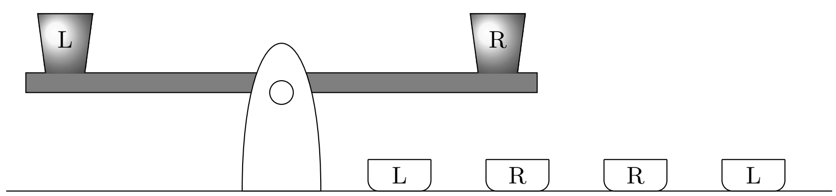
AUSTRALIAN MATHS TRUST

AMC Challenge
Junior: Years 7–8
Practice Problem

J5: Tipping Points

Two empty buckets are placed on a balance beam, one at each end. Balls of the same weight are placed in the buckets one at a time. If the number of balls is the same in each bucket, the beam stays horizontal. If there is a difference of only one ball between the buckets, the beam moves a little but the buckets and balls remain in place. However, if the difference between the number of balls is two or more, the beam tips all the way, the buckets fall off, and all the balls fall out.

There are several bowls, each containing some of the balls and each labelled L or R. If a ball is taken from a bowl labelled L, the ball is placed in the left bucket on the beam. If a ball is taken from a bowl labelled R, the ball is placed in the right bucket.



a. Julie arranges six labelled bowls in a row. She takes a ball from eachbowl in turn from left to right, and places it in the appropriate bucket. List all sequences of six bowls which do not result in the beam tipping.

b. Julie starts again with both buckets empty and with six bowls in a row. As before, she takes a ball from each bowl in turn, places it in the appropriate bucket, and the beam does not tip. She then empties both buckets and takes a ball from the 2nd, 4th, and 6th bowl in turn and places it in the appropriate bucket. Again the beam does not tip. Once more she empties both buckets but this time takes a ball from the 3rd and 6th bowl in turn and places it in the appropriate bucket. Yet again the beam does not tip. List all possible orders in which the six bowls could have been arranged.

For a large number of bowls, a ball could be taken from every bowl, or every second bowl, or every third bowl, and so on. If a ball is taken from bowl m, followed by bowl 2*m*, then bowl 3*m*, and so on (every *m*th bowl), we say an *m-selection* was used. For example, in Part b, Julie used a 1-selection, then a 2-selection, and finally a 3-selection.

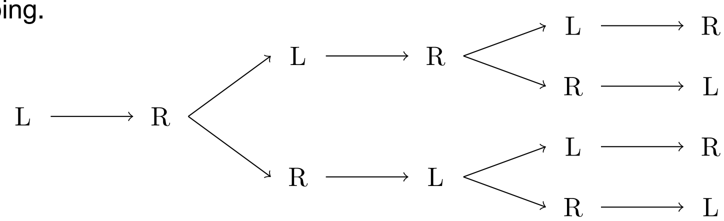
c. Find all sequences of 11 bowls for which the beam does not tip no matter what *m*-selection is used.

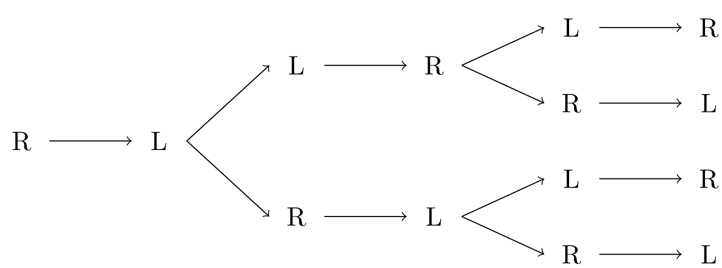
d. Show that it is impossible to have a sequence of 12 bowls so that every *m*-selection is non-tipping

Solutions

a. ***Alternative 1***

Moving through the sequence of bowls from the first to the last, the beam will tip if and only if the difference in the number of Ls and Rs is at any stage greater than 1. The following tree diagrams show the possible sequences, from left to right, of 6 bowls that avoid the beam tipping.





So the only sequences of bowls for which the beam does not tip are:

LRLRLR, LRLRRL, LRRLLR, LRRLRL,
RLLRLR, RLLRRL, RLRLLR, RLRLRL

Alternative 2

To avoid tipping the beam, the first two bowls in the sequence must be LR or RL. Either way, the beam remains perfectly balanced. So the next two bowls in the sequence must be LR or RL. Again, either way, the beam remains perfectly balanced. So the last two bowls in the sequence must be LR or RL. Hence the only sequences of bowls for which the beam does not tip are:

LRLRLR, LRLRRL, LRRLLR, LRRLRL,
RLLRLR, RLLRRL, RLRLLR, RLRLRL

b. Since Julie uses all bowls and does not tip the beam, the bowls must be in one of the eight sequences found in Part a:

LRLRLR, LRLRRL, LRRLLR, LRRLRL,
RLLRLR, RLLRRL, RLRLLR, RLRLRL.

Julie uses bowls 2, 4, 6 without tipping the beam. This eliminates the sequences LRLRLR, LRLRRL, RLRLLR, RLRLRL. So she is left with the sequences LRRLLR, LRRLRL, RLLRLR, RLLRRL.

Julie uses bowls 3 and 6 without tipping the beam. This eliminates the sequences LRRLLR and RLLRRL.

So the only sequences that work for all three procedures are: LRRLRL and RLLRLR.

c. ***Alternative 1***

The beam will not tip for any *m*-selection with $m \geq 6$ since, in those cases, a ball is drawn from only one bowl. For each *m*-selection with $m \leq 5$, the first 2 bowls must be RL or LR. For $m = 1$, let the first 2 bowls be LR.

Then, for $m = 2$, bowl 4 must be L. Hence bowl 3 is R.

Bowl	1	2	3	4	5	6	7	8	9	10	11
Letter	L	R	R	L							

For $m = 3$, bowl 6 must be L. Hence bowl 5 is R.

Bowl	1	2	3	4	5	6	7	8	9	10	11
Letter	L	R	R	L	R	L					

For $m = 2$, bowl 8 must be R. Hence bowl 7 is L.

Bowl	1	2	3	4	5	6	7	8	9	10	11
Letter	L	R	R	L	R	L	L	R			

For $m = 5$, bowl 10 must be L. Hence bowl 9 is R.

Bowl	1	2	3	4	5	6	7	8	9	10	11
Letter	L	R	R	L	R	L	L	R	R	L	

Bowl 11 can be L or R. Thus we have two sequences starting with LR such that no *m*-selection causes the beam to tip.

Similarly, there are two sequences starting with RL such that no *m*-selection causes the beam to tip.

So there are four sequences of 11 bowls such that no *m*-selection causes the beam to tip:

LRRLRLLRRLL, LRRLRLLRRLR,
RLLRLRRLLRR, RLLRLRRLLRL.

Alternative 2

As in the second solution to Part **a**, since the first two letters in any non-tipping sequence must be different, the next two letters in the sequence must be different, then the next two and so on.

If a sequence of 11 bowls is non-tipping for every *m*-selection, then for *m* = 1, the 5th and 6th bowls are different and the 7th and 8th bowls are different. So these four bowls are LRLR, LRRL, RLLR, or RLRL. For *m* = 2, the 6th and 8th bowls must be different. This eliminates LRLR and RLRL, and leaves us with:

Bowl	1	2	3	4	5	6	7	8	9	10	11
Seq. 1	R	L	L	R	L	R	R	L	L	R	
Seq. 2	L	R	R	L	R	L	L	R	R	L	

For *m* = 3, the 3rd and 6th letters must be different. For *m* = 4, the 4th and 8th letters must be different. So we have:

Bowl	1	2	3	4	5	6	7	8	9	10	11
Seq. 1					L	R	R	L			
Seq. 2					R	L	L	R			

For *m* = 2, the 2nd and 4th letters must be different. For *m* = 1, the 1st and 2nd letters must be different. So we have:

Bowl	1	2	3	4	5	6	7	8	9	10	11
Seq. 1			L	R	L	R	R	L			
Seq. 2			R	L	R	L	L	R			

For *m* = 5, the 5th and 10th letters must be different. For *m* = 1, the 9th and 10th letters must be different. So we have:

Bowl	1	2	3	4	5	6	7	8	9	10	11
Seq. 1	R	L	L	R	L	R	R	L			
Seq. 2	L	R	R	L	R	L	L	R			

Finally, the 11th bowl could be either L or R. So there are four sequences of 11 bowls for which every *m*-selection is non-tipping:

RLLRLRRLLRR, RLLRLRRLLRL,
LRRLRLLRRLL, LRRLRLLRRLR.

d. ***Alternative 1***

Suppose we have a sequence of 12 bowls for which every m-selection is non-tipping. Then, as in Part **c**, up to bowl 10 we have only two possible sequences:

Bowl	1	2	3	4	5	6	7	8	9	10	11	12
Letter	L	R	R	L	R	L	L	R	R	L		

Bowl	1	2	3	4	5	6	7	8	9	10	11	12
Letter	R	L	L	R	L	R	R	L	L	R		

For *m* = 6, bowl 12 must be R and L respectively. However, for *m* = 3, bowl 12 must be L and R respectively. So there is no sequence of 12 bowls for which every m-selection is non-tipping.

Alternative 2

As in the second solution to Part **a**, since the first two letters in any non-tipping sequence must be different, the next two letters in the sequence must be different, then the next two and so on.

Suppose we have a sequence of 12 bowls for which every m-selection is non-tipping. Then for $m = 1$, the 9th and 10th bowls are different and the 11th and 12th bowls are different. So the last 4 letters in the sequence are LRLR, LRRL, RLLR, or RLRL.

For $m = 2$, the 10th and 12th letters must be different. This eliminates LRLR and RLRL. For $m = 3$, the 9th and 12th letters must be different. This eliminates LRRL and RLLR. So there is no sequence of 12 bowls for which every m-selection is non-tipping.