Maths Enrichment

Gauss Student Notes

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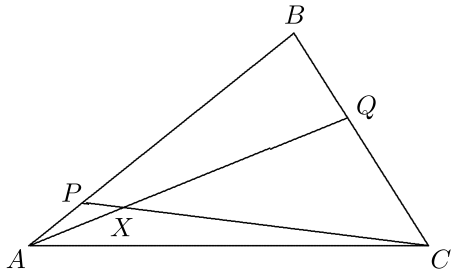
AUSTRALIAN MATHS TRUST

Contents

Chapter 2. Parallels



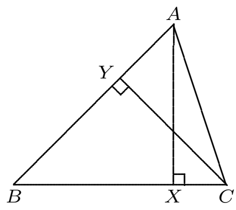
P and Q lie on the sides AB and BC respectively of the triangle ABC such that $\dfrac{AP}{PB} = \dfrac{1}{4}$ and $\dfrac{BQ}{QC} = \dfrac{2}{3}$. The area of triangle ABC is 85 cm^2. PC and QA intersect at X. Find the area of triangle AXC and the ratios $\dfrac{AX}{XQ}$ and $\dfrac{CX}{XP}$.

Comparing the Areas of Triangles

In this section we see that we can use areas to investigate certain situations. For example:

Any side of the triangle may be treated as the base. But we must use the corresponding perpendicular height or altitude.

If $AB = 20$, $BC = 24$ and $AX = 10$ in this diagram, we can determine CY by working out the area in two ways.



$\frac{1}{2}BC \times AX = \frac{1}{2}AB \times CY$.
$\frac{1}{2} \times 24 \times 10 = \frac{1}{2} \times 20 \times CY$.
Thus $CY = 12$.

The significant factor in comparing the areas of triangles is that we rarely have to compute the actual areas or even find the length of an altitude.

There are two particular situations in which this is the case:

(i) problems in which parallel lines are included in the figure,

(ii) problems in which triangles have a common vertex.

Chapter 3. Similarity

> Specify the dimensions of a set of seven standard paper sizes such that:
>
> - two pages of one size placed side by side fit exactly on the next higher size,
> - any design printed on one size, when enlarged by a suitable factor on a photocopier, fits exactly on the next higher size, and
> - the middle size 4 is approximately the size of a page of your exercise book.

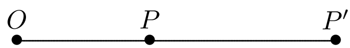
Enlargements

A **dilation** is a geometric transformation which moves each point, P, to an image, P', in relation to a fixed point, O, and a constant, k, according to the following conditions:

(i) O, P and P' are collinear.

(ii) $OP' = kOP$.

O is called the **centre of dilation**. k is the **scale factor**.



If $k > 1$, then this is an **enlargement** and O is the **centre of enlargement** while k is the **enlargement factor**.

Note that we adopt as standard practice that X' will be the label of the image of the point X.

Exercises

1. Use a full page for this question. Mark and label a point, O, and five other points, P, Q, R, S and T. Use O and a scale factor of two to determine the images, P', Q', R', S' and T' of the five points above.

Chapter 5. Pythagoras' Theorem

It is certainly possible to construct triangles with side lengths

(a) 6, 8 and 10 cm

(b) 3, 5 and 6 cm

(c) 4, 6 and 7 cm.

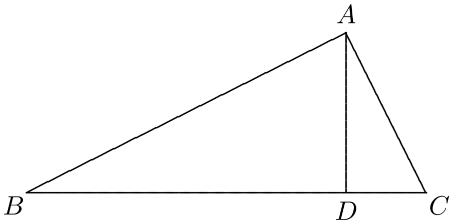
It turns out that one of these has an obtuse angle, one is right-angled and one has only acute angles. But which is which?

Is there a quick way to decide whether a triangle is obtuse-angled, right-angled or acute-angled?

Pythagoras' theorem states that *the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.* There are many ways to prove this most famous theorem of geometry. A whole book has been written on them. Here is one proof using similar triangles.

Given. The triangle ABC is right-angled at A. $AB = c$, $BC = a$ and $CA = b$.

Aim. To show that $a^2 = b^2 + c^2$.



Proof. Construct the altitude AD from A to BC.
$\angle BAD = 90° - \angle ABC$ (angle sum of triangle ABD).
$\angle ACB = 90° - \angle ABC$ (angle sum of triangle ABC).
Thus $\angle BAD = \angle ACB$ and each of the triangles ACB, DAB and DCA contains an angle of this size as well as a right angle.

So these three triangles are similar and thus the corresponding sides are in proportion:

$$\frac{c}{a} = \frac{AD}{b} = \frac{BD}{c} \quad \text{and} \quad \frac{b}{a} = \frac{DC}{b} = \frac{AD}{c}.$$

Chapter 6. Spreadsheets

> Find all the positive integers w, x, y and z such that
>
> $$w + x + y + z = 54 \quad \text{and} \quad w^2 + 9x^2 + 9y^2 + 4z^2 = 1981.$$

Spreadsheeting is a very useful computer tool to use in problem solving. By the generation of orderly arrays of numbers, the pattern we seek may become obvious. So while a spreadsheet will often provide an answer to a problem, it may not find all the solutions yet still provides a mechanism to complete the problem algebraically. In other circumstances, spreadsheets can provide all the solutions to a problem, because the number of possible values for the unknowns is limited in some way.

All spreadsheet programs will be different in the commands used and the formatting involved. Nonetheless, there are common features of all spreadsheet programs so that it will not matter which one you use in conjunction with the work in this chapter.

The most commonly used spreadsheet is probably Microsoft Excel, which is found on Windows and Macintosh systems.

You need to learn how to do a few simple things with the spreadsheet program you are using. You may need to consult the manual or a friend or teacher in order to find out the commands to use etc. Below is a list of the spreadsheet skills needed and this list is followed by some simple exercises to test your mastery of these skills.

Skills required

- Open a new spreadsheet.
- Save a spreadsheet.
- Retrieve a spreadsheet file, alter it and save the altered version as well as the new version.
- Fill up or down a column and fill left or right across a row.
- Write a formula in a cell and show the result of the application of the formula or the formula itself.
- Write formulae which involve cells.

Chapter 7. Diophantine Equations

> I wanted to mail a parcel overseas last Saturday which I knew would cost \$7.10. But the place where I buy my stamps when the Post Office is closed only had 45-cent and 20-cent stamps available. They had an unlimited supply of both of these types and my parcel was big enough to stick on all the stamps I was going to need to buy. I figured out how many of each stamp I needed and got my parcel in the mail on time. But back at home, I wondered if there were any other combinations of stamps which would have done the job.
>
> (i) List all the ways of making up the correct postage using 45- and 20-cent stamps.
>
> (ii) Could the problem be solved if, instead of 20-cent stamps, 30-cent stamps were available? Explain.
>
> (iii) List the ways of making up the postage if the stamps available were 45-cent, 20-cent and \$2.

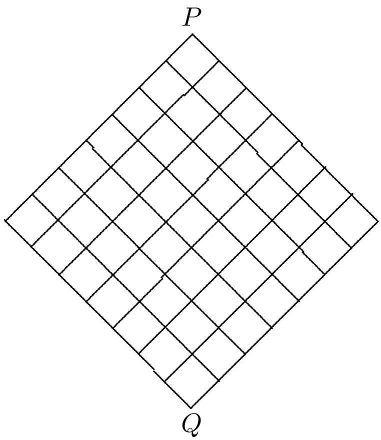
The questions above may be solved using Diophantine equations. This term describes equations whose solution or solutions must be integers. Often, though not always, the integers must be positive as well. They may be in several unknowns or only one. Their name pays tribute to Diophantus of Alexandria who, about 1700 years ago, got pretty involved in what you're about to study. Hopefully, you'll get as hooked on them as he did!

If you had any success with the above question, you will have noticed that, when formulated, what it produced looked rather like half of a pair of simultaneous equations in two unknowns. It is, of course, a simple linear relation which could be graphed on the Cartesian plane. But, instead of coming out as a straight line, the requirement for integer solutions means that the solution set is a series of points along that straight line. You will also have noticed that a bit of trial and error was required to get at least the first of your solutions. This is another feature of Diophantine equation solution technique—the more shrewd your guesses are, the less time you will spend flailing around in the dark!

Example 1. Your recipe calls for 1300 grams of flour. You have two measuring scoops: 250 g and 100 g. List the ways in which you can measure out the flour in fewer than eight scoops.

Chapter 9. Counting

If only downward motion along lines is allowed, what is the total number of paths from point P to point Q in the figure shown?



This kind of problem requires some sort of systematic counting, of which you have probably seen something before. A few examples and exercises will remind you of some important principles of systematic counting.

Principle 1: The Multiplication Principle

Example 1. If we were creating labels for items by selecting a letter of the alphabet A, B, C, . . . , Z, followed by a one-digit number 0, 1, 2, . . . , 9, we could list the possibilities:

$$\begin{array}{ccccc} A0, & A1, & A2, & \dots, & A9; \\ B0, & B1, & B2, & \dots, & B9; \\ \vdots & & & & \\ Z0, & Z1, & Z2, & \dots, & Z9 \end{array}$$

and get 26 rows of 10 to give $10 + 10 + 10 + \cdots + 10 = 26 \times 10 = 260$ different combinations.

However, using the multiplication principle, we can say that there are 26 ways of selecting a letter and 10 ways of selecting a digit, so there are $26 \times 10 = 260$ ways in total.

Chapter 10. Congruences

(a)	What are the last digits of $3^3 - 1$, $3^{33} - 1$ and $3^{333} - 1$?
(b)	What are the last 2 digits of $3^3 - 1$, $3^{33} - 1$ and $3^{333} - 1$?
(c)	What are the last 3 digits of $3^3 - 1$, $3^{33} - 1$ and $3^{333} - 1$?

Congruence

There is a variety of ways of tackling the above problem and we shall discuss methods which may help you solve a wide variety of problems in number theory.

If b is any integer and m is a known positive integer and b is divided by m, then we get an equation $b = q \times m + r$, where $0 \leq r < m$ and q is an integer.

The integers b and r have many properties in common relative to m. For example, b and m have the same highest common factor as m and r. If a also has the remainder r when it is divided by m, then $b - a$ is divisible by m.

For integers a and b we write:
$a \equiv b$ mod m when a and b have the same remainder on division by m.

For example, $63 \equiv 18 \bmod 5$ is a true statement, since we can divide 63 by 5, discard the quotient (12) and keep the remainder (3), similarly with 18, on division by 5, the remainder is 3, so $63 \equiv 18 \bmod 5$.

We say:
a is congruent to b modulo m, and we refer to $a \equiv b$ mod m by the name of 'congruence'.

It can be shown that $a \equiv b \bmod m$ if and only if $a - b$ is a multiple of m.

For example, notice that $63 - 18 = 45$ which is a multiple of 5.

Since the smallest non-negative integer remainders on division by 5 are 0, 1, 2, 3 and 4 and all integers must leave one of these remainders on division by 5, arithmetic modulo 5 effectively has only 5 numbers.

Examples

Following are some examples of addition, subtraction and multiplication, modulo 7, and you should construct some similar examples modulo another number. These demonstrate the properties of congruences which are discussed later.