Maths Enrichment

Noether Student Notes

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Contents

Chapter 2. Expansion and Factorisation

Find the value of the expression

$$\frac{a^{32} - b^{32}}{(a^2 + b^2)(a^4 + b^4)(a^8 + b^8)(a^{16} + b^{16})} + 12b$$

if the only thing which is known about a and b is that $a + b = 6$. Show that the answer does not depend on the way a and b satisfying $a + b = 6$ have been chosen.

This chapter is about how to expand and factorise algebraic expressions in order to simplify them or to make them more manageable when problem solving. Generally speaking, expanding means transforming an algebraic expression so that it does not contain any grouping symbols. Factorising is, in some sense, the reverse of expanding, that is, factorising means representing an algebraic expression as a product of two or more factors. Usually, expanding is easier to master than factorising, so expanding is the first thing we study in this chapter.

Getting Rid of Grouping Symbols

Here are two formulae that are most commonly used for expanding algebraic expressions.

Formula 1. $a(b + c) = ab + ac$

Formula 2. $a(b - c) = ab - ac$

Note. These two formulae are very helpful but they are very simple as well. Therefore, in order to use them effectively, one needs to learn how to apply the formulae to complicated algebraic expressions.

Example 1.

Expand the following algebraic expressions.

(a) $5p(3p - 2q)$;

(b) $x(y + z + t)$;

(c) $(2m - 3n)(4nm - n^2)$;

(d) $(1 + 5x + x^2)(1 - 5x)$.

Chapter 3. Further Expansion and Factorisation

Let α and β be the roots of the quadratic equation $x^2 + x - 1 = 0$. Find the value of $\alpha^6 + \beta^6$.

In this chapter, we shall continue to study expansion and factorisation of algebraic expressions but at a more advanced level.

More Formulae

Formula 6. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Formula 7. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

Example 9.
Simplify:

(a) $\dfrac{a^3 - 1}{(a+1)^2 - a}$;

(b) $\dfrac{x^3 - y^3}{x^3 + 8y^3} \times \dfrac{(x-2y)^2 + 2xy}{(x+2y)^2 - 3y(x+y)}$.

Solution.

(a) We have

$$a^3 - 1 = a^3 - 1^3 = (a-1)(a^2 + a \times 1 + 1^2) = (a-1)(a^2 + a + 1)$$

by **Formula 6**, and $(a+1)^2 - a = a^2 + 2a + 1 - a = a^2 + a + 1$ in view of **Formula 3**.
Hence

$$\frac{a^3 - 1}{(a+1)^2 - a} = \frac{(a-1)(a^2 + a + 1)}{a^2 + a + 1} = \frac{a-1}{1} = a - 1.$$

(b) According to **Formula 6**, $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$.
To see how **Formula 7** can be applied to the expression $x^3 + 8y^3$, we put $a = x$ and $b = 2y$.
Therefore,

$$\begin{aligned} x^3 + 8y^3 &= x^3 + (2y)^3 \\ &= a^3 + b^3 \\ &= (a+b)(a^2 - ab + b^2) = (x+2y)(x^2 - 2xy + 4y^2) \end{aligned}$$

Chapter 4. Sequences and Series

In a high school science class of 100 students, each student is given a piece of plasticine in the shape of a cube so that the nth student has the piece with an edge of length n cm. This means that the first student has the piece with edge length equal to 1 cm, the second student has the piece with edge length equal to 2 cm and so on. The teacher asks the students to combine all the pieces to make one large cuboid-shaped model with edge length being an integer number of centimetres. Can they succeed?

When a set consists of numbers ordered in some way, then this set can be considered as a sequence.

Description of Sequences

A **finite sequence** is a list of k numbers in a particular order. Thus there is a first number in the sequence, a second number in the sequence, and so on until the last, the kth number in the sequence. The numbers that form the sequence are called its **terms**. To emphasise the fact that each term of a sequence has its own unique number, the sequence is often written as $a_1, a_2, a_3, \ldots$ where a_1 is called the **first term** of the sequence, a_2 is called its **second term** and so on. Generally, a_n is called the nth term of the sequence.

There are three main ways of describing a sequence.

1. The simplest way of describing a finite sequence is to write down each of its terms as a separate number. This means that the sequence is represented in the form of a string of numbers where the first number is the first term of the sequence, the second number is the second term of the sequence and so on.

Example 1. Given the sequence $a_1 = 1$, $a_2 = 3$, $a_3 = 4$, $a_4 = -2$, $a_5 = 0$, find the value of $a_1^2 - a_2a_3 + a_4^3 - a_5$.

Solution.

Since we know what number each of the terms of the sequence is, we can easily find that

$$a_1^2 - a_2a_3 + a_4^3 - a_5 = 1^2 - 3 \times 4 + (-2)^3 - 0 = -19.$$

Note. The above method of describing a sequence is very convenient when the number of terms of the sequence is small. But it becomes very inefficient for sequences with a large number of terms.

Chapter 5. Number Bases

All digits in the representation of a positive integer m in base 3 are equal. Also each of the digits of the representation of m in base 9 is equal to the digits of the representation of m in base 3. Find all such numbers m.

When writing integers, we use the decimal system, that is, we represent integers in terms of powers of ten. But this is not the only way of representing integers. For example, the computations performed by computers are carried out in base 2. In this chapter we discuss how to use different number bases.

If an integer m is expressed as

$$m = a_k q^k + a_{k-1} q^{k-1} + \cdots + a_1 q + a_0,$$

where q is an integer greater than 1 and a_i is an integer such that $0 \leq a_i < q$ for $i = 0, 1, \ldots, k-1, k$ and $a_k \neq 0$, then this expression for m is called the **representation of m in base q** and is written as

$$m = (a_k a_{k-1} \ldots a_1 a_0)_q.$$

If $q = 10$, then m is said to be written in decimal system. If $q = 2$, then m is said to be written in binary system.

Note. Usually, numbers in base 10 are written without any special extra symbols, that is, a number $(a_k a_{k-1} \ldots a_1 a_0)_{10}$ is put down as $a_k a_{k-1} \ldots a_1 a_0$.

The following example shows how to use this definition.

Example 1. The number $(624)_7$ is written in base 7. Convert this number to base 3.

Solution.

Since we are used to writing numbers in base 10, the plan is to represent $(624)_7$ in base 10 first and then to write it in base 3.

According to the definition, we have

$$\begin{aligned} (624)_7 &= 6 \times 7^2 + 2 \times 7 + 4 \\ &= 294 + 14 + 4 \\ &= 312. \end{aligned}$$

Chapter 6. Inequalities

Prove that the product of all positive integers from 1 up to 1995 is less than 998^{1995}, that is, the following inequality holds.

$$1 \times 2 \times 3 \times \cdots \times 1994 \times 1995 < 998^{1995}$$

Quite often, while solving a problem, one needs to compare two numbers, that is, decide which of them is greater than the other. Generally, of two real numbers a and b, a is said to be greater than b if $a - b$ is positive, and a is said to be less than b if $a - b$ is negative. If a is greater that b, we write down $a > b$, and if a is less than b, we write down $a < b$. Obviously, $a > b$ implies $b < a$ and vice versa.

When dealing with inequalities, it is very important to remember that the rules inequalities obey differ from those for equations. This assumes that inequalities require special treatment and new approaches.

Rule 1. If $a > b$, then $a + x > b + x$ and $a - x > b - x$.
If $a < b$, then $a + x < b + x$ and $a - x < b - x$.

Rule 2. If $a > b$, then $-a < -b$.
If $a < b$, then $-a > -b$.

Rule 3. If $a > b$ and $x > 0$, then $ax > bx$ and $\frac{a}{x} > \frac{b}{x}$.
If $a > b$ and $x < 0$, then $ax < bx$ and $\frac{a}{x} < \frac{b}{x}$.
If $a < b$ and $x > 0$, then $ax < bx$ and $\frac{a}{x} < \frac{b}{x}$.
If $a < b$ and $x < 0$, then $ax > bx$ and $\frac{a}{x} > \frac{b}{x}$.

Example 1. Let x be a number such that $4x - 3 > 3x + 6$. Prove that $x > 9$.

Solution.

According to **Rule 1**, the inequality $4x - 3 > 3x + 6$ yields
$(4x - 3) + 3 > (3x + 6) + 3$.

Chapter 8. Methods of Proof

One of the terms of a sequence $a_1,\ a_2,\ a_3, \ldots$, whose terms satisfy the equation $a_{n+1} = 3a_n - 4$, does not exceed 2. Prove that none of the terms of the sequence $a_1,\ a_2,\ a_3, \ldots$ exceeds 2.

In mathematics, there are many different methods of proving statements. In this chapter, we shall consider two of the most common methods: proof by contradiction and proof by induction.

First we shall look at how proof by contradiction works.

As an illustration, consider the following situation. There are several balls in a box. Each ball can be one of two possible colours (blue or red), and one of two possible sizes (small or large). We are told that every red ball in the box is large. One ball is taken out of the box and it is small. What colour is this ball?

The answer is almost obvious: the ball must be blue. But can we explain why? Yes, we can and the explanation seems to be easy: every small ball in the box must be blue as otherwise it would be red and therefore large. This reasoning looks very simple but it contains an interesting twist: in order to prove that the ball is blue, we showed that it could not be red. Such reasoning by ruling out all other possibilities is the essence of the proof by contradiction method.

More formally, the proof by contradiction can be described as follows.

Suppose it is given that statement A is true and we are asked to prove that statement B is also true. Then we can use proof by contradiction which involves the following two steps.

1. Assume that statement B is not true.

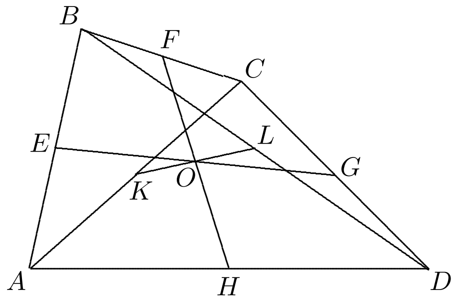
2. Using this assumption, show that statement A is not true either.

If we complete the above two steps, we show that statement A is not true. But it is given that statement A is true. Thus statement A is both true and false: this is a contradiction. The contradiction we obtained means that our assumption was wrong. Thus statement B is true.

Example 1. Prove that if one of two integers a and b is divisible by n and the other is not, then their sum $a + b$ cannot be divisible by n.

Chapter 9. Congruence

Prove that the segments EG and FH joining the midpoints of pairs of opposite sides of a quadrilateral $ABCD$ and the segment KL joining the midpoints of its diagonals are concurrent, that is, meet at one point, say O, and bisect one another, that is, $EO = OG$, $FO = OH$ and $KO = OL$.
This statement is called *Varignon's theorem.*



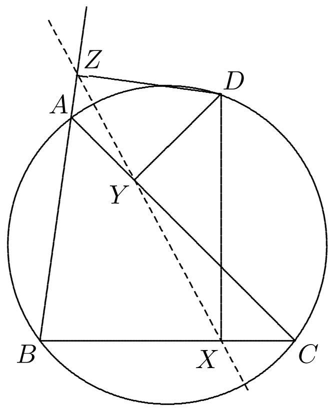
Generally speaking, two geometrical figures are called congruent if one of them can be put on the other so that they coincide in all details. However, it is quite difficult to give a rigorous definition of congruent figures. Therefore, we confine ourselves to congruent polygons. As triangles are the most basic polygons, the centre of our attention in this chapter will be congruent triangles.

Two polygons are called **congruent** if their corresponding sides are equal and their corresponding angles are equal. Two triangles are said to be **congruent** if three sides of one triangle are respectively equal to three sides of the other triangle and their corresponding angles are equal as well.

In the picture below, the triangles ABC and FED are congruent and the pentagons $LMNPQ$ and $VURST$ are congruent too, but the triangle GHK is congruent to none of the figures shown in the picture except itself.

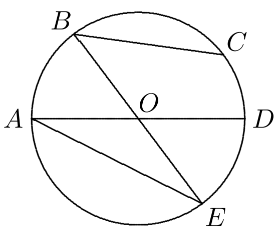
Chapter 10. Circles

Let A, B, C and D be four points placed on a circle and let X, Y and Z be the feet of the perpendiculars from D to the straight lines BC, AC and AB respectively, produced if necessary. Prove that X, Y and Z are collinear, that is, these points lie on a straight line.



First of all, we recall some terminology and give an illustration of the terms introduced.

A **circle** is the set of points equidistant from a fixed point, called the **centre**. A **chord** is any straight line segment joining two points on a circle. A **diameter** is a chord through the centre. A **radius** is any straight line segment from the centre to a point on the circle. The equal length of all such segments is also called the radius. An **arc** is a part of the circle cut off by a pair of points on the circle. The boundary of the circle is also called the **circumference** as is the length of this boundary.



O is the centre of the circle

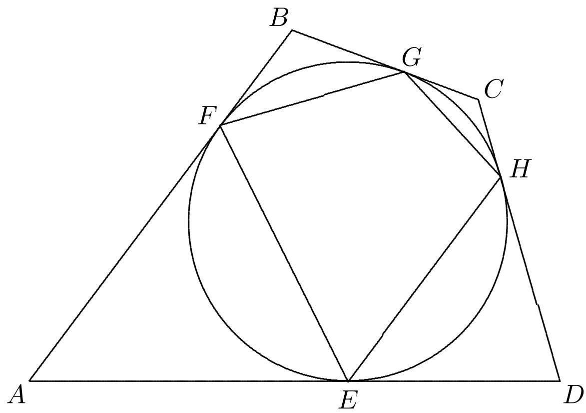
AD, BE are diameters

AE, BC, AD, BE are chords

OA, OB, OC (not drawn), OD, OE are radii

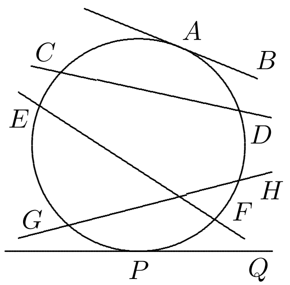
Chapter 11. Tangents

The sides AB, BC, CD and DA of a cyclic quadrilateral $ABCD$ touch a circle at the points F, G, H and E respectively.
Prove that the midpoints of the sides of the quadrilateral $EFGH$ form a rectangle.



In this section we discuss tangents to circles, tangential circles and their properties.

A line meeting a circle at two distinct points is called a **secant**. If it meets the circle at exactly one point, it is a **tangent**.



CD, EF and GH are secants

AB and PQ are tangents

When looking at tangents or trying to draw them, one can easily get a feeling that tangents are in a special position in respect to the corresponding circles. This suggests that tangents have interesting properties which could be very useful for problem solving.