LESSON CARD

The Frobenius Coin Problem

An activity suitable for Australian years 7–12

AUSTRALIAN MATHS TRUST

Learning areas: Number and place value, patterns and algebra, linear relationships, logic and enumeration.

Resources: For the links to the Australian Curriculum content descriptors, full solutions and additional resources related to this activity, including an interactive linear graphing tool and extension activities for senior Specialist Mathematics students, visit www.amt.edu.au/resources-for-the-classroom.

Frobenius Coins

In Australia we have coins valued at 5c, 10c, 20c, 50c, $1 and $2. So when we pay for something in cash, total costs need to be rounded to the nearest five cents, otherwise it may not be possible to make the correct change. (Your teacher might remember the good old days when we also had 1c and 2c coins, so rounding wasn't necessary.)

In the Republic of Pythagistan, they only have 5c and 8c coins. Actually, this is pretty convenient, because it means they can make most totals exactly, without needing any smaller denominations.

For example, to pay for something worth 73 Pythagistani cents exactly requires five 5c coins and six 8c coins.

Challenges

(a) Using Pythagistani 5c and 8c coins:

i. Find, if possible, a different way to make a total of 73 cents.

ii. Find the smallest total which can be made in two different ways. Explain why it is the smallest.

iii. Are there any other ways to make 73 cents? If so, find them all. If not, explain why not.

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iv. Find all totals less than your answer to ii. which are possible in at least one way.

v. What is the largest impossible total? Explain why you know that all totals greater than this are possible. [Hint: near the end of your list for part iv., how many consecutive possible totals are there? How does this guarantee that all larger totals are also possible?]

(b) In Euland they use 6c and 11c coins. Repeat parts i. to v. of (a) using Eulish coins.

(c) Make up your own countries with their own pairs of coins, and investigate which totals are possible and which are impossible.

(d) i. Give an example of a pair of coins for which there are infinitely many impossible totals. Explain why.

ii. Hence make a conjecture (guess) about when two coin values will result in only finitely many impossible totals.

(e) i. Returning to Pythagistan, if you have enough 5c coins it is possible to increase the total value by 1 cent by replacing some of them with 8c coins. Find the numbers of coins involved.

ii. Find the numbers of coins involved if the total increases by 1 cent when some 8c coins are replaced with 5c coins instead.

(f) Repeat (e) for Eulish coins and the other coins you made up in (c).

(g) Research the *Extended Euclidean Algorithm* online and investigate its connection with your results so far.

(h) Given a country X with coins worth A cents and B cents (which satisfy your conjecture in (d) ii.), find a rule in terms of A and B for the largest total which cannot be made with those coins.

(i) Research the *Frobenius Coin Problem* to check your answer to part (h).

(j) Investigate what happens if you allow three or more different coins.

(k) Research the *Water Jug Riddle* (as made famous by Bruce Willis in *Die Hard 3*). How are these problems related? How are they different?