $5s + 8t = 1$, and we can easily adapt these to find infinitely many solutions for $5s + 8t = 27$. It just so happens that each such solution has a negative value for either s or t, so it does not provide the non-negative integer solution required for the context of the coin problem.

One strategy to establish this is to exhaustively list all possibilities. For example, consider in turn each possible number of 5c coins: there cannot be zero 5c coins, because 27 is not a multiple of 8; there cannot be one 5c coin, because $27 - 5 = 22$ is not a multiple of 8, etc. This is relatively painless for such a small total, but for larger totals this approach quickly becomes impractical.

3.1 Divisibility and contradiction

Suppose that a non-negative integer solution (x, y) exists for the equation $5x + 8y = 27$. Adding five to both sides, rearranging and factorising, we have

$$\begin{aligned} 5x + 8y + 5 &= 32 \\ 5x + 5 &= 32 - 8y \\ 5(x + 1) &= 8(4 - y). \end{aligned}$$

From the uniqueness of prime factorisation (the *Fundamental Theorem of Arithmetic*), the fact that 5 and 8 have no common prime factors, and the condition $x + 1 \geq 1$, it follows that $(4 - y)$ must be a positive multiple of 5. In particular, $4 - y \geq 5$, but then $y \leq -1$, which contradicts the assumption that y is non-negative. It follows that no such non-negative solution (x, y) can exist, so 27 cents cannot be made from 5c and 8c coins.

Exercise J: Adapt the above argument to show that a total of 49 cents cannot be made from a combination of 6c and 11c coins, and also that 7787 cannot be made from 119c and 67c coins.

Exercise K: Suppose two coins have positive integer values a and b, where $\gcd(a, b) = 1$. Show that, in general, it is not possible to make a total value of $ab - a - b$ cents using these coins.

3.2 Extended Euclidean Algorithm revisited

Having established that, in general, a total of $ab - a - b$ is impossible, we are left with one final question: is the subsequent value of $ab - a - b + 1$ cents, or equivalently $(a - 1)(b - 1)$ cents, always possible? This expression corresponds to the initial cases that kickstarted the inductive arguments considered so far: the total of $4 \times 7 = 28$ cents for 5c and 8c coins, the total of $5 \times 10 = 50$ cents for 6c and 11c coins, and the total of $118 \times 66 = 7788$ cents for 119c and 67c coins. In the latter example, it is far from obvious how to actually make the total of 7788 cents from a combination of 119c and 67c coins. To do so, we return to the Extended Euclidean Algorithm.

Having found the solution $(s, t) = (-9, 16)$ of the equation $119s + 67t = 1$, we proceed as follows:

$$\begin{aligned} 7788 &= 118 \times 66 \\ &= (119 - 1) \times (67 - 1) \\ &= 119 \times 67 - 119 - 67 + 1 \\ &= 119 \times 67 - 119 - 67 + (119 \times (-9) + 67 \times 16) \\ &= 119 \times 67 + 119 \times (-10) + 67 \times 15. \end{aligned}$$

We now have a decision to make: how to regroup the first term 119×67? To ensure that the overall sum has non-negative coefficients, we choose to group it with the other multiple of 119:

$$\begin{aligned} 7788 &= [\,119 \times 67 + 119 \times (-10)\,] + 67 \times 15 \\ &= 119 \times 57 + 67 \times 15. \end{aligned}$$

Hence a total of 7788 cents can be made from fifty-seven 119c coins and fifteen 67c coins.

Exercise L: Use the method outlined above to show that it is possible to make a total of $6 \times 9 = 54$ cents from 7c and 10c coins. [Try to avoid guessing the solution first.]

To generalise the above, we need the following observation which follows from the general solution in Exercise G: *there exists an integer solution* (s, t) *of* $as + bt = 1$ *which satisfies* $0 < s < b$ *and* $-a < t < 0$.

Exercise M: With coins as in Exercise K, use the above observation to show that it is always possible to make a total of $(a-1)(b-1)$ cents.

4 Solution of the general coin problem

Theorem: *Suppose two coins have positive integer values* a *and* b*, respectively, where* $\gcd(a, b) = 1$. *Then making a total of* $ab - a - b$ *cents is impossible, but every larger total is possible.*

Exercise N: Complete the proof!

References

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