Maths Enrichment

Dirichlet Student Notes

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AUSTRALIAN MATHS TRUST
MATHS ENRICHMENT

Contents

Chapter 1
Mission Possible – Logic

It was 6:00 pm sharp when the team of four, Joanne, Leon, Andrew and Philippa, gathered to receive their new assignment. Joanne flicked the switch on the non-buggable satellite-connected speaker phone to connect them to their headquarters.

'We're all ready Bruce, fire away!'

'Good evening, my champion problem solvers! Your mission, should you choose to accept, is to save the Naccio Cephalopoda. It is a rare and exclusive marine mollusc, one of the last in the world and almost extinct.

'It has been used in a special breeding program and the research has shown it can be helpful to fight deadly viruses.

'It is worth millions and it was stolen two days ago. You will learn more during this assignment. The first task is to find a certain secret undercover agent who has details of the mollusc's whereabouts.

'The agent you are looking for will arrive as an international guest at a dinner being held tonight in town. The agent has worked as a deep-sea diver on the Naccio Cephalopoda research program.

'The dinner is being organised by a prominent person in the town. We do not know who, nor do we know exactly where it is to be held. Our agents have narrowed down the possibilities to the Governor-General, the Mayor, a businessman named Platt, and Captain Smythe, the captain of the research vessel in the port.

'The possible locations of the dinner are Government House, the Pier Mansion, the Hotel Savoy or the Banquet Room at the Town Hall.

'This mission is TOP SECRET. You must find this guest without making it obvious. It is vital you do not reveal yourselves to the wrong person as we don't know who they really are! Your cover is as follows . . . '

'Okay, let's spread out around the town and learn what we can. Be back here in an hour,' said Andrew, who usually led.

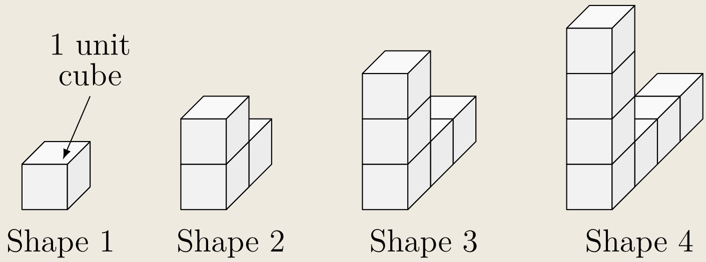
An hour later, they shared the information they had. 'I checked out the Hotel Savoy. I went to the kitchen with a false delivery and they told me there was no dinner there tonight,' said Philippa.

Chapter 2

Working with Patterns

Opening problem

The diagram shows a pattern of L-shapes formed by gluing together a number of unpainted unit cubes. After the cubes are glued, each shape is painted red on all of its faces.



How many squares are painted red on the 100th shape? Which shape is the first to have more than 1000 squares painted red?

You could try to picture the 100th shape to answer the first question, but how would you answer the second? By understanding the pattern!

Making the description simpler

In mathematics we use many symbols to represent words. These save us time and space. For example, the symbol ‘=’ saves us writing ‘equals’ or ‘is equal to’, often many times in the one problem. In the same way, it is much easier to write ‘+’, ‘−’, ‘×’ or ‘÷’ instead of ‘plus’, ‘minus’, and so on. But whether we use symbols or words, a mathematical statement should sound the same when you read it out loud.

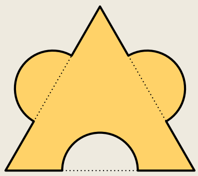
A *sequence* is just a list of objects called *terms*. The terms could be numbers, or shapes or anything else we like! The terms must stay in a fixed order, but they may or may not follow an obvious pattern.

The shortcut we often use to describe the terms of a sequence is ‘T_n’. Here ‘T’ stands for ‘term’ and n, a counting number, tells us the position of that term in the list. So T_1 means term 1 or the 1st term in the sequence, and T_5 means term 5 or the 5th term in the sequence.

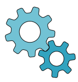
Chapter 3
Tessellations

Opening problem

This shape has been made by adding or removing identical semicircles in the middle of each side of an equilateral triangle.



Can you cover the whole plane without gaps or overlaps using identical tiles of this shape?

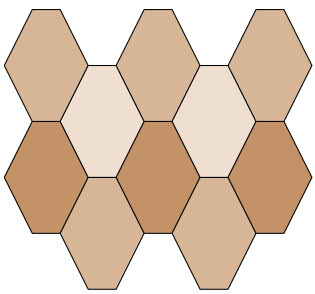


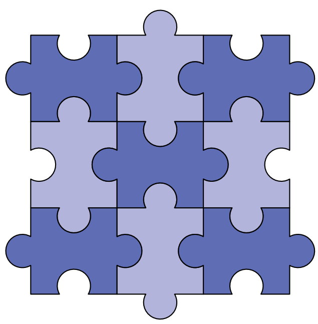
If your answer is ‘yes’, then you need to be sure it works forever in all directions! If your answer is ‘no’, you need to rule out every possibility!

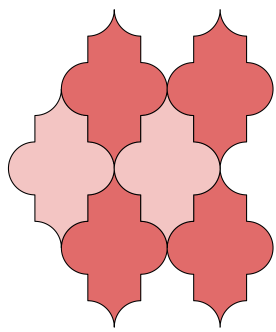
Some definitions

Tessellations: A *tessellation* is a pattern of 2D shapes with no gaps or overlaps which can be continued as far as you wish in all directions. If this is possible, we say that the shapes *tessellate*.

Think of patterns of tiles on a floor or jigsaw pieces in a puzzle:







In practice, the above tilings would need to stop eventually. Some of the pieces might even need to be cut off to make a straight edge, where the floor meets a

Chapter 4
Mission Possible – A Simpler Problem

'What an extraordinary dessert!' exclaimed Mr Lee. 'My compliments to the chef, Governor-General.'

'Excuse me, Miss Manego,' interrupted Leon. 'A phone call for you. You may take it in the study if you like.'

Leon directed Miss Manego to the study where the rest of the team awaited her.

'Miss Manego, there is no call for you,' Philippa explained, 'but we believe you have information for us.'

'Oh terrific! I've been waiting all night. I thought you'd approach me earlier.'

'Well, we wanted you to enjoy your meal first,' added Joanne quickly.

'A fantastic meal it was!' she exclaimed as the team turned and congratulated Andrew on his culinary skills. 'Let's get back to business now. The Naccio Cephalopoda has been kept captive in a warehouse behind the Pier Mansion. This map shows you how to get to the room it's in, using the secret passages. I have the key to get you through the doors. The main concern is the alarm system in the room which holds the Naccio Cephalopoda. Only those who know how it works are able to move through the room without sounding the alarm.'

'Do you have the code?' asked Philippa eagerly.

'No, but I do have a detailed account of how the alarm system works which I can tell you. After that it is up to you to find a way through,' said Miss Manego.

'We're all ears! Tell us what you know,' said Leon as the team of four gathered around to listen carefully to everything she said.

'The room is 100 metres long and has a door on both ends. The Naccio Cephalopoda is in a tank in the middle of the room.

- Following the map, you will enter through one door and exit from the other.
- At 1 metre intervals from the entrance you will encounter a thin wall of motion sensors. There is a separate alarm connected to each of the sensors.

Chapter 5

Counting Techniques

Opening problem

A *googol* is a 1 followed by 100 zeros. It is the number you get when you multiply 100 tens together. We write this as 10^{100} and say '10 to the power of 100'.

Neve just turned 12, so she says a number is *special* if each of its digits is a 1 or a 2. Show that the fraction of whole numbers from 1 to a googol that are special is

$$\frac{2}{5^{100}} - \frac{2}{10^{100}}.$$

Listing the special numbers is definitely not recommended! But there is enough structure in this problem to do the counting more efficiently.

In how many ways . . . ?

Problem solvers are often interested in questions such as

What proportion of . . .?

What is the probability that . . .?

In how many ways can . . .?

Very occasionally a problem such as this is small enough to be attacked with brute force: just list all of the possibilities you can think of and see what you get. For example, maybe you could list all of the outcomes for this question:

In how many ways can the first 2 *horses be placed if there are* 4 *in the race?*

But what if there are 20 horses and we are interested in the first 5 places?

Problems like this can quickly get out of hand and require more systematic thought.

Some solutions in this chapter may seem ingenious at first – and indeed many of them are! But, to paraphrase the eminent problem solver George Pólya,

> *when we can apply ingenious methods again and again to solve similar problems, that's when we have developed a 'technique'.*

Chapter 6
Primes and Composites

Opening problem

A code is devised by Secret Sam. The letter 'A' is represented by the smallest number with 1 factor, 'B' by the smallest number with 2 factors and so on.

Sam sets out the code as shown:

A	1
B	2
C	4
D	6
E	16
⋮	⋮
Z	12 288

Complete the table.

Maybe A–E could be found with a little guess and check, but where did Z come from? Certainly *not* by checking all 12 287 numbers before it!

Factors and primes

In this chapter we are only interested in *natural numbers* 1, 2, 3, 4, . . . You may also know these as 'whole numbers', 'counting numbers' or 'positive integers'.

The factors of 6 are 1, 2, 3 and 6.

The factors of 28 are 1, 2, 4, 7, 14 and 28.

The factors of 7 are 1 and 7.

In general, a natural number a is a *factor* of a natural number b if there exists a third natural number k such that

$$b = a \times k$$

(often written $b = ak$ without the multiplication symbol).

Chapter 7
Mission Possible – Working Backwards

As the team arrived at the Marine Sanctuary, Miss Manego met them there as planned.

‘You made it! You persisted with your problem-solving skills and you have achieved success. Well done!’ she exclaimed.

‘We’re nearly there Miss Manego. If you could please tell us what we need to do to get the Naccio Cephalopoda back into its shell, then when that’s done we can celebrate,’ said Leon.

‘Okay, here’s what you need to know. The Naccio Cephalopoda’s shell is built much like a Nautilus shell, in a continuing spiral starting from the centre. The spiral has 8 chambers, each larger than the one before it. That is, it makes 8 quarter turns. The lengths of the chambers fit a Fibonacci sequence.’

‘I know, like 1, 1, 2, 3, 5, 8, . . . ’ interrupted Leon excitedly.

‘That’s correct,’ added Miss Manego. ‘To find the next number you need to add the previous two numbers together. The last two you said were 5 and 8, Leon, so the next number is . . . ’

‘13,’ jumped in Andrew.

‘The problem is we can only measure the outer chamber. We have found it to be 118 mm. To accurately return the Naccio Cephalopoda to the inner part of the shell, this robotic arm needs to have the precise measurements of the lengths of the chambers, so it’s up to you to find them. You’ll need to be quick. By my calculations the Naccio Cephalopoda will not survive much longer than another five minutes out of its shell.’

The team gathered together.

‘Okay, let’s review the key information. We have 8 chambers. The length of the third is the sum of the lengths of the first and second. The length of the fourth is the sum of the lengths of the second and third, until we reach the eighth whose length is the sum of the lengths of the sixth and seventh. The only one we know is the eighth and this is 118 mm. How do we tackle this?’ asked Andrew.

‘Perhaps we could guess different numbers at the start and use a guess-and-check method to see if we get 118 by the eighth turn,’ suggested Joanne.

Chapter 8

Clock Arithmetic

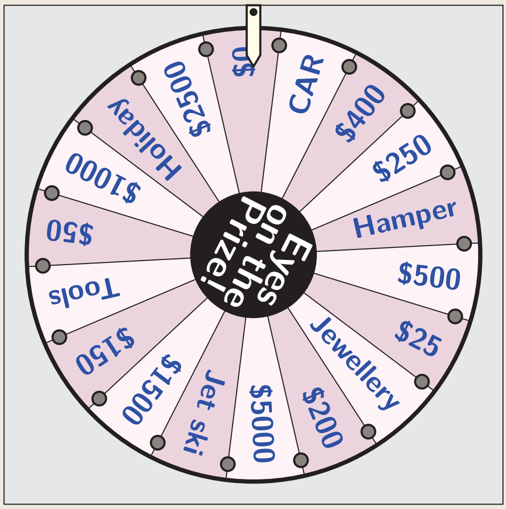
Opening problem

There are 18 prizes in the game show *Eyes on the Prize*, ranging from \$0 to the sports car.

Contestants earn spin tokens in the game. The final winner uses as many of their tokens as they like to land on a decent prize.

Daryl is a pro. He can spin the wheel for exactly 29 clicks every time, in either direction.

Starting on the \$0 prize, how many spins does Daryl need to win the car?



It might not take too long to work out what happens after each spin, but let's explore some techniques to simplify this type of problem.

Unusual clocks

Imagine a clock. This clock has 0 at the top instead of 12. If you know anything about railway or 24-hour time, maybe this isn't too strange. But now imagine this clock starts at zero o'clock and only goes up to six o'clock. Now *that's* strange! Why would we want such a thing?

The ideas in this chapter were first developed by the German mathematician and physicist Carl Friedrich Gauss (1777–1855). He invented a new kind of arithmetic which we call *clock arithmetic* or *modular arithmetic*. These ideas are important in many different areas of modern life, from calendars to internet security.

To warm up, suppose that today is a Friday. What day will it be in 57 days?