# Maths Enrichment

## Pólya Student Notes

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# Contents

# Chapter 1. Functions

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| Let $f : [0,1] \to [0,1]$ be a continuous function. Prove that there exists a number $c$ in $[0,1]$ such that $f(c) = c$. |

## What is a function?

You may have heard several possible definitions of a function. We'll discuss these, in order to clarify what we're talking about!

A function might be described in the following ways. But some of these descriptions are more accurate than others, and some are more correct than others!

A *function* $f : X \to Y$ is...

(i) a rule which takes an element of $X$ and gives you an element of $Y$

(ii) a 'machine' into which you can feed an input from $X$, and which gives an output from $Y$

(iii) a formula which, given an element $x$ of $X$, tells you what $f(x)$ is, like $f(x) = x^2$

(iv) a graph of $y = f(x)$

(v) a set of pairs of elements $(x, y)$ where $x \in X$ and $y \in Y$.

Some of these notions are more rigorous than others! Let's see how they stack up.

A function always has a *domain* and a *codomain*. Letting the function be $f$, the domain $A$ and the codomain $B$, we write $f : A \to B$.

We can define a function with domain $A$ and codomain $B$ as follows.

> *A function $f : A \to B$ is a rule which associates, to each element $a$ of $A$, a unique element of $B$.*

The element of $B$ which is associated to $a \in A$ is denoted by $f(a)$.

In this way you can think of a function $f$ as a 'machine' to which you can feed an input $a$ from the set $A$, and the 'machine' gives you as output $f(a)$, which is an element of $B$. But it is a 'predictable' machine because if you feed it the input $a$ you always get the same output $f(a)$!

# Chapter 2. Symmetric Polynomials

## A brief review of Vieta's formulas

From the theory of quadratic equations, Vieta's formulas are well known. For the quadratic case, they state the following:

If $x_1$ and $x_2$ are solutions of the equation

$$x^2 + px + q = 0$$

(with coefficients $p, q$ and variable $x$), we have $x_1 + x_2 = -p$ and $x_1x_2 = q$. This is an immediate consequence of the fact that we can write

$$x^2 + px + q = (x - x_1)(x - x_2) = x^2 - (x_1 + x_2)x + x_1x_2, \tag{1}$$

if $x_1$ and $x_2$ are indeed solutions as required. The formulas are then derived by simply comparing coefficients of the powers of $x$ in the left- and right-hand polynomials in (1).

A reasonably obvious idea is to use this method to derive formulas yielding similar expressions for the solutions of higher power (monic) equations. For instance, if $x_1$, $x_2$ and $x_3$ are solutions of the cubic equation

$$x^3 + px^2 + qx + r = 0,$$

we have

$$x^3 + px^2 + qx + r = (x - x_1)(x - x_2)(x - x_3)$$
$$= x^3 - (x_1 + x_2 + x_3)x^2 + (x_2x_3 + x_3x_1 + x_1x_2)x - x_1x_2x_3,$$

and therefore

$$x_1 + x_2 + x_3 = -p, \quad x_2x_3 + x_3x_1 + x_1x_2 = q \quad \text{and} \quad x_1x_2x_3 = -r.$$

If $x_1$, $x_2$, $x_3$ and $x_4$ are solutions of the quartic equation

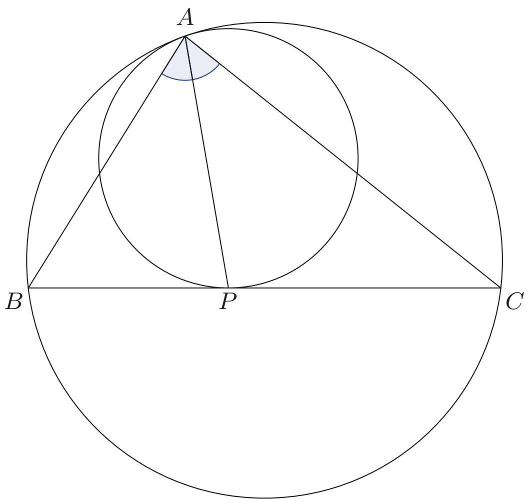
$$x^4 + px^3 + qx^2 + rx + s = 0,$$

a similar calculation gives us

$$x_1 + x_2 + x_3 + x_4 = -p, \quad x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 + x_1x_3 + x_2x_4 = q,$$

$$x_2x_3x_4 + x_1x_3x_4 + x_1x_2x_4 + x_1x_2x_3 = -r \quad \text{and} \quad x_1x_2x_3x_4 = s.$$

Let $ABC$ be a triangle. Let $P$ be the point on $BC$ so that $AP$ bisects $\angle BAC$. Consider the circle that is tangent to line $BC$ at point $P$ and that passes through point $A$. Prove that this circle is tangent to the circumcircle of triangle $ABC$.



# Chapter 4. Inequalities

An inequality is two expressions involving an unknown variable, say $x$, separated by an inequality symbol: $<$, $\leq$, $>$ or $\geq$. Often, we want to prove an inequality is true for all possible values $x$. Sometimes $x$ is constrained to a specific range (for example, $x > 0$) and sometimes inequalities contain two unknown variables or more.

## Squares are never negative

For any real number $a$, we know $a^2 \geq 0$. Similarly if an expression is a sum of squares then it cannot be negative. Look for square expressions like: $(x-1)^2$, $(2x-3)^2$, $(x-y)^2$, as part of the expression you are trying to prove is greater. Square expressions often appear in their expanded forms like: $x^2 - 2x + 1$, $4x^2 - 6x + 9$ and $x^2 - 2xy + y^2$. Also, a good trick is to to consider the difference between the LHS and the RHS of the equation. It is often much easier to prove a single expression is positive (or negative) rather than compare two different expressions directly. Here are some examples.

**Example 1**
Prove $x^2 + 25 \geq 10x$ for all real $x$.

**Solution**
It is sufficient to show that $x^2 + 25 - 10x \geq 0$. This expression factorises as

$$x^2 + 25 - 10x = (x-5)^2$$

which is greater than or equal to zero because it is square.

**Example 2**
For all positive $x$ and $y$ show that

$$x^2 - 7xy + 13y^2 \geq 0.$$

**Solution**
Multiply the equation by 4, and observe that it *almost* factorises as the square $(2x - 7y)^2$.

$$\begin{aligned} &4(x^2 - 7xy + 13y^2) \\ &= 4x^2 - 28xy + 52y^2 \\ &= \left(4x^2 - 28xy + 49\right) + 3y^2 \\ &= (2x - 7y)^2 + 3y^2 \end{aligned}$$

# Chapter 5. Functional Equations

Find all functions $f : \mathbb{R}\backslash\{0,1\} \longrightarrow \mathbb{R}$ such that, for all real numbers $x \neq 0, 1$,

$$f(x) + f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}.$$

## Introduction

A *functional equation*, as the name suggests, is nothing more than an equation involving functions!

So any equation involving a function like $f(x)$, for instance,

$$f(x+y) = f(x) + f(y) \quad \text{or} \quad f(x)^2 = 1 + xf(x) - f(x^2+3),$$

is a functional equation.

Functional equations can look very intimidating. But there are a few basic ideas that will usually get you started on solving them.

Of course, in order to solve a functional equation, you have to be familiar with functions! We will assume that you are comfortable with the idea of a function, and also with ideas such as *domain*, *range*, *(strictly) increasing*, *(strictly) decreasing*, *monotonic*, *continuous*, and *fixed points*. If you are not comfortable with all these concepts, review the *Functions* chapter.

We'll take a look at several examples of functional equation problems, illustrating some useful ideas that you can use to solve them.

## The first step: substitute values

When you're given a functional equation problem, there's one thing you can do immediately, and which will likely help you understand the problem: *substitute some values.*

In general, by substituting values which are as *simple* as possible, or which *simplify* the equation, you can obtain some interesting information.

**Example 1**
Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that, for all real numbers $x$ and $y$,

$$f(x^2 + y) = xf(x) + y.$$

# Chapter 6. Number Theory

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|---|
| Is $2^{100} + 3^{100} + 6^{100}$ divisible by 7? |

## The fundamental theorem of arithmetic

Problems concerning the properties of integers are known collectively as *Number Theory*. As you will know from primary school, the fundamental building blocks of the integers are the prime numbers. You should be familiar with the idea of breaking down a number into its prime factors. The *fundamental theorem of arithmetic* tells us that any positive integer greater than 1 can be broken down into a product of prime factors. Furthermore, apart from the ordering of these numbers, this factorisation is unique.

**Example 1**
Factorise 720.

**Solution**
We may start by seeing that 720 is a multiple of 10 so that $720 = 10 \times 72$ and then breaking down 10 and 72 so that we get $720 = 2 \times 5 \times 8 \times 9 = 2 \times 5 \times 2 \times 2 \times 2 \times 3 \times 3$.
Alternatively, we might be more systematic and keep dividing by 2 until we can no longer do so, getting $720 = 2 \times 360 = 2 \times 2 \times 2 \times 2 \times 45 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5$. In both cases (and any other way we try), we finish up with the same factors. A neat and systematic way to present such a factorisation is to write the number as powers of primes in order. So that $720 = 2^4 \times 3^2 \times 5^1$. Of course, we don't really need to show the powers which are just 1 but we do need to remember that they are there!
A useful bonus when we express numbers in this way is that we can tell how many factors the number has by adding 1 on to each exponent and then multiplying the exponents. So that 720 has $5 \times 3 \times 2 = 30$ factors. Can you explain why this works?

## Divisibility

When an integer $a$ divides exactly into an integer $b$, we can use the notation $a|b$ to mean $a$ divides $b$. If $m$ is the largest integer which divides into both $a$ and $b$, we say that $m$ is the highest common factor of $a$ and $b$. The notation used in number theory for this is $(a, b) = m$.
Two integers $a$ and $b$ are said to be coprime if $(a, b) = 1$.

Prove that for all $n \geq r \geq 0$:

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}.$$

## Permutations

In how many ways can three people form a queue?

An obvious way to answer this question is to simply list all possibilities. However, we need to do this systematically so that we can be sure no queue is missed and we don't repeat a queue. We could first choose who goes at the head of the queue. Then, for each person we chose to go first, we choose the person to go second. The only person left must then go last. Let's call the people $R$, $G$, $B$. Thus we get exactly six queues:

$$RGB,\ RBG,\ GRB,\ GBR,\ BRG,\ BGR.$$

When we change the order of objects, we say that we are *permuting* those objects. A *permutation* of objects is an arrangement of those objects in a line. We have just seen that there are six permutations of any three distinct objects. How many permutations of a larger number, say nine objects, are there? Again, we could list them all systematically, but that would take a lot of time.

Instead, let's use some notation to keep track of things. Suppose the number of permutations of $n$ objects is $p_n$. We have $n$ choices for the first object. After each such choice, there are $n-1$ objects left to arrange and that can be done in $p_{n-1}$ ways. So $p_n = np_{n-1}$. This applies for all $n \geq 2$ and of course $p_1 = 1$. So $p_2 = 2p_1 = 2$, $p_3 = 3p_2 = 6$, $p_4 = 4p_3 = 24$, and so on. In general $p_n = n(n-1)(n-2)\cdots(2)(1)$. Hence $p_9 = (9)(8)(7)(6)(5)(4)(3)(2)(1) = 362\,880$.

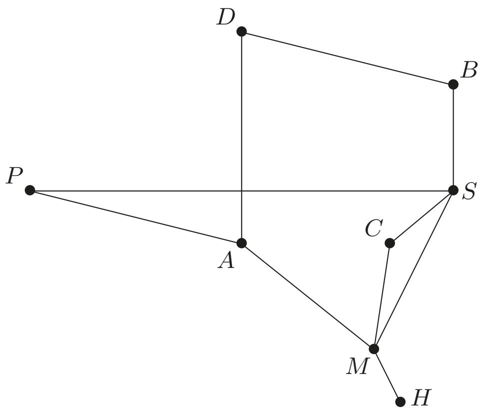
The product $n(n-1)(n-2)\cdots(2)(1)$ often occurs in counting problems so we give it a special symbol and a special name. It is denoted by $n!$ and called $n$ *factorial*.

$$n! = n(n-1)(n-2)\cdots(2)(1)$$

# Chapter 8. Graph Theory

Prove that amongst any six high school students there are three who were born in the same year or three that were born in three different years.

Graphs are used to display and understand relations between elements of a set: for example, atoms in a molecule, websites on the internet, DNA amongst species, cities on airline routes. They consist of *vertices* (or points or nodes) joined by *edges* (or lines or curves). Here is a simple example. Perhaps you can guess what the letters stand for.




Notice that the location of the vertices on the page is not important as long as the edges between vertices remain the same. The next graph is regarded as the same as the one above.