Maths Enrichment

Dirichlet Student Notes

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AUSTRALIAN MATHS TRUST

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Chapter 1

Mission Possible — Logic

It was 6:00 pm sharp when the team of four, Joanne, Leon, Andrew and Philippa, gathered to receive their new assignment. Joanne pressed the satellite-connected non-buggable speaker phone to connect them to their headquarters.

'We're all ready Bruce, fire away!' They turned their attention to Bruce.

'Good evening, my champion problem solvers! Your mission, should you choose to accept, is to save the Naccio Cephalopoda! It is a rare and exclusive marine mollusc, one of the last in the world and almost extinct. It has been used in a special breeding program and the research has shown it can be used to fight deadly viruses. It is worth millions and it was stolen two days ago. You will learn more during this assignment. The first part of the assignment is to find a certain secret undercover agent who has details of the mollusc's whereabouts. The agent you are looking for will arrive as an international guest at a dinner being held tonight in town. The agent has worked as a deep sea diver on the Naccio Cephalopoda research program.

'The dinner is being organised by a prominent person in the town; we do not know who, nor do we know exactly where it is to be held. Our agents have narrowed down the possibilities to the Governor-General, the Mayor, a businessman named Platt, and Captain Smythe, the captain of the research vessel in the port. The possible locations of the dinner are Government House, the Pier Mansion, the Hotel Savoy or the Banquet Room of the Town Hall.

'This mission is Top Secret; you must find this guest without making it obvious that you are looking for him or her. It is vital you do not reveal yourselves to the wrong person as we don't know who they really are! Your cover is as follows . . . '

'OK, let's spread out around the town and learn what we can. Be back here in an hour', said Andrew, who usually led.

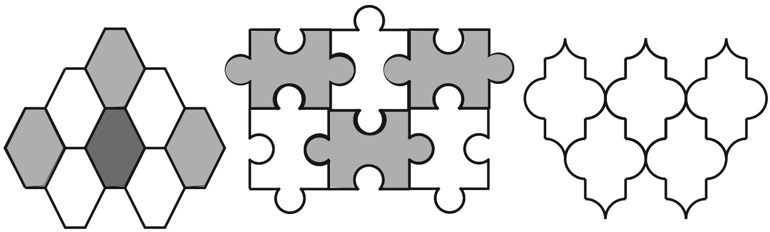
An hour later, they shared the information they had. 'I checked out the Hotel Savoy. I went to the kitchen with a false delivery and they told me there was no dinner there tonight', said Philippa.

'I chatted to Mr. Platt's secretary', said Joanne. 'He is going out tonight – he had asked her to check the tram times, so it must be at Government House or the Pier Mansion – there is no tram near the others. She also said she overheard his wife asking who would be hostess, so it cannot be him organising the dinner.'

Chapter 2

Tessellations

A tessellation is a pattern of shapes with no gaps or overlaps which can be continued as far as you wish. We say that a shape (or combination of shapes) tessellates if a tessellation of these shapes can be made. Tessellations are common in floor tilings and designs.

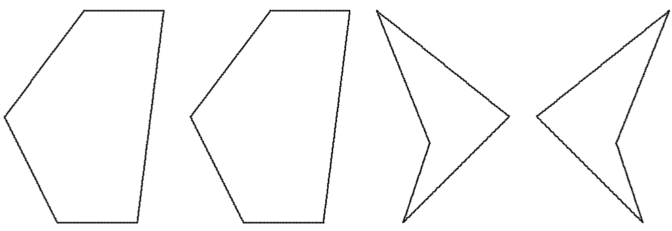


Regular Polygon

Definition: We are going to look at tessellations formed from polygons (planar, straight-edged figures such as triangles, squares and hexagons). A **regular** polygon is a polygon with all sides equal in length and all angles equal. A square is a regular four-sided polygon.

Congruent

Definition: We say that two objects are **congruent** if they are exactly the same size and shape. This means that one of the objects can be moved so that it exactly covers the other one (it may need to be flipped over for this to happen). For example, the shapes in these pairs are congruent. One of the second pair must be flipped to fit on top of the other.



Note that *any* triangle will tessellate:

Chapter 3

One-handed Arithmetic

The number system we use is called the decimal system. This means it uses base 10.

We have ten symbols, 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, which are called digits. When we want to go to the next number past 9, instead of having a new symbol, we combine the ten 1s or units into one group and call it 1 in the next column to the left (the tens column) and start the units off again, 10, 11, 12, 13, . . . , 19, 20, 21, . . . If we keep counting on, we will come to 99.

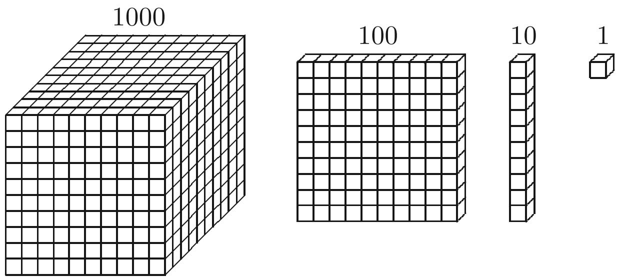
Adding 1 more makes ten 10s or one hundred. We put 1 in the next column to the left (the hundreds column), and the units start off again. Every time we use up the ten digits in one column, we need another column to the left. This happens at 10, $100 = 10 \times 10 = 10^2$, $1000 = 10 \times 10 \times 10 = 10^3$, 10 000 $=10^4$, and so on.

We can break down a number into units, tens, hundreds, etc. Thus 43 means $4 \times 10 + 3$, 2735 means $2 \times 1000 + 7 \times 100 + 3 \times 10 + 5$.

It is believed that the base 10 system developed because early counting was likely to have been done on the fingers of both hands, and we have ten fingers.

Two illustrations of how the decimal or base 10 system works are given below.

1. Base 10 blocks



Ten unit blocks are replaced by one 10 ‘strip’. Ten 10 strips are replaced by one 100 ‘flat’. Ten 100 flats are replaced by one 1000 ‘block’.

Chapter 4

Mission Possible — A Simpler Problem

'What an extraordinary dessert!' said Mr Lee. 'My compliments to the chef, Governor-General!'

'Excuse me, Miss Manego', interrupted Leon, 'A phone call for you. You may take it in the study if you like'.

Leon directed Miss Manego to the study where the rest of the team awaited her.

'Miss Manego, there is no call for you', Philippa spoke first, 'but we believe you have information for us'.

'Oh terrific! I've been waiting all night. I thought you'd approach me earlier.'

'Well, we wanted you to enjoy your meal first', added Joanne quickly.

'A fantastic meal it was!' she stated as the team turned and congratulated Andrew on his culinary skills. 'Let's get back to business now. The Naccio Cephalopoda has been kept captive in a warehouse behind the Pier Mansion. This map shows you how to get to the room it's in, using the secret passages, and I have the key to get you through the doors. The main concern is the alarm system in the room which holds the Naccio Cephalopoda. Only those who know how it works are able to move through the room without sounding the alarm.'

'Have you been able to find the code?' asked Philippa eagerly.

'I have a detailed account of how the alarm system works which I can tell you. After that it is up to you to find a way through!' said Miss Manego.

'We're all ears! Tell us what you know' said Leon and the team of four gathered around to listen carefully to everything she said.

'The room is 100 metres long and has a door on both the east and west ends. The Naccio Cephalopoda is in an aquarium in the middle of the room.

- Following the map, you will enter through the east door and exit from the west.
- At 1 metre intervals from the entrance you will encounter a thin wall of motion sensors. There is a separate alarm connected to each of the sensors.

Chapter 5

Time, Distance and Speed

Average Speed

Speed is a measure of how fast something is moving. Car speeds are usually measured in kilometres per hour, or km/h. Other units may be used such as metres per second (m/s) or centimetres per minute (cm/min). Speed is always distance per unit time.

A speed of 50 km/h means that it takes 1 hour to travel 50 kilometres. If a car travels 50 kilometres in 1 hour at a constant speed, then its speed is 50 km/h for the whole trip. In fact, it is unlikely that it would keep the same speed for the whole 50 kilometres. At times it may travel faster and at other times more slowly. So we say that its **average speed** is 50 km/h.

If a car travels 210 kilometres in 3 hours, then on average, this is equivalent to travelling $210 \div 3 = 70$ km every hour. So its average speed is 70 km/h.

If you walk 16 km in $2\frac{1}{2}$ hours, this is equivalent to walking $16 \div 2\frac{1}{2}$ km every hour, or

$$\text{average speed} = 16 \div 2\frac{1}{2}\,\text{km/h}.$$

But 16 kilometres in $2\frac{1}{2}$ hours is equivalent to 32 kilometres in 5 hours. So $16 \div 2\frac{1}{2}$ is the same as $32 \div 5 = 6\frac{2}{5}$ km/h or 6.4 km/h. So your average speed is $6\frac{2}{5}$ km/h or 6.4 km/h.

You can see that if you travel a distance in a given time, then your average speed will be the distance travelled divided by the time taken. We may write this as a formula:

$$\text{average speed} = \text{distance} \div \text{time taken},$$

where distance and time taken are measured in the units with which you want to express the speed.

If a car travels 60 kilometres in 3 hours then

$$\text{average speed} = \text{distance} \div \text{time taken} = 60 \div 3\,\text{km/h} = 20\,\text{km/h}.$$

Exercises

1. A student walks 28 km in 4 hours. What is her average speed in kilometres per hour?

2. A car travels 120 km in $1\frac{1}{2}$ hours. What is its average speed in kilometres per hour?

Chapter 6

Working with Patterns

Making the description simpler

In mathematics we use many symbols to represent words. These save us time and space. For example, the symbol '=' saves us writing 'equals', or 'is equal to', often many times in a single problem. In the same way it is much easier to write '×' or '÷', than to use the words and the meaning is the same whether we use the words or the symbols. Whether symbols or words are used, a mathematical statement should sound the same when you read it out loud.

In **sequences**, consisting of terms in a fixed order, we will describe a term number as T_n, where n is a counting number.

When $n = 1$, T_1 means term 1, or the first term in the sequence.

When $n = 5$, T_5 means term 5, or the fifth term in the sequence, and so on.

We may use expressions containing n to describe the pattern of the sequence.

Describing patterns

Some sequences involve a number multiplied by the term number, with another number added to, or subtracted from, the result.

Look at the number patterns below and see if you can work out how the differences between the numbers in each sequence relate to the pattern description:

Term number					Pattern description	T_n
1	2	3	4	5		
1	2	3	4	5	Term number	n
2	3	4	5	6	Term number + 1	$n + 1$
5	6	7	8	9	Term number + 4	$n + 4$
0	1	2	3	4	Term number − 1	$n - 1$

Term number					Pattern description	T_n
1	2	3	4	5		
2	4	6	8	10	2 × term number	$2 \times n = 2n$
3	5	7	9	11	2 × term number + 1	$2 \times n + 1 = 2n + 1$
1	3	5	7	9	2 × term number − 1	$2 \times n - 1 = 2n - 1$
7	9	11	13	15	2 × term number + 5	$2 \times n + 5 = 2n + 5$

Chapter 7
Mission Possible — Working Backwards

As the team on a mission arrived at the Marine Sanctuary, Miss Manego met them there as planned.

'You made it! You persisted with your problem-solving skills and you have achieved success. Well done!' she exclaimed.

'We're nearly there Miss Manego. If you could please tell us what we need to do to get the Naccio Cephalopoda back into its shell, then when that's done we can celebrate!' said Leon.

'Okay, here's what you need to know. The Naccio Cephalopoda's shell is built much like a Nautilus shell, in a continuing spiral starting from the centre. The spiral has 8 chambers, each larger than the one before it. That is, it makes 8 quarter turns. There is a passage, which flows inside. The eight lengths of the chambers fit a Fibonacci sequence.'

'I know, like 1, 1, 2, 3, 5, 8, ...', interrupted Leon excitedly.

'That's correct', added Miss Manego. 'To find the next number you need to add the previous two numbers together. The last two you said were 5 and 8, Leon, so the next number is ...'

'13', jumped in Andrew.

'The only problem is we can only measure the outer chamber. We have found it to be 118 mm. To accurately return the Naccio Cephalopoda to the inner part of the shell, this robotic arm needs to have the precise measurements of the lengths of the chambers, so it's up to you to find them. You'll need to be quick. By my calculations the Naccio Cephalopoda will not survive much longer than another 5 minutes out of its shell.'

The team gathered together.

'Okay, let's gather the key information. We have 8 chambers. The length of the third is the sum of the lengths of the first and second. The length of the fourth is the sum of the lengths of the second and third, until we reach the eighth whose length is the sum of the lengths of the sixth and seventh. The only one we know is the eighth and this is 118 mm. How do we tackle this?' asked Andrew.

'Perhaps we could guess different numbers at the start and use a guess-and-check method to see if we get 118 by the eighth turn', suggested Joanne.

'I know what you mean. For example, start with 2 and 5 and we get 2, 5, 7, 12, 19, 31, 50, 81. The eighth is not 81 so we must try again', said Leon.

Chapter 8

Recurring Decimals

If you were asked to divide 62 by 8 then the answer would be 7 and you would have a remainder of 6. How you manage this remainder depends a fair bit on the way in which the question was asked. If you were told that 62 bricks were to be stacked into piles of 8, then you would probably say that there would be 7 piles with 6 bricks left over.

If you were told that 62 pieces of paper were to be shared exactly among 8 people, you would give each person $7\frac{6}{8}$ or $7\frac{3}{4}$ pieces each. And if the question involved sharing \$62 among those people, you would almost certainly give the answer in decimal form as \$7.75. Calculators normally use decimals when the result of a division involves a remainder, but it isn't hard to do examples on paper.

$$\begin{array}{r} 8\overline{)\,62.{}^{6}0{}^{4}0} \\ 7.\ 7\ 5 \end{array}$$

The general method is to put a point after the whole number and tack on as many zeros as you need. Then use the remainders as you normally would in a division question. This trick always works, even when the number you are dividing by is larger than the one you are dividing it into. Just as a warm-up exercise, try doing a few of these on paper. (Remember, we're trying for decimal answers only.)

Exercises

1. $135 \div 12$

2. $7 \div 10$

3. Convert $\dfrac{21}{15}$ into a decimal. (In other words divide 15 into 21 and give the answer as a decimal.)

4. Write $13 \div 20$ as a decimal.

5. What is $\dfrac{23}{16}$ in decimal form?

The answers to these five questions are all decimals which stop after a little while. They are called **terminating decimals**. You can expect to get a terminating decimal if the fraction's denominator (or the number you have divided by) has only 2s and/or 5s as factors when it is cancelled down as far as possible.