Maths Enrichment

Euler Student Notes

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Contents

Chapter 2. Primes and Composites

A code is devised by Secret Sam. Sam has decided that he will denote A as the smallest number with 1 factor, B the smallest number with 2 factors and so on. He sets out his code as shown in the table.

A	1
B	2
C	4
D	6
E	16
⋮	
Z	12288

(a) Complete the table for the code.

(b) Secret Sam decides that he would like to punctuate his code properly and use some mathematical symbols. For example, he denotes '=' as the smallest number with 39 factors. What is this number?

Factors and Primes

The factors of 6 are 1, 2, 3 and 6.
The factors of 28 are 1, 2, 4, 7, 14 and 28.
The factors of 7 are 1 and 7.

A natural number a is a *factor* of a natural number b if there exists a natural number k such that:

$$b = ak.$$

If a number greater than 1 has only factors 1 and itself it is said to be a *prime*. Among the first 100 numbers, the following 25 are prime: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

To consider whether a number n is prime, one method is to consider all the numbers less than $\sqrt{n}$ and test if they are factors of n.

For example, if $n = 97$, $9 < \sqrt{n} < 10$. Therefore, we could test the numbers: 2, 3, 4, 5, 6, 7, 8 and 9. However, we do not need to test numbers which are not prime, so we only need to test 2, 3, 5 and 7.

Chapter 3. Least Common Multiple

A girl had been collecting 20c coins for many years. She had carefully counted them but had forgotten the total. However, she remembered that if the number of coins was divided by 2, 3, 4, 5 or 6 there was always 1 coin left over. A 'nice' property of the number was that it was divisible by 7 and she had between 700 and 800 coins. How many 20c coins had the girl collected?

Least Common Multiple

The least common multiple of natural numbers m and n is the smallest natural number which is a multiple of both m and n.

The least common multiple of 2 and 3 is 6.
The least common multiple of 4 and 6 is 12.

Note: The prime factorisation of two numbers will help us find the least common multiple of these two numbers.
e.g. $1080 = 2^3 \times 3^3 \times 5$ and $25200 = 2^4 \times 5^2 \times 3^2 \times 7$.

To find the least common multiple we choose the highest power of a prime occuring in either number and take the product of these powers.

In the above example, the least common multiple

$$\text{LCM} = 2^4 \times 3^3 \times 5^2 \times 7 = 75600.$$

This definition can be extended to more than 2 numbers. For example, the least common multiple of natural numbers k, m and n is the smallest natural number which is a multiple of k, m and n.

Example 1
Find the least common multiple of the first ten natural numbers.

Solution

These numbers and their prime factorisations are:

$$2 = 2,\ 3 = 3,\ 4 = 2^2,\ 5 = 5,\ 6 = 3\times 2,\ 7 = 7,\ 8 = 2^3,\ 9 = 3^2,\ 10 = 2\times 5.$$

Hence, the least common multiple of these numbers

$$\begin{aligned} &= 2^3 \times 3^2 \times 5 \times 7 \\ &= 2520. \end{aligned}$$

Chapter 4. Highest Common Factor and the Euclidean Algorithm

A button manufacturer finds he has produced 388 800 yellow buttons, 1 244 160 red buttons and 3 542 940 brown buttons. He wants to package them so that:

(i) All packages have the same number of buttons.

(ii) All packages contain buttons of only one colour.

(iii) The packages are to contain the largest number of buttons possible while satisfying properties (i) and (ii).

How many buttons are there in each package?

Highest Common Factor

A *common factor* of two natural numbers a and b is a natural number which is a factor of both a and b. For example:
6 is a common factor of 24 and 54.
3 is a common factor of 81 and 333.

The *highest common factor* of two natural numbers a and b is the largest number which divides both a and b, that is, the largest common factor. The highest common factor of two natural numbers a and b will be denoted (a, b). For example:
The factors of 8 are 1, 2, 4, 8.
The factors of 12 are 1, 2, 4, 6, 12.
The highest common factor of 8 and 12 is 4, i.e. $(8, 12) = 4$.

Consider the numbers 140 and 110. The prime factorisations of these numbers are:

$$140 = 2^2 \times 5 \times 7 \text{ and } 110 = 2 \times 5 \times 11.$$

The highest number which is a factor of 140 and 110 must have prime factors which occur in both factorisations. The exponent (power) of each of these prime factors will be the smaller of the two exponents occurring in the prime factorisation of both 140 and 110.
Thus $(140, 110) = 2 \times 5 = 10$.

Example 1
Find the highest common factor of 588 and 441.

Solution $588 = 2^2 \times 3 \times 7^2$ and $441 = 3^2 \times 7^2$.
Hence $(588, 441) = 3 \times 7^2 = 147$.

Chapter 5. Arithmetic Sequences

> The King of Nanastam decided to have a reception for the marriage of his daughter Hulu to the Prince of the Northland. Many invitations were sent out to the four corners of the land. The guests arrived in a strange way. The first time the drawbridge to the Lungi castle was lowered three people crossed, on the second eight people, on the third thirteen. The king noted that the number crossing the drawbridge increased by five people with each lowering. When the drawbridge had been lowered fifty times the king decided he would have to refuse entrance to any more guests.
> How many guests were in the castle?

Arithmetic Sequences

$1, 2, 4, 8, \ldots$ is an example of a *sequence* of numbers. The dots indicate that the numbers continue.

Each number in a sequence is called a *term*. In the sequence $1, 2, 4, 8, \ldots$, 1 is the first term, 2 is the second term and 4 is the third term.

Another sequence is $1, 3, 5, 7, \ldots$.

This is an example of an *arithmetic sequence*. A sequence is arithmetic if the difference between any two consecutive terms is always the same (constant).

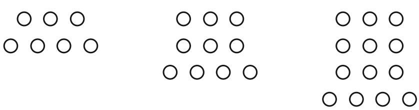
The sequence $1, 3, 5, 7, \ldots$ is arithmetic as

$$\begin{aligned} 3-1 &= 2 \\ 5-3 &= 2 \\ 7-5 &= 2 \\ &\vdots \end{aligned}$$

Here are some examples of arithmetic sequences.

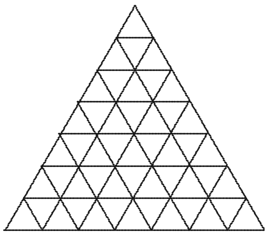
Example 1
Each of the diagrams below represent rows of beads.

1 2 3 . . .



7 beads 10 beads 13 beads

Chapter 6. Figurate Numbers



Here is an equilateral triangle. Each side is seven units long. It is divided as shown into smaller triangles.

(a) How many triangles are there with all sides one unit long?

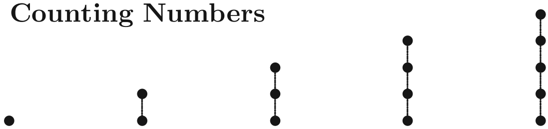
(b) How many triangles are there with all sides two units long?

(c) For an equilateral triangle of side length n units, how many triangles are there with all sides one unit long?

Patterns of Dots

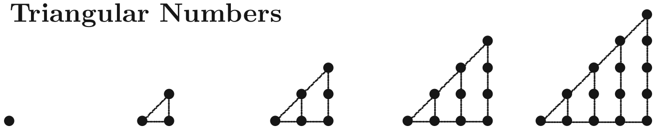
We consider the following patterns:

Counting Numbers



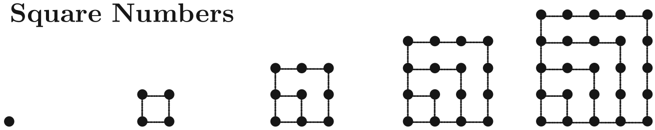
1 2 3 4 5

Triangular Numbers



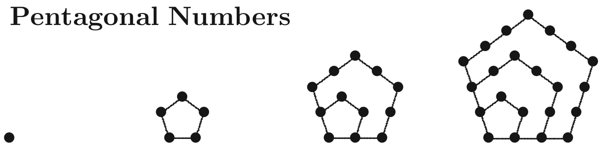
1 3 6 10 15

Square Numbers



1 4 9 16 25

Pentagonal Numbers



1 5 12 22

Chapter 7. Congruences

On a rainy Sunday afternoon Laura felt like developing a mathematical result. Needing somewhere to begin her investigation, she decided to look at powers of 2. The first thing she did was to prepare a table of powers.

n	2^n
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024
11	2048
12	4096

Laura noticed that there was a pattern in the last digits:
the sequence 2, 4, 8, 6, 2, 4, 8, 6, 2, 4, 8, 6.

(a) Find a formula for the powers of 2 with last digit:
  (i) 2
  (ii) 4
  (iii) 8
  (iv) 6

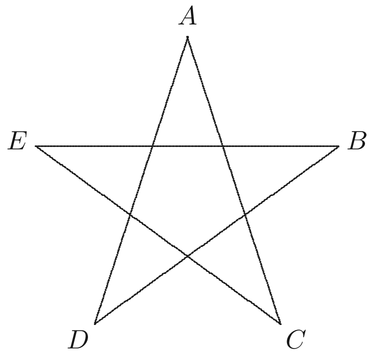
(b) Laura, finding a little satisfaction with this, decided to extend the table and see if there was a similar pattern for the last two digits. She found the pattern. Can you?

(c) Laura then investigated larger powers of 2 and found she could predict the last two digits. She decided that finding the last two digits of 2^{2013} would be impressive. She managed to find them. Can you?

Congruent Integers

In the previous chapters we have considered the natural numbers 1, 2, 3, ... It is desirable at this stage to enlarge our scope to include all integers, i.e. $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$.

Chapter 9. Find That Angle



A regular pentagon is a five-sided figure with all its sides equal in length and all its interior angles equal. The regular five-pointed star shown consists of a regular pentagon with its sides extended until they meet. It can be drawn without removing the pen from the paper. It poses a number of interesting questions:

(i) Are the angles at A, B, C, D and E equal?

(ii) Is it possible to draw other such regular stars with more points without removing the pen from the paper? Indeed, is it possible to draw all such stars without removing the pen from the paper?

(iii) Is the angle at each point of the star an integer number of degrees?

(iv) For which, if any, regular star polygons are the angles at each point an integer number of degrees?

In order to be able to answer such questions, one needs:

(a) a knowledge of facts concerning relationships between the angles of various figures;

(b) 'angle chasing' persistence, by which is meant the tenacity to follow leads which enable one to exploit observed angle properties in order to deduce others.

Chapter 10. Counting Techniques

> Show that the fraction of integers between 1 and 10^{1000} inclusive which contain no digits other than 2 and 5 is
>
> $$2\left(\frac{1}{5^{1000}} - \frac{1}{10^{1000}}\right).$$

Introduction

We are often confronted with questions of the type:

What proportion of . . .?
What is the probability that . . .?
In how many ways can . . .?

Many cases require some systematic thought and a little bit of extra information, e.g. *How many different routes can I drive from Newcastle to Mudgee? In how many ways can the first three horses be placed if there are six horses in the race?*

While there are several mathematical principles and techniques that are useful in a variety of situations, many questions can be answered directly by applying some systematic counting techniques. By systematic counting, we mean listing the possible outcomes in some systematic order and then counting these or developing counting patterns.

Some solutions may seem ingenious when first seen (and indeed many of them are!) but, to paraphrase the eminent problem solver George Pólya, when we can apply these ingenious methods again in similar and related situations, we have then developed a technique.

Counting techniques are also important in a variety of probability situations, which can be solved by simply comparing the different numbers of ways that certain events can occur.

There is a variety of ways of tackling the problem above: we will state four principles which help students solve a wide class of counting problems, give examples of the use of these and then look at some less routine and more ingenious methods.

Chapter 11. The Pigeonhole Principle

> Six people sit down to dinner at a circular table. It is soon discovered that nobody is sitting in his/her correct place. Show that, by rotating the table, it is always possible to place at least 2 people correctly.

Some Examples

Example 1

If 5 pigeons fly into 4 pigeonholes, then at least one hole contains two or more pigeons.

In general, if $n+1$ pigeons are in n pigeonholes, at least one of the holes will contain two or more pigeons.

Example 2

I select some socks from a drawer containing socks of three different colours. If I make a 'hole' for each colour sock then I will have 3 'holes'. If I select 4 socks then at least one of my holes contains two or more socks. So if I select 4 socks I will always have one pair.

This is a very simple example and it can help us solve some simple problems.

Example 3

If I have 13 pigeons in 4 holes, then at least one hole will contain 4 or more pigeons.

In general, if I have more than k times as many pigeons as pigeonholes, then at least one hole will contain $k+1$ or more pigeons.

Exercises

1. What is the least number of people that must be chosen to be sure that at least 2 have the same first initial?

2. In a group of 8 people show that at least 2 have their birthday on the same day of the week.

3. A consumer organiser selects 11 phone numbers from the phone book. Show that at least 2 have the same last digit.

4. If I put more than 100 marbles in two bags, show that at least one bag contains more than 50 marbles.