LESSON CARD

The Frobenius Coin Problem

An activity suitable for Australian years 7–12

AUSTRALIAN MATHS TRUST

Learning areas: Number and place value, patterns and algebra, linear relationships, logic and enumeration.

Resources: For additional resources related to this activity, including an interactive linear graphing tool and extension activities for senior Specialist Mathematics students, visit www.amt.edu.au/resources-for-the-classroom. Links to the Australian Curriculum content descriptors are on page 4.

Frobenius Coins

In Australia we have coins valued at 5c, 10c, 20c, 50c, \$1 and \$2. So when we pay for something in cash, total costs need to be rounded to the nearest five cents, otherwise it may not be possible to make the correct change. (Your teacher might remember the good old days when we also had 1c and 2c coins, so rounding wasn't necessary.)

In the Republic of Pythagistan, they only have 5c and 8c coins. Actually, this is pretty convenient, because it means they can make most totals exactly, without needing any smaller denominations.

For example, to pay for something worth 73 Pythagistani cents exactly requires five 5c coins and six 8c coins.

Challenges

(a) Using Pythagistani 5c and 8c coins:

i. Find, if possible, a different way to make a total of 73 cents.

ii. Find the smallest total which can be made in two different ways. Explain why it is the smallest.

iii. Are there any other ways to make 73 cents? If so, find them all. If not, explain why not.

iv. Find all totals less than your answer to ii. which are possible in at least one way.

v. What is the largest impossible total? Explain why you know that all totals greater than this are possible. [Hint: near the end of your list for part iv., how many consecutive possible totals are there? How does this guarantee that all larger totals are also possible?]

(b) In Euland they use 6c and 11c coins. Repeat parts i. to v. of (a) using Eulish coins.

(c) Make up your own countries with their own pairs of coins, and investigate which totals are possible and which are impossible.

(d) i. Give an example of a pair of coins for which there are infinitely many impossible totals. Explain why.

ii. Hence make a conjecture (guess) about when two coin values will result in only finitely many impossible totals.

(e) i. Returning to Pythagistan, if you have enough 5c coins it is possible to increase the total value by 1 cent by replacing some of them with 8c coins. Find the numbers of coins involved.

ii. Find the numbers of coins involved if the total increases by 1 cent when some 8c coins are replaced with 5c coins instead.

(f) Repeat (e) for Eulish coins and the other coins you made up in (c).

(g) Research the *Extended Euclidean Algorithm* online and investigate its connection with your results so far.

(h) Given a country X with coins worth A cents and B cents (which satisfy your conjecture in (d) ii.), find a rule in terms of A and B for the largest total which cannot be made with those coins.

(i) Research the *Frobenius Coin Problem* to check your answer to part (h).

(j) Investigate what happens if you allow three or more different coins.

(k) Research the *Water Jug Riddle* (as made famous by Bruce Willis in *Die Hard 3*). How are these problems related? How are they different?

Some answers

(a) i. Thirteen 5c coins and one 8c coin.

ii. The smallest total in two ways is 40c. To have two different ways of getting a certain total, it must be possible to swap some 8c coins with some 5c coins, or vice versa, without changing that total. The value of the coins in the swap must be a multiple of both 5c and 8c, and the smallest such multiple is 40c. To make the total involved as small as possible, we just make sure no other coins are involved, so that the total is itself 40c (either eight 5c coins or five 8c coins).

iii. No, there are no other ways to make 73c. As in ii., if we try to swap some 5c and 8c coins, we are dealing with a swap of 40c at a time. However, it is not possible to further adapt the two known solutions for 73c without one of the numbers of coins taking on negative values, which is not allowed.

iv. The following totals under 40c are possible with 5c and 8c coins:

5c, 8c, 10c, 13c, 15c, 16c, 18c, 20c, 21c, 23c, 24c, 25c, 26c,
28c, 29c, 30c, 31c, 32c, 33c, 34c, 35c, 36c, 37c, 38c, 39c.

v. The largest impossible total is 27c. In iv. we see that, in particular, each total from 28c to 32c is possible. Adding 5c to each of these, it follows that 33c to 37c are also possible. Repeatedly applying this strategy, we then see that 38c to 42c, 43c to 47c, 48c to 52c, etc. are all possible. (This argument can be formalised by a method of proof called the *Principle of Mathematical Induction*.)

(b) i. Three 6c coins and five 11c coins.

ii. The smallest total which is possible in two ways is $6 \times 11 = 66$c.

iii. There is no other way since a swap of 66c is too large to apply to the solution in part i.

iv. The possible totals less than 66c are: 6c, 11c, 12c, 17c, 18c, 22c–24c, 28c–30c, 33c–36c, 39c–42c, 44c–48c, 50c–65c.

v. The largest impossible total is 49c. After this there are at least six consecutive possible totals, to which we can successively add 6c coins to achieve all larger totals.

(d) i. For example, use 4c and 6c coins. Since both values are even, it is only possible to make an even total. Hence there are infinitely many impossible totals, namely 2c and all odd totals. (Similarly for any other pair of values with a common factor greater than 1.)

ii. We claim that there are finitely many impossible totals if, and only if, the coin values have highest common factor equal to 1 (that is, they are *relatively prime* or *coprime*).

(e) i. Replace three 5c coins (15c) with two 8c coins (16c).

ii. Replace three 8c coins (24c) with five 5c coins (25c).

(f) i. Replace one 11c coin (11c) with two 6c coins (12c).

ii. Replace nine 6c coins (54c) with five 11c coins (55c).

(g) The *Euclidean Algorithm* is an efficient method for calculating the highest common factor (or *greatest common divisor*) of two numbers. When the answer is 1, the *Extended* Euclidean Algorithm gives a way to write 1 as the difference of multiples of the two numbers; this results in one of the solutions to (e) and (f), and the other solution can be found by swapping the 40c or 66c referred to in part ii. of (a) and (b).

(h) Given A cent and B cent coins, with highest common factor 1, the largest impossible total is $AB - A - B$ cents. (For example, with 5c and 8c coins we have $5 \times 8 - 5 - 8 = 27$c, as in part vi. of (a).)

For further hints and tips, contact mail@amt.edu.au.

Australian Curriculum content descriptors

The following is not intended to be an exhaustive list, but indicates how the above activity aligns with various stages of the mathematics curriculum. Follow the links to the ACARA website for elaborations.

- Year 7, ACMNA149 Investigate index notation and represent whole numbers as products of powers of prime numbers
- Year 8, ACMNA183 Carry out the four operations with rational numbers and integers, using efficient mental and written strategies and appropriate digital technologies
- Year 9, ACMNA208 Plot linear relationships on the Cartesian plane with and without the use of digital technologies
- Year 10, ACMNA235 Solve problems involving linear equations, including those derived from formulas