Find all real numbers $r$ for which there exists exactly one real number $a$ such that when

$$(x + a)(x^2 + rx + 1)$$

is expanded to yield a cubic polynomial, all of its coefficients are greater than or equal to zero.

For each positive integer $n$, the $n$th triangular number is the sum of the first $n$ positive integers. Let $a, b, c$ be three consecutive triangular numbers with $a < b < c$.

Prove that if $a + b + c$ is a triangular number, then $b$ is three times a triangular number.

Let $A, B, C, D, E$ be five points in order on a circle $K$. Suppose that $AB = CD$ and $BC = DE$. Let the chords $AD$ and $BE$ intersect at the point $P$.

Prove that the circumcentre of triangle $AEP$ lies on $K$.

Let $Q$ be a point inside the convex polygon $P_1P_2 \cdots P_{1000}$. For each $i = 1, 2, \ldots, 1000$, extend the line $P_iQ$ until it meets the polygon again at a point $X_i$. Suppose that none of the points $X_1, X_2, \ldots, X_{1000}$ is a vertex of the polygon.

Prove that there is at least one side of the polygon that does not contain any of the points $X_1, X_2, \ldots, X_{1000}$. 
AUSTRALIAN MATHEMATICS TRUST

Australian Mathematical Olympiad
2019

DAY 2
Wednesday, 6 February 2019
Time allowed: 4 hours
No calculators are to be used.
Each question is worth seven points.

5. A fancy triangle is an equilateral triangular array of integers such that the sum of the three numbers in any unit equilateral triangle is a multiple of 3. For example,

\[
\begin{array}{ccc}
1 & 0 & 2 \\
0 & 2 & 0 \\
5 & 7 & 3 \\
\end{array}
\]

is a fancy triangle with three rows because the sum of the numbers in each of the following four unit equilateral triangles is a multiple of 3.

\[
\begin{array}{ccc}
1 & 0 & 0 \\
0 & 2 & 0 \\
2 & 5 & 7 \\
\end{array}
\]

Suppose that a fancy triangle has ten rows and that exactly \( n \) of the numbers in the triangle are multiples of 3.
Determine all possible values for \( n \).

6. Let \( K \) be the circle passing through all four corners of a square \( ABCD \). Let \( P \) be a point on the minor arc \( CD \), different from \( C \) and \( D \). The line \( AP \) meets the line \( BD \) at \( X \) and the line \( CP \) meets the line \( BD \) at \( Y \). Let \( M \) be the midpoint of \( XY \).
Prove that \( MP \) is tangent to \( K \).

7. Akshay writes a sequence \( a_1, a_2, \ldots, a_{100} \) of integers in which the first and last terms are equal to 0. Except for the first and last terms, each term \( a_i \) is larger than the average of its neighbours \( a_{i-1} \) and \( a_{i+1} \).
What is the smallest possible value for the term \( a_{19} \)?

8. Let \( n = 16^r - 4^r + 1 \) for some positive integer \( r \).
Prove that \( 2^n - 1 \) is divisible by \( n \).
1. Find all real numbers $r$ for which there exists exactly one real number $a$ such that when

$$(x + a)(x^2 + rx + 1)$$

is expanded to yield a cubic polynomial, all of its coefficients are greater than or equal to zero.

**Solution 1** (Angelo Di Pasquale)

Answer: $r = -1$.

Expanding the brackets, we see that we want the following three inequalities to be true.

1. $a \geq 0$ (constant term)
2. $ar + 1 \geq 0$ (coefficient of $x$)
3. $a + r \geq 0$ (coefficient of $x^2$)

If $r \geq 0$, then any $a \geq 0$ satisfies (1), (2), and (3).

It remains to address $r < 0$. In this case note that (3) immediately implies (1). So we only need to consider (2) and (3). Since $r < 0$, inequalities (2) and (3) are equivalent to the following.

4. $a \leq \frac{-1}{r}$
5. $a \geq -r$

Hence, we seek all values of $r < 0$ such that there is exactly one real number $a$ satisfying

$$-r \leq a \leq \frac{-1}{r}.$$  

(6)

Thus $-r = \frac{-1}{r}$, which implies $r = \pm 1$. Since $r < 0$ we have $r = -1$. This implies that the only corresponding value of $a$ is $a = 1$.

It only remains to observe that

$$(x + 1)(x^2 - x + 1) = x^3 + 1,$$

which has no negative coefficients.

**Solution 2** (Alan Offer)

Expanded, the cubic is

$$x^3 + (a + r)x^2 + (ar + 1)x + a.$$

Plotted on a Cartesian plane with $a$ on the horizontal axis and $r$ on the vertical axis, the condition that $a + r \geq 0$ is satisfied by the points in the region above and right of the line $a + r = 0$. Similarly, the condition that $ar + 1 \geq 0$ corresponds to the region between the two branches of the hyperbola $r = -1/a$. 

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The intersection of these two regions is then where both of the coefficients $a + r$ and $ar + 1$ are non-negative, so we are being asked for the horizontal coordinates $r$ at which a vertical line meets this region in exactly one point, and this occurs at $r = -1$, where the line and the hyperbola meet at $(-1, 1)$. (Notice that the coefficient $a$ is then also non-negative.)
2. For each positive integer $n$, the $n$th triangular number is the sum of the first $n$ positive integers. Let $a, b, c$ be three consecutive triangular numbers with $a < b < c$.

Prove that if $a + b + c$ is a triangular number, then $b$ is three times a triangular number.

**Solution 1** (Mike Clapper)

Let $T_m = T_{n-1} + T_n + T_{n+1}$.

Then $\frac{m}{2}(m+1) = \frac{n}{2}(n-1) + \frac{n}{2}(n+1) + \frac{n+1}{2}(n+2)$ which simplifies to $3(n^2 + n) + 2 = m^2 + m$.

Considering this equation modulo 3, we see that the LHS $\equiv 2 \pmod{3}$.

This is only possible if $m \equiv 1 \pmod{3}$ so we can let $m = 3s + 1$ for some integer $s$.

Hence, $3(n^2 + n) + 2 = (3s + 1)(3s + 2)$ giving $n^2 + n = 3(s^2 + s)$ and $T_n = 3T_s$.

**Solution 2** (Ivan Guo)

Instead of triangular numbers, it suffices to double everything and work only with numbers of the form $n(n+1)$ where $n \geq 1$. The required condition can be rewritten as

$$n(n-1) + n(n+1) + (n+1)(n+2) = (m+1)(m+2) \iff 3n^2 + 3n = m^2 + 3m.$$ 

So $3 | m$. Writing $m = 3s$ yields $n^2 + n = 3(s^2 + s)$, as required. (Note that we need to check $s \geq 1$ but this is clear since both sides are positive here.)
3. Let $A, B, C, D, E$ be five points in order on a circle $K$. Suppose that $AB = CD$ and $BC = DE$. Let the chords $AD$ and $BE$ intersect at the point $P$. 

Prove that the circumcentre of triangle $AEP$ lies on $K$.

**Solution 1** (Angelo Di Pasquale)

Since $AB = CD$ and $ABCD$ is cyclic, it follows that $ABCD$ is an isosceles trapezium with $AD \parallel BC$. Similarly, $BE \parallel CD$.

Let $O$ be the midpoint of arc $AE$ of circle $ABCDE$. Thus $OA = OE$. Let $Q$ be second intersection point of line $AO$ with circle $AEP$. Let $x = \angle BPA$. We calculate the following angles.

$$
\angle CDA = x \quad \text{(} BE \parallel CD \text{)}
$$
$$
\angle DAB = x \quad \text{(isosceles trapezium } ABCD\text{)}
$$
$$
\angle EQA = x \quad \text{(} AQEP \text{ cyclic)}
$$
$$
\angle ABP = 180^\circ - 2x \quad \text{(angle sum } \triangle ABP\text{)}
$$
$$
\angle QOE = 180^\circ - 2x \quad \text{(} ABEO \text{ cyclic)}
$$
$$
\angle OEQ = x \quad \text{(angle sum } \triangle OQE\text{)}
$$

Hence, $\triangle OQE$ is isosceles with $OQ = OE$. Since $OQ = OE = OA$, it follows circle $AEQ$ has centre $O$. Since $P$ also lies on this circle, we may conclude that $O$ is the circumcentre of $\triangle AEP$.

**Solution 2** (Alice Devillers)

We need to prove that the centre $O$ of the circumcircle to $AEP$ satisfies $\angle AOE = 180 - \angle ADE$.

We will repeatedly use the angles intercepting arcs of the same length are the same: for instance $\angle ABC = \angle BCD = \angle CDE$. 

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Since the sum of the angles in a pentagon is 540°, so
\[
540° = \angle ABC + \angle BCD + \angle CDE + \angle DEA + \angle EAB
= 3\angle CDE + \angle DEB + \angle BEA + \angle EAD + \angle DAB
= 3\angle CDE + 180° - \angle DCB + \angle BEA + \angle EAD + 180° - \angle DCB
= 360° + \angle CDE + 180° - \angle APE.
\]
Thus, \(\angle CDE = \angle APE\).

Because of the circumcircle, \(\angle AOE = 360° - 2\angle APE = 360° - 2\angle CDE\). On the other hand,
\[\angle ADE = \angle CDE - \angle ADC = \angle CDE - (180° - \triangle ABC) = 2\angle CDE - 180°.\]
Hence, \(\angle AOE + \angle ADE = 180°\) and we are done.

**Solution 3** (Angelo Di Pasquale)

With notation in solution 1, we have \(\angle APE = 180° - x\) and \(\angle EOA = 2x\). Thus \(\angle AEO\) (reflex) = 360° – 2x = 2\angle APE. Consider any point X that satisfies the following.

- X and P lie on opposite sides of line AE.
- X lies on the perpendicular bisector of AE.
- \(\angle AXE\) (reflex) = 2\angle APE.

There is only one point X that has these properties. This is because as X moves on the perpendicular bisector of AE away from (closer to) AE, the reflex angle AXE gets larger (smaller). The circumcentre of \(\triangle AEP\) and point O both have the above properties. Hence, O is the circumcentre of \(\triangle AEP\).

**Solution 4** (Angelo Di Pasquale)

(Variation on the alternative solution) Let \(x = \angle BPA\). Then \(\angle CDA = x\) since \(CD \parallel BE\) from isosceles trapezium \(BCDE\). Also \(\angle DAB = x\) from isosceles trapezium \(ABCD\). From the angle sum in \(\triangle ABP\), we deduce \(\angle ABE = 180° - 2x\).

We also have \(\angle APE = 180° - x\). Let O be the circumcentre of \(\triangle AEP\). Thus reflex angle \(\angle AOE = 2\angle APE = 360° - 2x\), and so \(\angle EOA = 2x\). Since \(\angle ABE + \angle EOA = 180°\), it follows that \(ABEO\) is cyclic. Thus O lies on circle(\(ABE\)) = circle(\(ABCDE\)).

**Solution 5** (Ivan Guo)

The given length conditions imply that \(ABCD\) and \(BCDE\) are isosceles trapezia, while \(BCDP\) is a parallelogram. Hence, let \(\angle ABC = \angle BCD = \angle CDE = \angle EPA = \theta\). Construct O to be the midpoint of the arc AE. Since in a cyclic hexagon, the three non-adjacent angles add up to 360°, we have 360° – \(\angle EOA = 2\theta = 2\angle EPA\). Therefore, O is the circumcentre of \(EPA\).

**Solution 6** (Daniel Mathews)
Let the given circle be $\Gamma$, with centre $O$, and let $a = \angle DAE$ and $b = \angle BEA$. Then $a$ and $b$ are the angles subtended by the arcs $DE$ and $AB$ respectively; note $a, b < 90^\circ$. As $AB = CD$ then $\angle CED = b$, and as $BC = DE$ then $\angle BEC = a$.

Now $AE$ subtends $\angle APE = 180 - \angle AEP - \angle EAP = 180 - a - b$ at $P$, which is obtuse. Hence, $AE$ subtends $a + b$ at points of $\Gamma$ on the other side of $AE$ from $P$, and subtends $2a + 2b$ at $O$. Thus $O$ lies on the other side of AE from P, and satisfies $\angle AOE = 2a + 2b$.

On the other hand, $AE$ subtends an angle of $180^\circ - 2a - 2b$ at $D$, since $\angle ADE = 180 - \angle DAE - \angle AED = 180 - \angle DAE - \angle AEP - \angle BEC - \angle CED = 180 - 2a - 2b$,

and hence, subtends $2a + 2b$ at points of $\Gamma$ on the other side of $AE$ from $P$. Thus $O$ lies on $\Gamma$.

**Solution 7** (Kevin McAvaney)

From isosceles trapezia $ABCD$ and $BCDE$, triangles $ABP$ and $EDP$ are isosceles and equiangular. Let $DO$ be the perpendicular bisector of $EP$ with $O$ on the circumcircle of $ABCDE$. Then $DO$ bisects angle $PDE$. Angles $ODE$ and $OBE$ are equal. Hence, $BO$ bisects angle $ABP$. Therefore, $BO$ is the perpendicular bisector of $AP$. Hence, $O$ the circumcentre of triangle $AEP$.

**Solution 8** (Alan Offer)

Let $O$ be the centre of the circumcircle of triangle $AEP$. In terms of directed angles, it follows that $\angle AOE = 2\angle APE$. Now $O$ is on the circumcircle of $ABCDE$ if $\angle AOE = \angle ABE$, so it suffices to show that $\angle ABE = 2\angle APE$. With this in mind, we have

$$2\angle APE = 2\angle ABE + 2\angle DAB$$

(1) (exterior angle of $\triangle ABP$)

$$= \angle ABE + \angle ACE + 2\angle DAB$$

(2) ($ABCE$ cyclic)

$$= \angle ABE + \angle ACE + (\angle DAC + \angle CAB) + \angle DAB$$

(3)

$$= \angle ABE + \angle DAB + (\angle ACE + \angle BCA + \angle ECD)$$

(4) (arcs $DC = BA$ and $CB = ED$)

$$= \angle ABE + (\angle DAB + \angle BCD)$$

(5)

$$= \angle ABE$$

(6) ($ABCD$ cyclic).
4. Let $Q$ be a point inside the convex polygon $P_1P_2 \cdots P_{1000}$. For each $i = 1, 2, \ldots, 1000$, extend the line $P_iQ$ until it meets the polygon again at a point $X_i$. Suppose that none of the points $X_1, X_2, \ldots, X_{1000}$ is a vertex of the polygon.

Prove that there is at least one side of the polygon that does not contain any of the points $X_1, X_2, \ldots, X_{1000}$.

Solution 1

Since $Q$ does not lie on a diagonal of the polygon, each of the points $X_1, X_2, \ldots, X_{1000}$ lies on the interior of a side of the polygon. Suppose that $X_1$ lies on the side $P_iP_{i+1}$. Without loss of generality, we may assume that $i \leq 500$; otherwise, we could relabel the vertices in the opposite orientation instead.

Then the points $P_2, P_3, \ldots, P_i$ lie on one side of the line $P_1Q$, which means that the points $X_2, X_3, \ldots, X_i$ must lie on the other side of the line $P_1Q$. So the $i$ points $X_1, X_2, X_3, \ldots, X_i$ must lie on the 1001 $- i$ sides $P_iP_{i+1}, P_{i+1}P_{i+2}, \ldots, P_{1000}P_1$. Furthermore, no other point $X_j$ can lie on one of these sides, since they lie on the other side of the line $P_1Q$. However, since $i \leq 500$, we have $i < 1001 - i$. It follows that there must be at least one of the sides $P_iP_{i+1}, P_{i+1}P_{i+2}, \ldots, P_{1000}P_1$ that does not contain any of the points $X_1, X_2, \ldots, X_{1000}$.

Solution 2 (Angelo Di Pasquale)

Define a butterfly to be the region formed by the two triangles cut out by a pair of consecutive main diagonals of the polygon. If $Q$ lies inside a butterfly, then it is easy to see that the conclusion of the problem is true since a line that enters a triangle must exit it somewhere.

To finish, it suffices to prove that the point $Q$ lies inside a butterfly. For any directed line $AB$, we define its positive side to be the half-plane of points $X$ such that $0 < \angle BAX < 180^\circ$. We also define its negative side to be the half-plane of points $X$ such that $180^\circ < \angle BAX < 360^\circ$. In both cases, the angle is directed anticlockwise modulo $360^\circ$.

Without loss of generality, suppose that $Q$ lies on the positive side of the directed line $P_0P_{500}$, where we consider all subscripts modulo 1000. Then $Q$ lies on the negative side of the directed line $P_{500}P_0$. Hence, there exists an integer $i$ with $0 \leq i \leq 499$ such that $Q$ lies on the positive side of $P_iP_{i+500}$ but on the negative side of $P_{i+1}P_{i+501}$. Thus, $Q$ lies inside the butterfly defined by $P_iP_{i+500}$ and $P_{i+1}P_{i+501}$.

Solution 3 (Kevin McAvaney)

We will prove the statement more generally for a convex $2m$-gon. Suppose that each side of the polygon contains at least one of the points $X_1, X_2, \ldots, X_{2m}$ on its interior. Then
each side contains exactly one of the points $X_1, X_2, \ldots, X_{2m}$ on its interior. Otherwise, one of the lines through $Q$ passes through an interior point of at least two polygon sides and this contradicts the convexity of the polygon.

Label the lines through $Q$ in clockwise order $L_1, L_2, L_3, \ldots, L_{2m}$. Label the vertices of the polygon in clockwise order $P_1, P_2, P_3, \ldots, P_{2m}$. Without loss of generality, suppose that $L_1$ passes through $P_1$. Then $L_2$ passes through an interior point of the side $P_1P_2$ and $L_3$ passes through $P_2$. Hence, $L_4$ passes through an interior point of side $P_2P_3$ and $L_5$ passes through $P_3$. Continuing in the same manner, we see that only the lines indexed by odd integers pass through vertices of the polygon and this produces the desired contradiction.

**Solution 4** (Chaitanya Rao)

For notational convenience let $P_{1000+i} = P_i$ for $i \in \{1, 2, \ldots, 1000\}$. The diagonal $P_1P_{i+500}$ joining opposite vertices divides the convex polygon into the following two 501-gons:

$P_1P_2 \cdots P_iP_{i+500}P_{i+501} \cdots P_{1000}$ and $P_1P_{i+1}P_{i+2} \cdots P_{i+500}$. Since $Q$ is not on a diagonal and the original polygon is convex, $Q$ lies inside one of these polygons and outside the other.

Define a function $f : \{1, 2, \ldots, 1000\} \rightarrow \{0, 1\}$ by

$$f(i) = \begin{cases} 1, & \text{if } Q \text{ lies inside } P_1P_2 \cdots P_iP_{i+500}P_{i+501} \cdots P_{1000}, \\ 0, & \text{otherwise}. \end{cases}$$

Note that $f(i) = 1 - f(i + 500)$, since $Q$ is inside exactly one of the two 501-gons $P_1P_2 \cdots P_iP_{i+500}P_{i+501} \cdots P_{1000}$ and $P_1P_{i+1}P_{i+2} \cdots P_{i+500}$. Hence, the function $f$ is not constant and there exists some $j$ for which $f(j) = 0$ and $f(j + 1) = 1$, as shown in the following diagram.

We then find that segment $P_jP_{j+1}$ contains both $X_{j+500}$ and $X_{j+501}$ since they are the bases of internal cevians of triangles $P_jP_{j+500}P_{j+1}$ and $P_jP_{j+501}P_{j+1}$, respectively. Note that both triangles have $Q$ in the interior of their intersection. Since $Q$ is not on a diagonal, none of the points $X_i$ is a vertex of the the polygon and we conclude that there exists another side of the polygon that does not contain any of the points $X_1, X_2, \ldots, X_{1000}$.

**Solution 5** (Ian Wanless)

Consider the diagonal $d = P_1P_m$. By assumption, $Q$ does not lie on $d$. Assume that $Q$ lies on the same side of $d$ as $P_{2m}$ does, since the other case is equivalent after relabelling the vertices. Let $1 \leq i \leq m + 1$ and note that the ray from $P_i$ to $Q$ hits $d$ before it hits $Q$. This means that the point $X_i$ is on the same side of $d$ as $Q$. But this means that the
$m + 1$ points $X_1, X_2, \ldots, X_{m+1}$ lie on the $m$ sides $P_{m+1}P_{m+2}, \ldots, P_{2m-1}P_{2m}, P_{2m}P_1$. By the pigeonhole principle, at least two of points $X_1, X_2, \ldots, X_{m+1}$ lie on the same side of the polygon. By a second application of the pigeonhole principle, it follows that there is some side of the polygon that contains none of the points $X_1, X_2, \ldots, X_{2m}$.
5. A fancy triangle is an equilateral triangular array of integers such that the sum of the three numbers in any unit equilateral triangle is a multiple of 3. For example,

\[
\begin{array}{ccc}
1 \\
0 & 2 \\
5 & 7 & 3
\end{array}
\]
is a fancy triangle with three rows because the sum of the numbers in each of the following four unit equilateral triangles is a multiple of 3.

\[
\begin{array}{cccc}
1 & 0 & 2 \\
0 & 2 & 0 \\
5 & 7 & 7 & 3
\end{array}
\]

Suppose that a fancy triangle has ten rows and that exactly \( n \) of the numbers in the triangle are multiples of 3. Determine all possible values for \( n \).

**Solution** (Angelo Di Pasquale)

Answers: \( n = 0, 18, 19, \) or 55

Consider the four numbers in any two unit equilateral triangles that share a common edge as shown in the diagram.

\[
\begin{array}{ccc}
u \\
v & w \\
x
\end{array}
\]

Since \( u + v + w \equiv 0 \equiv v + w + x \pmod{3} \), it follows that \( u \equiv x \pmod{3} \). Using this observation we deduce that if we reduce the entries of the triangle modulo 3, it takes the following form.

\[
\begin{array}{cccc}
u & v & w \\
w & u & v \\
u & v & w & u \\
v & w & u & v & w \\
w & u & v & w & u & v \\
u & v & w & u & v & w & u \\
v & w & u & v & w & u & v \\
w & u & v & w & u & v & w & u \\
u & v & w & u & v & w & u & v \\
u & v & w & u & v & w & u & v
\end{array}
\]

Note that the triangular array is fancy if and only if \( 3 \mid u + v + w \). Reducing modulo 3, we have the cases \( (u,v,w) = (0,0,0), (1,1,1), (2,2,2) \), or any permutation of \( (0,1,2) \).

- If \( (u,v,w) = (0,0,0) \), then \( n = 55 \).
- If \( (u,v,w) = (1,1,1) \) or \( (2,2,2) \), then \( n = 0 \).
- If \( (u,v,w) = (0,1,2) \) or \( (0,2,1) \), then \( n = 19 \).
- If \( (u,v,w) = (1,0,2) \) or \( (1,2,0) \) or \( (2,1,0) \) or \( (2,0,1) \), then \( n = 18 \).
6. Let $\mathcal{K}$ be the circle passing through all four corners of a square $ABCD$. Let $P$ be a point on the minor arc $CD$, different from $C$ and $D$. The line $AP$ meets the line $BD$ at $X$ and the line $CP$ meets the line $BD$ at $Y$. Let $M$ be the midpoint of $XY$.

Prove that $MP$ is tangent to $\mathcal{K}$.

Solution 1

By the converse of the alternate segment theorem, it suffices to prove that $\angle MPA = \angle ABP$.

Let $\angle AXB = \angle MXP = \theta$. Since $AC$ is a diameter of the circle, $\angle APC = \angle APY = 90^\circ$. So $M$ is the midpoint of the hypotenuse of the right-angled triangle $XYP$. It follows that $MX = MP$, so $\angle MPA = \angle MPX = \angle MXP = \theta$.

Now observe that $\angle AXD = 180^\circ - \theta$ and $\angle XDA = 45^\circ$, so $\angle DAP = \angle DAX = \theta - 45^\circ$. By cyclic quadrilateral $ABPD$, we have $\angle DBP = \angle DAP = \theta - 45^\circ$. Therefore, $\angle ABP = \angle ABD + \angle DBP = 45^\circ + (\theta - 45^\circ) = \theta$.

So we have shown that $\angle MPA = \angle ABP = \theta$, as required.

Solution 2 (Alice Devillers)

Pick coordinates such that $A = (0, -1)$, $B = (-1, 0)$, $C = (0, 1)$, $D = (0, 1)$, so $P = (\cos \theta, \sin \theta)$ where $\theta$ is between $0$ and $\pi/2$. We easily compute the equations of $AP$: $y + 1 = \frac{\sin \theta + 1}{\cos \theta}x$ and $CP$: $y - 1 = \frac{\sin \theta - 1}{\cos \theta}x$, while $BD$ is just $y = 0$. Thus $X = \left(\frac{\cos \theta}{\sin \theta + 1}, 0\right)$ and $Y = \left(-\frac{1}{\sin \theta}, 0\right)$. The middle point $M$ is $X = \left(\frac{1}{\sin \theta}, 0\right)$ (here we used $\sin^2 \theta - 1 = -\cos^2 \theta$ and $\cos \theta \neq 0$). If we take the dot product of the vectors $OP$ and $MP$, we get $0$ so $MP$ is tangent to the circle of radius $1$ centred at $O$.

Solution 2 (Angelo Di Pasquale)

Let $O = AC \cap BD$. Note that $AC \perp BD$, and so $\angle AOX = 90^\circ$. Also $\angle AP \perp PC$ because $AC$ is a diameter of $\mathcal{K}$. Thus $\angle APY = 90^\circ = \angle AOX$, and so $AOPY$ is cyclic.
As $\angle XPY = 90^\circ$ and $M$ is the midpoint of $XY$, we have $M$ is the centre of circle $PXY$. Thus $MX = MP = MY$.

From $MY = MP$ and cyclic $AOPY$, we find $\angle MPY = \angle PYO = \angle PXO = \angle PXC$, and so by the alternate segment theorem, $MP$ is tangent to $K$ at $P$.

**Solution 3** (Ivan Guo)

Let $Q$ be the reflection of $P$ about $AC$, so $PQ \parallel BD$. Then since $APCQ$ is a cyclic kite, the points $A, P, C, Q$ are harmonic. Projecting them from $P$ onto $BD$ yields $X, M', Y, \infty$ where $M'$ is the intersection of the tangent at $P$ with $BD$. Since $X, M', Y, \infty$ are harmonic, then $M'$ must be the midpoint of $XY$.

**Solution 4** (Ivan Guo)

Since $ABCD$ is a square, $A, B, C, D$ are harmonic. Projecting from $P$ onto $BD$ yields the harmonic points $B, X, D, Y$. Via a standard length calculation on the line $BD$, we immediately get $MX^2 = MD \times MB$. Since $AP \perp PY$, $MX = MP$ and the required tangency follows by power of a point.

**Solution 5** (Ivan Guo)

Since $ABCD$ is a square, $AP$ and $CP$ are internal and external angle bisectors of $\angle BPD$. By the angle bisector theorem, we see that the circles $DPB$ and $XPY$ are circles of Apollonius. It is well-known that circles of Apollonius are orthogonal, hence the required tangency.

**Solution 6** (Ivan Guo)

Let $AY$ meet the circle at $R$. Apply the central projection that sends the line through $Y$ perpendicular to $BD$ to infinity while maintaining the circle. Then $A'R'P'C'$ is rectangle, hence $X'$ is the new centre of the circle. Furthermore $\infty'C'$ is a tangent to the circle. But since harmonic points are preserved under central projections, $X'$ is the midpoint of $M'\infty'$. By symmetry and $B'D'||P'C'$, we must have $M'P'$ being a tangent to the circle.

**Solution 7** (Kevin McAvaney)

Let $O$ be the centre of the circle. We show that $OP$ and $MP$ are perpendicular.

Angle $CPA = \angle CDA = 90$ degrees. So $XYP$ is a right-angled triangle and $M$ is therefore the centre of its circumcircle. Hence $MP = MY$. Since $ABCD$ is a square, $O$ is the intersection of its diagonals and they are perpendicular.

So we have angle $MPY = \angle MYP = \angle OAP = \angle OPA$. Hence angle $OPM = \angle OPA + \angle APM = \angle MPY + \angle APM = 90$ degrees, as required.

**Solution 8** (Alan Offer)

This problem can be handled fine with coordinates. Choose coordinates so that $A = (-1,0)$, $B = (0,-1)$, $C = (1,0)$, and $D = (0,1)$. Then $P = (u,v)$ with $u^2 + v^2 = 1$. Also,
X = (0, s) and Y = (0, t) for some numbers s and t. As the slope of AX is equal to the slope of AP, we obtain s = v/(1 + u). As the slope of CY is equal to the slope of CP, we have t = v/(1−u). The midpoint of XY is then at M = (0, 1/2(s+t)) = (0, v/(1−u^2)) = (0, 1/v).

Calling the origin O, the product of the slopes of MP and OP is

\[
\frac{v - 1/v}{u} \times \frac{v}{u} = \frac{v^2 - 1}{u^2} = -\frac{u^2}{u^2} = 1.
\]

Hence MP is perpendicular to the radius OP and so is tangent to the circle.

### Solution 9 (Chaitanya Rao)

As in the official solution we have ∠MPA = ∠DXP. If O is the centre of K, then by the angle between intersecting chords theorem, ∠DXP = 1/2(∠AOB + ∠DOP) = 45° + 1/2∠DOP = ∠ABD + ∠DBP = ∠ABP. Hence ∠MPA = ∠ABP and by the converse of the alternate segment theorem MP is tangent to K.
7. Akshay writes a sequence $a_1, a_2, \ldots, a_{100}$ of integers in which the first and last terms are equal to 0. Except for the first and last terms, each term $a_i$ is larger than the average of its neighbours $a_{i-1}$ and $a_{i+1}$.

What is the smallest possible value for the term $a_{19}$?

**Solution**

Let $d_i = a_{i+1} - a_i$ for $i = 1, 2, 3, \ldots, 99$, so that $a_j = d_1 + d_2 + \cdots + d_{j-1}$ for $j = 2, 3, \ldots, 100$. The conditions of the problem are equivalent to the fact that $d_1 > d_2 > \cdots > d_{99}$ are integers and

$$d_1 + d_2 + \cdots + d_{99} = (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{100} - a_{99}) = a_{100} - a_1 = 0.$$

Observe that we can take $d_i = 50 - i$ for $i = 1, 2, 3, \ldots, 99$, which yields

$$a_{19} = (a_{19} - a_{18}) + (a_{18} - a_{17}) + \cdots + (a_2 - a_1)$$

$$= d_{18} + d_{17} + \cdots + d_1$$

$$= (50 - 18) + (50 - 17) + \cdots + (50 - 1)$$

$$= 18 \times 50 - \frac{18 \times 19}{2}$$

$$= 729.$$

We will now show that this is the smallest possible value for $a_{19}$. For the sake of contradiction, suppose that $a_{19} < 729$. Then

$$729 > a_{19} = d_{18} + d_{17} + \cdots + d_1$$

$$\geq (d_{18}) + (d_{18} + 1) + \cdots + (d_{18} + 17) = 18d_{18} + \frac{17 \times 18}{2}.$$  

This leads to $d_{18} < 32$.

However, we also have

$$-729 < -a_{19} = -(d_1 + d_2 + \cdots + d_{18}) = d_{19} + d_{20} + \cdots + d_{99}$$

$$\leq (d_{18} - 1) + (d_{18} - 2) + \cdots + (d_{18} - 81) = 81d_{18} - \frac{81 \times 82}{2}.$$  

This leads to $d_{18} > 32$, which contradicts the inequality obtained earlier. Therefore, we can conclude that the smallest possible value for the term $a_{19}$ is 729.
8. Let \( n = 16^3r - 4^{3r} + 1 \) for some positive integer \( r \).

Prove that \( 2^{n-1} - 1 \) is divisible by \( n \).

**Solution 1** (Angelo Di Pasquale)

Observe that \( n \) has the form \( y^2 - y + 1 \), where \( y = 4^{3r} \). Thus, \( 4^{3r+1} + 1 = y^3 + 1 = n(y+1) \). Therefore,

\[
4^{3r+1} + 1 \equiv 0 \pmod{n} \\
\Rightarrow 2^{2\cdot 3r+1} \equiv -1 \pmod{n} \\
\Rightarrow 2^{4\cdot 3r+1} \equiv 1 \pmod{n}.
\]

(1)

To show that \( 2^{n-1} - 1 \equiv 1 \pmod{n} \), it suffices to show that \( 4 \cdot 3r+1 | n-1 \), since if \( n-1 = 4 \cdot 3r+1m \), then raising both sides of (1) to the power of \( m \) yields the result.

Since \( n - 1 = 4^{3r}(4^r - 1) \), it suffices to show that \( 3^{r+1} | 4^{3r} - 1 \). This can be done either by induction or by repeatedly factoring using the difference of perfect cubes.

**Variant 1.** (By induction)

For \( r = 1 \), it is easily verified that \( 3^2 | 4^3 - 1 \).

Assume that \( 3^{r+1} | 4^{3r} - 1 \). Then

\[
4^{3r+1} - 1 = (4^{3r})^3 - 1 = (4^{3r} - 1)(16^{3r} + 4^{3r} + 1).
\]

The inductive assumption tells us that \( 3^{r+1} \) divides the first bracket. The second bracket is congruent to \( 1^{3r} + 1^{3r} + 1 \equiv 0 \) modulo 3. Thus, \( 3^{r+2} \) divides \( 4^{3r+1} - 1 \) and this completes the induction.

**Variant 2.** (By repeatedly factoring using the difference of perfect cubes)

\[
4^{3r} - 1 = (4^{3r-1} - 1)(16^{3r-1} + 4^{3r-1} + 1) \\
= (4^{3r-2} - 1)(16^{3r-2} + 4^{3r-2} + 1)(16^{3r-1} + 4^{3r-1} + 1) \\
\vdots \\
= (4 - 1) \prod_{i=0}^{r-1} (16^i + 4^i + 1)
\]

Each bracket in the above factorisation is divisible by 3. Since there are \( r + 1 \) brackets, it follows that \( 3^{r+1} \) divides \( 4^{3r} - 1 \).

**Solution 2** (Ivan Guo)

In the last part of the official solution, in order to prove \( 3^{r+1} | 4^{3r} - 1 \), it suffices to note that \( \phi(3^{r+1}) = 2 \times 3^r \), thus \( 4^{3r} = 2^{\phi(3^{r+1})} \equiv 1 \pmod{3^{r+1}} \) by Euler’s theorem.