LESSON CARD

Scaly Tiles

An activity suitable for Australian years 5–12

AUSTRALIAN MATHS TRUST

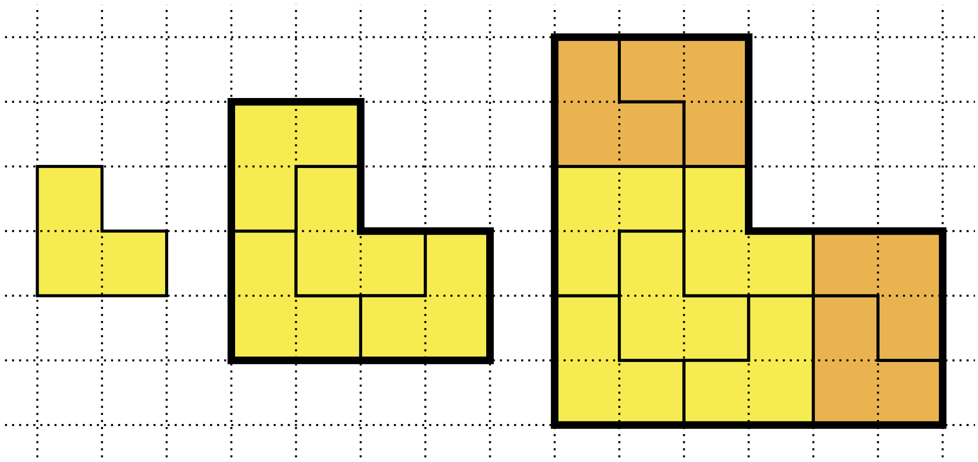
Learning areas: Shape, geometric reasoning, similarity, scale factor, transformations, patterns and algebra, surds, Pythagoras' theorem.

Resources: Square grid paper and isometric grid paper.

Visit www.amt.edu.au/resources-for-the-classroom for additional resources, including downloadable templates. Links to the applicable Australian Curriculum content descriptors are on page 8.

Scaly Tiles

The L-shaped tile on the left is formed by joining three identical squares. This type of tile is called *2-scaly* because a number of them can be arranged, without gaps or overlaps, into an enlarged copy of itself which is twice as big (it has been scaled by a factor of 2). It is also *3-scaly*, because a number of them can be arranged into an enlarged copy which is 3 times as big.



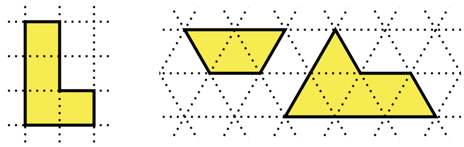
original tile **2-scaly** (2 × original size) **3-scaly** (3 × original size)

Sometimes there is more than one way to arrange the tiles to make the enlarged copy. For example, in the 3-scaly arrangement on the right, either of the two dark rectangles could be tiled with small tiles in a different way.

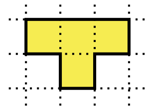
Also note that when making the enlarged copy, the original tiles can be flipped over as well as rotated.

Challenges

(a) Show that each of these tiles is both 2-scaly and 3-scaly. Remember that tiles can be flipped over, as well as rotated.



(b) Show that this T-shaped tile is 4-scaly. Explain why it is *not* 2-scaly or 3-scaly.

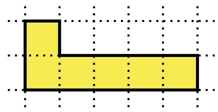


(c) How many **original** tiles fit into a 2-scaly arrangement? How many fit into a 3-scaly arrangement? What is the rule? Can you explain it?

(d) Explain why every 2-scaly tile is also 4-scaly. Show examples using the tiles in (a). For which other values of n is a 2-scaly tile also n-scaly?

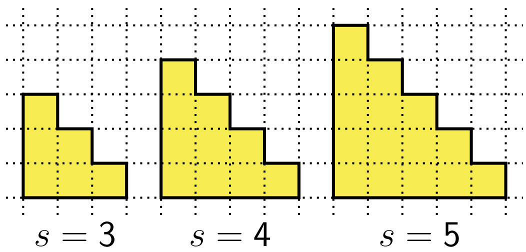
(e) Explain why every tile which is both 2-scaly and 3-scaly is also 6-scaly. Show examples using the tiles in (a). For which other values of n is a 2- and 3-scaly tile also n-scaly? What is the general rule?

(f) Find the smallest n for which this long L-shaped tile is n-scaly.

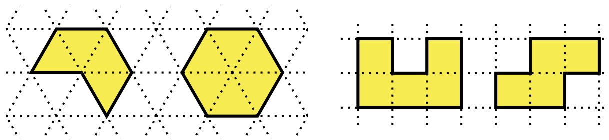


(g) Explain why every triangle is n-scaly for all n.

(h) A staircase tile can be made with any number of steps s, as shown. Explain why every staircase tile is n-scaly for some n, and find a rule for n in terms of s.

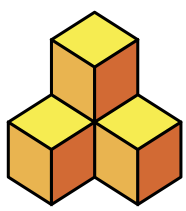


(i) Explain why none of these tiles is n-scaly for any n.



(j) Find an example of a triangle which is $\sqrt{2}$-scaly, one which is $\sqrt{5}$-scaly, and one which is $\sqrt{10}$-scaly. (Hint: by (c), how many tiles are needed?)

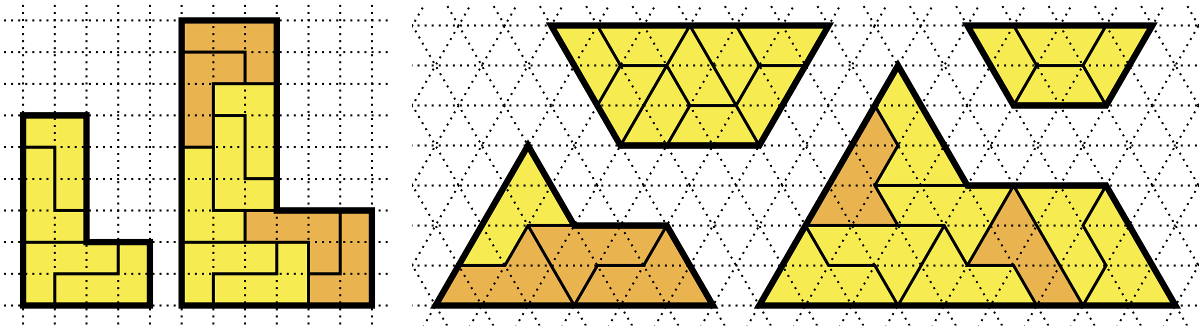
(k) This 3D block is made of four identical cubes. The visible cubes are all joined to one face of the hidden cube. Is this block 2-scaly? Is it 3-scaly? (Hint: how does (c) change?)



(l) Research how scaly tiles are related to the concepts of non-periodic tilings and self-similarity in fractals.

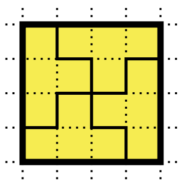
Solutions and Extensions

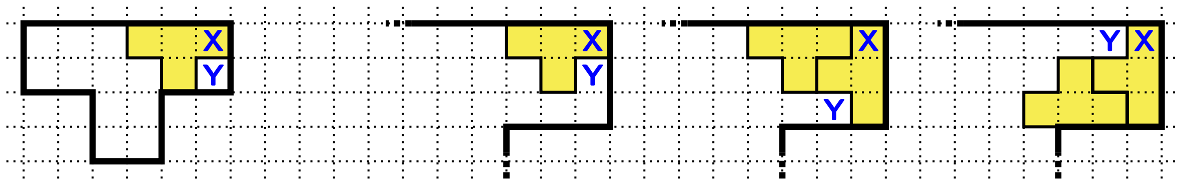
(a) The darker shading indicates which tiles have been flipped over. There are other possible solutions for most of these tilings.



Extensions: Find other examples of tiles which are 2-scaly or 3-scaly. Investigate whether the tiles above are 4-scaly, 5-scaly, 6-scaly, etc.

(b) Four T-shaped tiles can form a 4 × 4 square, as shown, so four copies of this arrangement can then be used to make a 4-scaly version of the T-shape. On the other hand, if we attempt to make a 2- or 3-scaly version of this shape, then the square in the top-right corner, marked **X**, can only be covered in the ways shown below. But none of these tilings can be completed because, among others, the square marked **Y** cannot be covered.



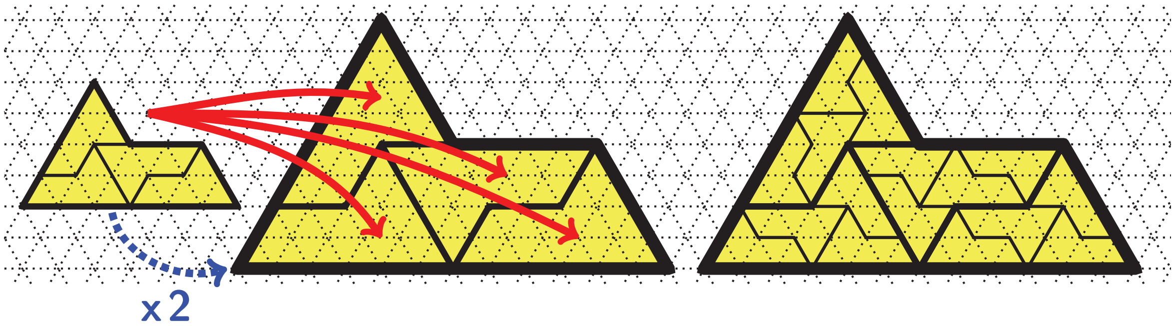


Extensions: Investigate whether this T-shaped tile is 5-scaly, 6-scaly, 7-scaly, etc. Investigate other tetrominos and pentominos.

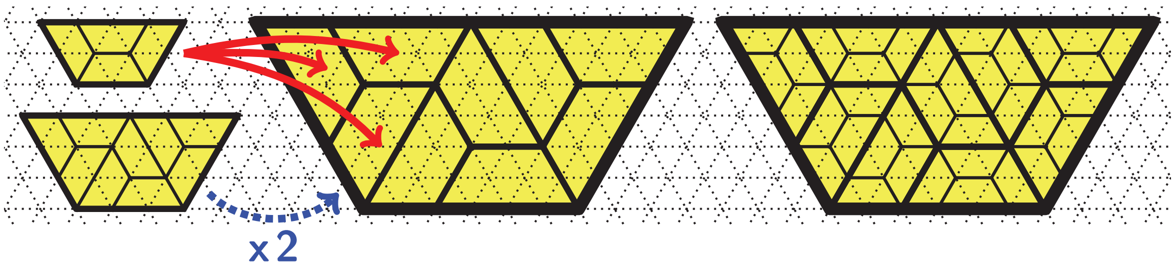
(c) Every 2-scaly arrangement has 4 tiles and every 3-scaly arrangement has 9 tiles. In general, every n-scaly arrangement has n^2 tiles. Since the linear scale factor from the original tile to the enlarged copy is n, the area scale factor is n^2. That is, the enlarged copy has n^2 times the area of the tile. Hence, that number of tiles is needed to exactly cover the same area without gaps or overlaps. (While this explanation uses some advanced concepts and terminology, it is essentially based on the generalisation of this fact: if you double the side lengths of a square, you get four times the area.)

(d) Imagine the four tiles in a 2-scaly arrangement are glued together to make a double-sized 'supertile'. Four supertiles can then be arranged in the same pattern to make a new enlarged copy which has doubled in size again. This enlarged copy is made up of 16 original tiles, with an overall scale factor of 4. Therefore the tile is also 4-scaly.

In effect, we are substituting the arrangement into an enlarged version of itself to get a more complicated arrangement; see the example below. Repeating this process, we can also get a scale factor of 8, 16, 32, ..., so the tile is 2^k-scaly for any k.



(e) Adapting the strategy of (d), we can substitute the 2-scaly arrangement into the 3-scaly one, or vice versa, to get a 6-scaly arrangement; see the example below. Repeating this process, the tile is n-scaly for any n which can be formed by multiplying 2 and 3 some number of times. That is, $n = 2^k \times 3^\ell$ for some k and ℓ.



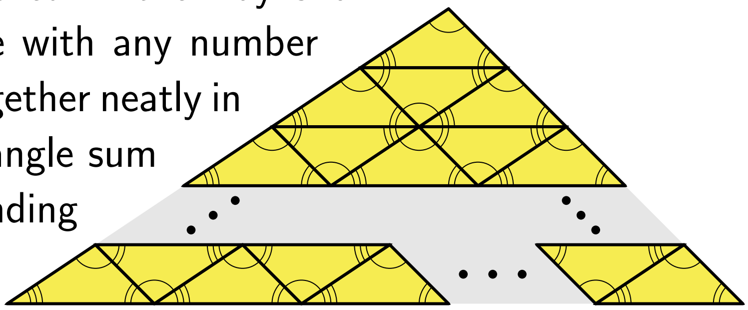
In general, we can multiply any combination of the known scale factors as many times as we like to get a new valid scale factor. More formally, if a tile is n_1-scaly, n_2-scaly, n_3-scaly, . . . , for some collection of positive numbers n_1, n_2, n_3, . . . , then it is also $n_1^{k_1} n_2^{k_2} n_3^{k_3} \cdots$-scaly for any collection of non-negative indices k_1, k_2, k_3, ...

Extensions: If a tile is p-scaly for all primes p, what can you deduce? If a tile is n-scaly and n has a prime factor p, is the tile necessarily p-scaly?

(f) Put two L-shaped tiles together to form a 6×2 rectangle, then place three of these rectangles together to form a 6×6 square. Now we can use six of these squares to form an L-shaped tile which has been scaled by a factor of 6 (and is covered by $6^2 = 36$ tiles). Therefore the tile is 6-scaly. Using similar arguments to (b), we can rule out the possibility that this tile is 2-, 3-, 4- or 5-scaly, so the least n is 6.

Extension: Investigate L-shaped tiles with different dimensions.

(g) Any triangle can be stacked in the way shown to form a similar triangle with any number of rows. The stack fits together neatly in parallel rows due to the angle sum in a triangle and corresponding angles. Similarity follows from the AAA test.

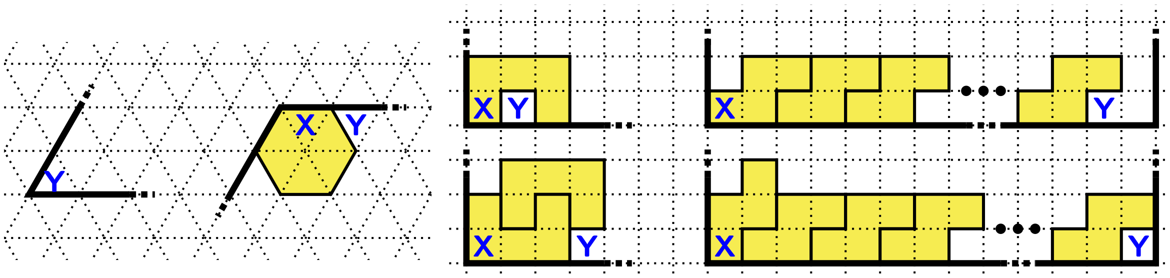


Extensions: Investigate the number of tiles in each row and compare with (c). Investigate which quadrilaterals are n-scaly for all n.

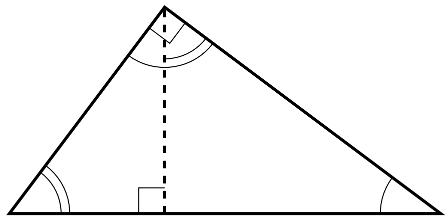
(h) Consider the 3-step staircase tile. Two such tiles can be put together to form a 3×4 rectangle. A 4×3 arrangement of 12 such rectangles then forms a 12×12 square. Finally, six copies of this square can be arranged into a 3-step staircase which has been scaled by a factor of 12, so the original tile is 12-scaly. Following the same procedure, it follows that any s-step staircase is n-scaly where $n = s(s+1)$.

Extension: Does this rule give the smallest possible value of n?

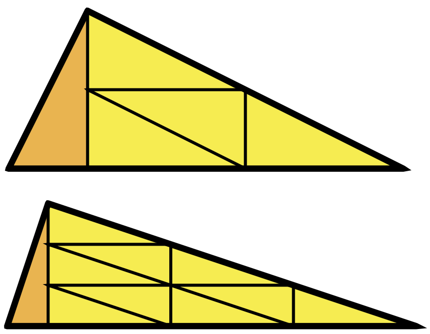
(i) As in (b), each diagram shows part of the enlarged copy and a certain block **Y** which cannot be covered by a tile. This fact is either obvious or follows from the fact that neighbouring blocks, starting at **X**, can only be covered in a handful of ways. Hence the tiling cannot be completed.



(j) By (c), for a $\sqrt{2}$-scaly triangle we expect $(\sqrt{2})^2 = 2$ tiles to fit into the enlarged copy. If two congruent triangular tiles can be put together to form a larger triangle, there is at least one pair of angles, one from each tile, which form a straight angle. Clearly this means the original tile must have a right angle, otherwise it would need to have two different angles which add to 180°, which is impossible. Placing two right-angled triangles 'back-to-back' along an equal length side forms an isosceles triangle, but since this needs to be similar to the original tile, that tile must also be isosceles. Thus a $\sqrt{2}$-scaly triangle must be both right-angled and isosceles, with side lengths in the ratio $1 : 1 : \sqrt{2}$.



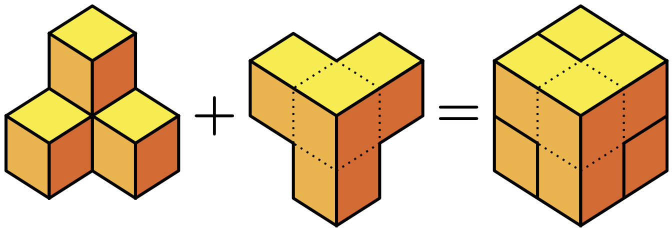
Motivated by the above results, we focus on searching for right-angled triangles with the required scaling properties. Given any right-angled triangle, a line perpendicular to the hypotenuse and passing through the right-angled vertex (called an *altitude*) divides it into two triangles which are both similar to the original, as illustrated; this follows from the AAA similarity test via the fact that the two acute angles add to 90°. If we can further divide the larger of those two triangles into a number of copies of the smaller one, then we will have a solution to the problem, provided the correct total number of tiles is used overall.



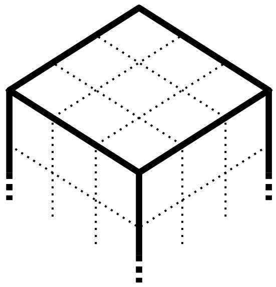
As above, the $\sqrt{5}$- and $\sqrt{10}$-scaly triangles require 5 and 10 tiles, respectively. Notice that both of these totals are 1 more than a perfect square, namely 2^2 and 3^2. By (g), any triangle can be tiled by a square number of tiles which are similar to it, thus we get potential solutions from the arrangements shown. In order for this to work in the first diagram, the longer of the two short sides must align with two of the shortest sides, so the ratio of the three sides is $1 : 2 : \sqrt{5}$. Similarly, in the second diagram the ratio is $1 : 3 : \sqrt{10}$.

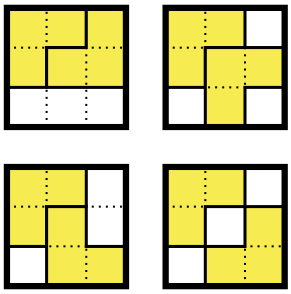
Extensions: Adapt the method above to find a $\sqrt{13}$-scaly triangle. Generalise to other scale factors of the form $\sqrt{m}$ for certain m.

(k) By rotating a second block in 3D space, we can see that two blocks can be combined to make a $2 \times 2 \times 2$ cube, as shown. Putting four such cubes together makes a 2-scaly version with $2^3 = 8$ original blocks in total, as expected via the volume version of the argument in (c).



Suppose the block is 3-scaly. Then it must be possible to make the top 3×3 layer of cubes shown. Notice that each block will contribute one or three of its cubes to the top layer and, respectively, the remaining three or one of its cubes to the second layer, but there can be no additional cubes protruding above or to the sides. It is not possible for six or more blocks to all contribute only one cube to the top layer, since they would then contribute too many to the second layer. On the other hand, it is not possible for three blocks to contribute three cubes each to the top layer since, viewing the arrangement from the top, it is not possible for three 3-square L-shaped tiles to tile a 3×3 square. Hence it must be that two blocks, A and B say, contribute three cubes each to the top layer, and the remaining three cubes are from three other blocks. Up to rotation and reflection, there are only four ways that blocks A and B can be placed in the top layer, as in the top views shown. We can now check that none of these tilings can be completed without at least two of the additional blocks clashing on the second layer. Hence this block is not 3-scaly.





Extensions: Explain why this block is n-scaly for all even numbers n. Investigate whether it is n-scaly for any odd n.

For further hints and tips, contact mail@amt.edu.au.

Australian Curriculum content descriptors

The following is not intended to be an exhaustive list, but indicates how the above activity aligns with various stages of the mathematics curriculum. Follow the links to the ACARA website for elaborations.

- Year 5, ACMMG114 Describe translations, reflections and rotations of two-dimensional shapes, and identify line and rotational symmetries
- Year 5, ACMMG115 Apply the enlargement transformation to familiar two dimensional shapes and explore the properties of the resulting image compared with the original
- Year 6, ACMMG142 Investigate combinations of translations, reflections and rotations
- Year 7 ACMMG161 Draw different views of prisms and solids formed from combinations of prisms
- Year 7 ACMMG163 Identify corresponding, alternate and co-interior angles when two straight lines are crossed by a transversal
- Year 7 ACMMG166 Demonstrate that the angle sum of a triangle is 180° and use this to find the angle sum of a quadrilateral
- Year 7 ACMMG181 Describe translations, reflections in an axis and rotations of multiples of 90° on the Cartesian plane using coordinates
- Year 8, ACMMG200 Define congruence of plane shapes using transformations
- Year 8, ACMMG201 Develop the conditions for congruence of triangles
- Year 9, ACMMG220 Use the enlargement transformation to explain similarity and develop the conditions for triangles to be similar
- Year 9, ACMMG221 Solve problems using ratio and scale factors in similar figures
- Year 9, ACMMG222 Investigate Pythagoras' Theorem and its application to solving simple problems involving right-angled triangles
- Year 10, ACMMG243 Formulate proofs involving congruent triangles and angle properties
- Year 10, ACMMG244 Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes