AUSTRALIAN MATHS TRUST

Maths Challenge
Junior: Years 7–8
Practice Problem

J6: Tossing Counters

Ms Smartie told all the students in her class to write four different integers from 1 to 9 on the four faces of two counters, with one number on each face. She then asked them to toss both counters simultaneously many times and write down the sum of the numbers that appeared on the upper faces each time.

J6: Questions

a. The only sums that Jack was able to get were 8, 9, 10, and 11. Find all five possible combinations of four numbers on the counters.

b. Jill wrote 4 and 5 on opposite sides of one counter. The only sums she was able to get were three consecutive integers. Find all possible ways the second counter could be numbered.

c. Ben was only able to get sums that were four consecutive numbers. Show that either one or three of the numbers he wrote on the counters were even.

d. Show that it is possible to number four counters with 8 different positive integers less than 20, one number on each face, so that the sums that appear are 16 consecutive numbers.

Solutions

a. ***Alternative 1***

The smallest sum, 8, is the sum of the smaller numbers on the two counters. Since 1 + 7 = 8, the smaller numbers on the two counters are 1 and 7.

Since the sum of the larger numbers on the two counters is 11, the sum of 7 and the larger number on the first counter is 9 or 10. So the larger number on the first counter is 2 or 3.

a.

If the first counter is 1/2, then the second counter must be 7/9. If the first counter is 1/3, then the second counter must be 7/8.

Alternative 2

The smallest sum, 8, is the sum of the smaller numbers on the two counters. The largest sum, 11, is the sum of the larger numbers on the two counters. So we have either of the following addition tables for the four numbers on the counters.

+	1	2
7	8	9
9	10	11

+	1	3
7	8	10
8	9	11

There is only one way to complete each table:

+	1	?
7	8	9
?	10	11

+	1	?
7	8	10
?	9	11

Thus the two counters are 1/2 and 7/9 or 1/3 and 7/8.

b. ***Alternative 1***

The smallest sum, 7, is the sum of the smaller numbers on the two counters. The largest sum, 10, is the sum of the larger numbers on the two counters. Since 7 = 1 + 6 = 2 + 5 = 3 + 4, the two smaller numbers on the counters are 1 and 6 or 2 and 5 or 3 and 4. Since
10 = 1 + 9 = 2 + 8 = 3 + 7 = 4 + 6, the two larger numbers on the counters are 1 and 9 or 2 and 8 or 3 and 7 or 4 and 6. The four numbers on the two counters are all different. The table shows the only combinations we need to consider.

Smaller numbers	Larger numbers	Give sums 7, 8, 9, 10?
1 and 6	2 and 8	yes
	3 and 7	yes
2 and 5	1 and 9	no
	3 and 7	yes
	4 and 6	yes
3 and 4	1 and 9	no
	2 and 8	no

So the two counters are:
1/2 and 6/8, 1/3 and 6/7, 2/3 and 5/7, or 2/4 and 5/6.

Alternative 2

Let the numbers on one counter be a and $a + r$.
Let the numbers on the other counter be b and
$b + s$.
Then $a + b = 7$ and $a + r + b + s = 10$. So $r + s = 3$.
Hence (a,b) = (1,6), (2,5), (3,4), (4,3), (5,2), or
(6,1),
and (r,s) = (1,2) or (2,1).

From symmetry we may assume $r = 1$ and $s = 2$.
So the two counters are:
1/2 and 6/8, 2/3 and 5/7, 3/4 and 4/6 (disallowed),
4/5 and 3/5
(disallowed), 5/6 and 2/4, or 6/7 and 1/3.

Alternative 3

The smallest sum, 7, is the sum of the smaller numbers on the two counters. So these numbers are 1 and 6, 2 and 5, or 3 and 4. The largest sum, 10, is the sum of the larger numbers on the two counters. So we have the following addition tables for the four numbers on the counters.

+	1	?
6	7	8
?	9	10

+	1	?
6	7	9
?	8	10

+	2	?
5	7	8
?	9	10

+	2	?
5	7	9
?	8	10

+	3	?
4	7	8
?	9	10

+	3	?
4	7	9
?	8	10

There is only one way to complete each table:

+	1	2
6	7	8
8	9	10

+	1	3
6	7	9
7	8	10

+	2	3
5	7	8
7	9	10

+	2	4
5	7	9
6	8	10

+	3	4
4	7	8
6	9	10

+	3	5
4	7	9
5	8	10

We must exclude the last two tables because the four numbers on the counters must be different. So the two counters are:
1/2 and 6/8, 1/3 and 6/7, 2/3 and 5/7, or 2/4 and 5/6.

c. ***Alternative 1***
Suppose the numbers on the second counter are c and d with $c < d$. The minimum sum is $4 + c$ and the maximum sum is $5 + d$. The other two sums, $4 + d$ and $5 + c$, are between these two. If these four sums form three consecutive integers, then $4 + d = 5 + c$ and $d - c = 1$.

Since d is less than 10 and the numbers 4 and 5 already appear on the first counter, the second counter is 1/2, 2/3, 6/7, 7/8, or 8/9.

Alternative 2
Suppose the three sums are s, $s + 1$, $s + 2$. The smallest sum, s, is the sum of the smaller numbers on the two counters. The largest sum, $s + 2$, is the sum of the larger numbers on the two counters. Let the smaller number on the second counter be x. Then the addition table for the four numbers on the counters is:

$+$	4	5
x	s	$s+1$
?	?	$s+2$

So the larger number on the second counter is $x + 1$. Since each of x and $x + 1$ is less than 10 and is neither 4 nor 5, x is one of the numbers 1, 2, 6, 7, 8. So the second counter is 1/2, 2/3, 6/7, 7/8, or 8/9.

d. ***Alternative 1***

Suppose the numbers on the first counter are a and b with $a < b$, and on the second counter c and d with $c < d$.

The minimum sum is $a + c$ and the maximum sum is $b + d$. The other two sums, $a + d$ and $b + c$, are between these two.

Suppose $a + d < b + c$. Since the four sums are consecutive, we have
$a + d = a + c + 1$ and $b + c = a + d + 1$. Hence
$d = c + 1$ and $b = a + 2$.
So only one of c and d is even and a and b are either both odd or both even.
Similarly, if $b + c < a + d$, then only one of a and b is even and c and d are either both odd or both even.

Thus either one or three of a, b, c, d are even.

Alternative 2

Suppose the numbers on the first counter are a and b with $a < b$, and on the second counter c and d with $c < d$.

The minimum sum is $a + c$ and the maximum sum is $b + d$. Since the four sums are consecutive, we have $b + d = a + c + 3$.

If $a + c$ is odd, then $b + d$ is even. So one of a and c is even and neither or both of b and d are even.

If $a + c$ is even, then $b + d$ is odd. So one of b and d is even and neither or both of a and c are even.

In both cases one or three of a, b, c, d are even.

Alternative 3

Suppose the four sums are *s*, *s* + 1, *s* + 2, *s* + 3. The smallest sum, *s*, is the sum of the smaller numbers on the two counters. The largest sum, *s* + 3, is the sum of the larger numbers on the two counters. So we can arrange the addition table for the four numbers on the counters as follows:

$+$	?	?
?	s	$s+1$
?	$s+2$	$s+3$

Since s is either even or odd, we have:

$+$	x	?
?	even	odd
?	even	odd

$+$	x	?
?	odd	even
?	odd	even

Since x is either even or odd, we can complete each of these tables in two ways:

$+$	even	odd
even	even	odd
even	even	odd

$+$	even	odd
odd	odd	even
odd	odd	even

$+$	odd	even
odd	even	odd
odd	even	odd

$+$	odd	even
even	odd	even
even	odd	even

In each case either one or three of the four numbers on the counters are even.

Alternative 4
If all four numbers on the counters were even or all four were odd, then all sums would be even and therefore not consecutive.

Suppose two numbers on the counters were even and the other two odd.

If the two even numbers were on the same counter, then all sumswould be odd and therefore not consecutive. So each counter must have an even and an odd number.

If the two smaller numbers on the counters were odd, then the two larger numbers would be even. Then the lowest and highest sums would both be even and the four sums would not be consecutive. Similarly, the two smaller numbers on the counters cannot be even.

If one of the smaller numbers on the counters was odd and the other smaller number was even, then one of the larger numbers would be even and the other odd. Then the lowest and highest sums would both be odd and the four sums would not be consecutive.

So either one or three of the four numbers on the counters are even.