Noether Enrichment Stage

Table of Contents

Chapter 1. Problems I Enjoy - George Harvey

1. Andrew arranged the numbers $1, 2, 3, \ldots, 11, 12$ into six pairs so that the sum of any two numbers in a pair is prime and no two of these primes are equal. Find the primes and the pairs.

2. A circus performance was attended by 120 people who paid a total of \$120. The men paid \$5, the women \$2 and the children 10 c. How many men, how many women and how many children went to the circus?

3. There are three boxes, containing respectively:

two black marbles (BB)
two white marbles (WW)
one black and one white marble (BW).

The boxes are initially correctly labelled according to their contents: BB, WW and BW. The labels are then switched in such a way that every box is now incorrectly labelled. The contents of all three boxes is to be determined by drawing one marble at a time (noting its colour). What is the smallest number of drawings needed to do this?

4. "How old are your three children?" the mathematics master asks a former student. He is told their ages add to 13, and multiply to give the number on his study door (which they both can see). "I will need to know more", the master says, after a few moments reflection. "The eldest one is learning to play the violin", replies his former pupil. "Ah! In that case I can now give you their ages", the master tells him, and does so correctly.

How does he know? What are their ages? What is the number on the door?

5. A number is written on a blackboard. You may perform the following operations one after the other in any order as often as you wish.

(i) Replace the current number on the blackboard with twice this number.
(ii) Delete the last digit of the current number.

Can you obtain 14 from a starting number of 458?

6. Using each of the ten digits 0, 1, 2, ..., 9 just once, is it possible to form positive integers whose sum is exactly 100?

7. The only timepiece owned by a man was a wall clock which had stopped because he had forgotten to wind it. In the afternoon, he wound it while it was showing (incorrectly) 1 o'clock, then walked to a friend's place and noted that the (correct) time was 3 pm. He left his friend's place at $5:30$ pm that evening, taking the same route home and walking at the same average speed. On arrival, his clock showed $5:30$ pm. At what (correct) time did he arrive home?

8. How many differently shaped rectangles, with positive integer dimensions, have a perimeter equal to their area?

9. Let x be any number less than 1, and let y be any number greater than 1. Let S be the sum of x and y, and let P be the product of x and y. Prove that the difference between S and P must be greater than 1.

10. Prove that it is impossible to find four distinct numbers p, q, r and s which satisfy the equation

$$pq + rs = ps + qr .$$

Now try Problems 1 and 2 in the Noether Student Problems Book.

Chapter 2. Expansion and Factorisation

Find the value of the expression

$$\frac{a^{32} - b^{32}}{(a^2 + b^2)(a^4 + b^4)(a^8 + b^8)(a^{16} + b^{16})} + 12b$$

if the only thing which is known about a and b is that $a + b = 6$. Show that the answer does not depend on the way a and b satisfying $a + b = 6$ have been chosen.

This chapter is about how to expand and factorise algebraic expressions in order to simplify them or to make them more manageable when problem solving. Generally speaking, expanding means transforming an algebraic expression so that it does not contain any grouping symbols. Factorising is, in some sense, the reverse of expanding, that is, factorising means representing an algebraic expression as a product of two or more factors. Usually, expanding is easier to master than factorising, so expanding is the first thing we study in this chapter.

Getting Rid of Grouping Symbols

Here are two formulae that are most commonly used for expanding algebraic expressions.

Formula 1. $a(b + c) = ab + ac$

Formula 2. $a(b - c) = ab - ac$

Note. These two formulae are very helpful but they are very simple as well. Therefore, in order to use them effectively, one needs to learn how to apply the formulae to complicated algebraic expressions.

Example 1. Expand the following algebraic expressions.

(a) $5p(3p - 2q)$;

(b) $x(y + z + t)$;

(c) $(2m - 3n)(4nm - n^2)$;

(d) $(1 + 5x + x^2)(1 - 5x)$.

Solution.

(a) To see how **Formula 2** can be used for expanding the expression $5p(3p-2q)$, we put $5p = a$, $3p = b$ and $2q = c$.
Then $5p(3p-2q) = a(b-c) = ab - ac = 5p \times 3p - 5p \times 2q$.
But $5p \times 3p = 5 \times p \times 3 \times p = 5 \times 3 \times p \times p = 15p^2$.
And also $5p \times 2q = 5 \times p \times 2 \times q = 5 \times 2 \times p \times q = 10pq$.
Therefore, $5p(3p-2q) = 5p \times 3p - 5p \times 2q = 15p^2 - 10pq$.

(b) Now we put $x = a$, $y = b$ and $z + t = c$.
Then, by **Formula 1**, $x(y+z+t) = a(b+c) = ab + ac = xy + x(z+t) = xy + xz + xt$.
Thus, $x(y+z+t) = xy + xz + xt$.

(c) By **Formula 2**,

$$\begin{aligned}
&(2m-3n)(4mn-n^2)\\
&\qquad = (2m-3n)4mn - (2m-3n)n^2\\
&\qquad = 2m \times 4mn - 3n \times 4mn - 2m \times n^2 + 3n \times n^2\\
&\qquad = 8m^2n - 12mn^2 - 2mn^2 + 3n^3 .
\end{aligned}$$

But $-12mn^2 - 2mn^2 = (-12) \times mn^2 + (-2) \times mn^2 = ((-12)+(-2)) \times mn^2 = -14mn^2$ in view of **Formula 1**.
Hence $(2m-3n)(4mn-n^2) = 8m^2n - 12mn^2 - 2mn^2 + 3n^2 = 8m^2n - 14mn^2 + 3n^3$.

(d) Briefly, the process of expanding and then simplifying the expression $(1+5x+x^2)(1-5x)$ can be shown in the following way.

$$\begin{aligned}
&(1+5x+x^2)(1-5x)\\
&\qquad = (1+5x+x^2)1 - (1+5x+x^2)5x\\
&\qquad = 1 + 5x + x^2 - 1 \times 5x - 5x \times 5x - x^2 \times 5x\\
&\qquad = 1 + 5x + x^2 - 5x - 25x^2 - 5x^3\\
&\qquad = 1 + (5x - 5x) + (x^2 - 25x^2) - 5x^3\\
&\qquad = 1 - 24x^2 - 5x^3 .
\end{aligned}$$

Exercises

1. Expand each of the following:

(a) $-3x(1+6y)$;

(b) $(a-2b)(a+2b)$;

(c) $(2-9x+x^2)(1+x)$;

(d) $(a-c)(c-2a)(3a+c)$;

(e) $(2+x-x^2)^2$.

Now we can start studying the main ways of factorising algebraic expressions.

Using Common Factors

If all terms of an algebraic expression have a common factor, then we can put the common factor outside a pair of parentheses surrounding the rest of the expression without the common factor according to **Formula 1** and **Formula 2**. It is important to remember that any mathematical formula can be read in two different ways, that is "from left to right" as well as "from right to left". For example, when **Formula 1** is read as $a(b+c)=ab+ac$, it provides a rule for expanding algebraic expressions, but when it is read as $ab+ac=a(b+c)$, it shows how to factorise by using common factors.

Example 2. Factorise the following algebraic expressions.

(a) $4m-6n$;

(b) $14x^2y^4+21x^3y^3$;

(c) $10u^4v^2w^5+15u^2v^3w-25u^5v^4w^2$;

(d) $(c-3d)b-(c-3d)4a$;

(e) $(x+y)a-(x+y)b^2+2(2x+2y)c$.

Solution

(a) First of all, we notice that $4m-6n=2\times 2m-2\times 3n$.
Now we see that both terms $4m$ and $6n$ of the expression in question have common factor 2.
Therefore, putting $a=2$, $b=2m$ and $c=3n$, we have
$4m-6n=2\times 2m-2\times 3n=ab-ac=a(b-c)=2(2m-3n)$ by **Formula 2.**
Thus $4m-6n=2(2m-3n)$.

(b) $14x^2y^4+21x^3y^3=7x^2y^3\times 2y+7x^2y^3\times 3x$, therefore $7x^2y^3$ is a common factor for all terms of the expression $14x^2y^4+21x^3y^3$ and we can complete the factorising in the following manner.
$14x^2y^4+21x^3y^3=7x^2y^3\times 2y+7x^2y^3\times 3x=7x^2y^3(2y+3x)$.

(c) $10u^4v^2w^5=5u^2v^2w\times 2u^2w^4$, $15u^2v^3w=5u^2v^2w\times 3v$, $25u^5v^4w^2=5u^2v^2w\times 5u^3v^2w$ and we see that $5u^2v^2w$ is a common factor for all terms of the expression $10u^4v^2w^5+15u^2v^3w-25u^5v^4w^2$.